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Scattering Theory of Lax-Phillips for Isotropic Elastic Wave Equations in a Perturbed Half Space

広島大・理　川下　美潮 (Kawashita, Mishio)
Graduate School of Science, Hiroshima University

広島大・総合（非常勤）川下　和日子 (Kawashita, Wakako)
Faculty of Integrated Arts and Sciences(temporary)
Hiroshima University

茨城大・教育　曽我　日出夫 (Soga, Hideo)
Faculty of Education, Ibaraki University

1 Introduction

There are many works discussing formulations of the scattering theories. For hyperbolic problems, we have two types of the formulations, i.e. the Lax and Phillips type and the Wilcox type. The Wilcox type formulation is based on the spectral theory of the self-adjoint operators which is derived from scattering problems arizing quantum mechanics. Hence this formulation is called "Schrödinger methods for the acoustic scattering" (cf. [**16**], [**13**]).

On the other hand Lax and Phillips [**7**] proposed a different formulation of scattering theories for hyperbolic differential equations. In this theory, considering one parameter family of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ on some Hilbert space H, they introduced concept of the outgoing subspace D_+ and incoming subspace D_- of $U(t)$ and showed that $U(t)$ can be regarded as translation. Namely, corresponding to $D_\pm$, there exist a separable Hilbert space N and a unitary operator $T^\pm$ from H to $L^2(\mathbb{R};N)$ such that

$$\begin{aligned}&T^\pm U(t)(T^\pm)^{-1}=\tau_t,\\ &T^\pm(D_\pm)=\{k(s)\in L^2(\mathbb{R};N); k(s)=0 \text{ in } \pm s<0\},\end{aligned}$$

where $\tau_t : k(s)\mapsto k(s-t)$. Conversely, for a translation representation $T^\pm$ with respect to $\{U(t)\}_{t\in\mathbb{R}}$ stated in the above, we can define the corresponding subspaces $D_\pm$. From these operators $T^\pm$, the scattering operator is defined by $T^+(T^-)^{-1}$, which is considered as a mathematical representation of scattering states. Thus, "translation" and "outgoing (resp. incoming) subspace" are the basis of the scattering theory of Lax and Phillips.

Furthermore, Lax and Phillips applied their theory to the (scalar-valued) wave equation. They expressed the solutions as a unitary group $U(t)$ on the energy space H. They made the translation representations such that the corresponding outgoing and incoming subspaces $D_\pm$ are characterized by

$$D_\pm = \{\vec{\mathbf{f}} = {}^t(f_1, f_2) \in H\,;\; U(t)\vec{\mathbf{f}} = 0 \text{ in } |x| \le \pm t\}. \tag{1.1}$$

This plays an essential role in the Lax and Phillips theory for the wave equation. They made concretely the translation representations $T^\pm$ with (1.1), based on the Radon transform $f(x) \mapsto \int_{x\cdot\omega=s} f(x)\,dS_x$.

For other hyperbolic equations or systems, many authors have formulated the scattering theories of the Lax and Phillips type (cf. Lax and Phillips **[8]**, Soga **[15]**, Petkov **[11]**, Yamamoto **[17]**, Shibata and Soga **[14]**, etc). All their theories are treated scattering theories for perturbed problem of hyperbolic differential equations in the whole Euclidian space $\mathbb{R}^n$. In these cases, the outgoing (resp. incoming) subspace has the similar property to (1.1), and the translation representations are given by convination of the variants of the Radon transform.

In this article we consider the formulation of the scattering theory of Lax and Phillips for the isotropic elastic wave equation with the Neumann boudnary condition in a perturbed domain $\Omega \subset \mathbb{R}^3$ from the half space $\mathbb{R}^3_+ = \{\mathbf{x} = (x_1, x_2, x_3) = (\mathbf{x}', x_3) \in \mathbb{R}^3; x_3 > 0\}$. Thus we assume that the boundary $\partial\Omega$ is smooth and satisfies that $\partial\Omega \cap \{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x}| > R_0\} = \{\mathbf{x} \in \partial\mathbb{R}^3_+; |\mathbf{x}| > R_0\}$ for some R_0. Let $\rho(\mathbf{x})$ be the density of the solid, $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ be the Lamé functions satisfying

$$\rho(\mathbf{x}) > 0, \quad \lambda(\mathbf{x}) + 2\mu(\mathbf{x})/3 > 0, \quad \mu(\mathbf{x}) > 0 \qquad \text{for any } \mathbf{x} \in \bar{\Omega},$$

where $\lambda(\mathbf{x}) = \lambda_0 + \tilde{\lambda}(\mathbf{x})$, $\mu(\mathbf{x}) = \mu_0 + \tilde{\mu}(\mathbf{x})$, $\rho(\mathbf{x}) = \rho_0 + \tilde{\rho}(\mathbf{x})$ and $\tilde{\lambda}(\mathbf{x}), \tilde{\mu}(\mathbf{x}), \tilde{\rho}(\mathbf{x}) \in C_0^\infty(\bar{\Omega} \cap \{|\mathbf{x}| < R_0\})$, and λ_0, μ_0 and ρ_0 are the constants. For the displacement field $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) = (u_1(t, \mathbf{x}), u_2(t, \mathbf{x}), u_3(t, \mathbf{x}))$ of the solid at time t and position $\mathbf{x}$, elasticity gives the following mixed problem:

$$\begin{cases} (\rho(\mathbf{x})\partial_t^2 - \mathcal{A}(x, \partial_\mathbf{x}))\mathbf{u}(t, \mathbf{x}) = \mathbf{0} & \text{in } \mathbb{R} \times \Omega, \\ \mathcal{N}(\mathbf{x}, \partial_\mathbf{x})\mathbf{u}(t, \mathbf{x}) = \mathbf{0} & \text{on } \mathbb{R} \times \partial\Omega, \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{f}_1(\mathbf{x}), \;\; \partial_t\mathbf{u}(0, \mathbf{x}) = \mathbf{f}_2(\mathbf{x}) & \text{on } \Omega, \end{cases} \tag{1.2}$$

where

$$\mathcal{A}(\mathbf{x}, \partial_\mathbf{x})\mathbf{u} = \sum_{i,j=1}^{3} \partial_{x_i}\left(a_{ij}(\mathbf{x})\partial_{x_j}\mathbf{u}\right) \qquad \mathcal{N}(\mathbf{x}, \partial_\mathbf{x})\mathbf{u} = \sum_{i,j=1}^{3} \nu_i(\mathbf{x})a_{ij}(\mathbf{x})\partial_{x_j}\mathbf{u}\,\big|_{\partial\Omega}\,.$$

In $\mathcal{A}(\mathbf{x}, \partial_{\mathbf{x}})$ the coefficients $a_{ij}(\mathbf{x})$ are $n \times n$-matrices and their (p,q)-components $a_{ipjq}(\mathbf{x})$ are given by the Lamé functions of the forms: $a_{ipjq}(\mathbf{x}) = \lambda(\mathbf{x})\delta_{ip}\delta_{jq} + 2\mu(\mathbf{x})(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp})$, where δ_{ij} are Kronecker's delta. The boundary operator $\mathcal{N}(\mathbf{x}, \partial_{\mathbf{x}})$ is the conormal derivative of $\mathcal{A}(\mathbf{x}, \partial_{\mathbf{x}})$ and $\nu(\mathbf{x}) = (\nu_1(\mathbf{x}), \nu_2(\mathbf{x}), \nu_3(\mathbf{x}))$ is the unit outer normal vector at $\mathbf{x} \in \partial\Omega$.

For our perturbed problem, we choose the case of the half space as the free problem in the scattering theory. Even in the free case we have surface waves which are different from waves traveling inside of the elastic medium. In our case, there are two types of the surface waves. One is coressponding to the total refrection phenomena, which are called the evanesent waves. These are caused by the existence of the several waves having different speeds. The other waves are called the Rayleigh surface waves which are concrete ones to the Neumann boundary condition in elastic wave equation. This is one of the main differences between the whole space case and our one.

Even though there are the surface waves, the scattering theory of the Wilcox type are developed by the similar methods to the various perturbed problem from the whole space $\mathbb{R}^n$ (cf. Dermenjian and Guillot [**1**]). Contrary, for formulating the scattering theory of Lax and Phillips, existence of the surface waves makes differences between the whole space case and our case, and also causes new difficulty. Since as is in [**5**] we have the both scattering theories are the same each other in an abstract sense, we can construct some translation representations using generalized Fourier transforms which provides one of the key concepts of the theory of Wilcox type (cf. Theorems 4.1 in [**5**]). Using these facts, we also have the translation representation of the concrete forms for the free case in our problem (cf. §3 or, §6 in [**5**]). The obtained translation representation consists of not only the terms of the Radon transform but also the other terms. These additional terms are written using the Poisson integrals and the Hilbert transforms. They are ones of the influence of the surface waves and of the differeces between the free case of the half space and the one of the whole space.

The other difference which causes difficulty is that the corresponding outgoing (resp. incoming) subspace does not have the similar property to (1.1) (cf. Theorem 3.2 and Proposition 3.4). This is rather serious since in the theory of Lax and Phillips the property (1.1) plays an essential role to make the translation representation for the perturbed case. Thus it seemd to be difficult to develop the theory according Lax and

Phillips straightforwardly. As Phillips [12] pointed out, this kind of difficulty arized even in a short range perturbation from the whole Euclidian space though the free case has the property (1.1). He overcomed this considering the relation between the wave operators and the subspaces $D_{\pm}$.

Recently in [2] Ikawa gave a proof of the completeness of the wave operator using a weak version of the decomposition given by Morawetz. The original decomposition was proposed in the famous argument of Morawetz to obtain the rate of the uniform decay of the energy of the solution of the wave equation (cf. [10]). The argument by Ikawa has an advantage that we can avoid contradiction arguments and gives a procedure for constructing the preimage of the wave operator. Unfortunately, we can not apply Ikawa's argument directly since the decomposition of Morawetz requires Huygens Principle which holds in the case of the odd (≥ 3) dimensional space. In our case, we have neither Huygens Principle nor the similar property to (1.1). Still we can obtain some modification of the decomposition of Morawetz which is useful to apply the argument of Ikawa. Hence we also have the completeness of the wave opertor which is the same as the existence of the outgoing (resp. incoming) subspaces. Thus in this sense we can develop the scattering theory of Lax and Phillips to our problem.

To obtain the decomposition similar to the one of Morawetz type we use essentially the translation representation for the free case. This idea was first introduced by Melrose [9] to show the uniform energy decay of the solutions of the wave equation for nontrapping obstacle case. For the original decaying problem considered by Morawetz we also have some good decomposition even in the case of elastic wave equation (cf. [3]). Hence using the translation representations we can determine the rate of the uniform decay if we assume that the uniform decay property holds. Thus we can say the property (1.1) is more essential than Huygens priciple to develop the Morawetz argument. But these arguments can not be applied to show the completeness of the wave opeator since the completeness result are used in [9] and [3] essentially. Hence we need to modify the arguments according to showing the completeness. In this article we give only the outline. The details are disussed in the forthcoming paper.

2 Generalized Fourier tranform for the free case

Before going to the construction of the trasnlation representation, we consider the generalized Fourier transform of the free space problem. The generalized Fourier transform is one of the basis of the scattering theory of Wilcox type. Since our purpose is to construct the translation representation, we use the generalized Fourier transform modified the one obtained by Dermenjian and Guillot [**1**] (cf. see [**5**]).

The free space problem for our problem is given by the following mixed problem:

$$
(2.1) \qquad \begin{cases} (\rho_0\partial_t^2 - \mathcal{A}_0(\partial_{\mathbf{x}}))\mathbf{u}(t,\mathbf{x}) = \mathbf{0} & \text{in } \mathbb{R}\times\mathbb{R}^3_+, \\ \mathcal{N}_0(\partial_{\mathbf{x}})\mathbf{u}(t,\mathbf{x}) = \mathbf{0} & \text{on } \mathbb{R}\times\partial\mathbb{R}^3_+, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{f}_1(\mathbf{x}),\ \partial_t\mathbf{u}(0,\mathbf{x}) = \mathbf{f}_2(\mathbf{x}) & \text{on } \mathbb{R}^3_+, \end{cases}
$$

where $\mathcal{A}_0(\partial_{\mathbf{x}})\mathbf{u} = \sum_{i,j=1}^3 \partial_{x_i}(a^0_{ij}\partial_{x_j}\mathbf{u})$ and $\mathcal{N}_0(\partial_{\mathbf{x}})\mathbf{u} = \sum_{i,j=1}^3 \nu^0_i a^0_{ij}\partial_{x_j}\mathbf{u}|_{\partial\mathbb{R}^3_+}$ ($\nu^0 = {}^t(\nu^0_1,\nu^0_2,\nu^0_3) = {}^t(0,0,-1)$). In the above, the coefficients a^0_{ij} are 3×3-matrices whose (p,q)-components a^0_{ipjq} are given by $a^0_{ipjq} = \lambda_0\delta_{ip}\delta_{jq} + 2\mu_0(\delta_{ij}\delta_{pq}+\delta_{iq}\delta_{jp})$. In this case the operator $\mathcal{A}_0(\partial_{\mathbf{x}})$ is of the following well-known form: $\mathcal{A}_0(\partial_{\mathbf{x}})\mathbf{u} = (\lambda_0+\mu_0)\nabla_{\mathbf{x}}(\mathrm{div}\ \mathbf{u}) + \mu_0\Delta\mathbf{u}$. The velocity $c_P = \sqrt{(\lambda_0+2\mu_0)/\rho_0}$ of P-waves is larger than that $c_S = \sqrt{\mu_0/\rho_0}$ of S-waves and the velocity c_R of R-waves is smaller than c_P and c_S.

We formulate the problem (2.1) as an abstract form by introducing the Hilbert space $\mathcal{H}_0 = L^2(\mathbb{R}^3_+;\mathbb{C}^3,\rho_0 d\mathbf{x})$ with the norm $\|\mathbf{f}\|_{\mathcal{H}_0} = \{\int_{\mathbb{R}^3_+}|\mathbf{f}(\mathbf{x})|^2\rho_0 d\mathbf{x}\}^{1/2}$ and the self-adjoint operator A_0 with the domain $D(A_0)$ on $\mathcal{H}_0$ defined by

$$
(2.2) \qquad A_0\mathbf{u} = -\rho_0^{-1}\mathcal{A}_0(\partial_{\mathbf{x}})\mathbf{u} \quad \text{for } \mathbf{u}\in D(A_0) = \{\mathbf{u}\in H^2(\mathbb{R}^3_+;\mathbb{C}^3);\ \mathcal{N}_0(\partial_{\mathbf{x}})\mathbf{u} = 0\}.
$$

For the operator A_0 we introduce the generalized eigenfunctions consisting of the five parts ϕ_0^P, ϕ_0^{SV}, ϕ_0^{SVO}, ϕ_0^{SH} and ϕ_0^R. Each ϕ_0^α ($\alpha\in\Lambda' = \{P, SV, SVO, SH\}$) is sum of the incident wave $\phi_0^{\alpha,i}$ and the reflected (or the totally reflected) wave $\phi_0^{\alpha,r}$. Note that We choose these ϕ_0^α different from the ones in Dermenjian and Guillot [**1**]. This differences is important to make the translation representation in the theory of Lax and Phillips. The eigenfunctions ϕ_0^α are defined as the following way:

We set $S^2_P = S^2_{SH} = S^2_+ = \{\omega = {}^t(\omega',\omega_3)\in S^2\,;\ \omega_3\geq 0\}$, $S^2_{SV} = \{\omega\in S^2_+\,;\ |\omega'|\leq \frac{c_S}{c_P}\}$, $S^2_{SVO} = \{\omega\in S^2_+\,;\ |\omega'|\geq\frac{c_S}{c_P}\}$ and $S^2_R = \{\zeta\in R^2\,;\ |\zeta| = 1\}$. We define the functions $\phi_0^{\alpha,i} = \phi_0^{\alpha,i}(\mathbf{x};\sigma,\omega)$ ($\alpha\in\Lambda'$, $\sigma\in\mathbb{R},\omega\in S^2_\alpha$) by

$$
\begin{aligned}
\phi_0^{P,i}(\mathbf{x};\sigma,\omega) &= e^{i\sigma c_P^{-1}\breve{\omega}\cdot\mathbf{x}}\mathbf{a}_P(\breve{\omega}),\ \phi_0^{SVO,i}(\mathbf{x};\sigma,\omega) = \frac{\Delta_+^{SVO}(\sigma,\omega)}{\Delta^{SVO}(\omega)}e^{i\sigma c_S^{-1}\breve{\omega}\cdot\mathbf{x}}\mathbf{a}_{SV}(\breve{\omega}),\\
\phi_0^{SV,i}(\mathbf{x};\sigma,\omega) &= e^{i\sigma c_S^{-1}\breve{\omega}\cdot\mathbf{x}}\mathbf{a}_{SV}(\breve{\omega}),\quad \phi_0^{SH,i}(\mathbf{x};\sigma,\omega) = e^{i\sigma c_S^{-1}\breve{\omega}\cdot\mathbf{x}}\mathbf{a}_{SH}(\breve{\omega}),
\end{aligned}
$$

where $\check{\omega} = {}^t(\omega_1, \omega_2, -\omega_3)$, $\omega' = {}^t(\omega_1, \omega_2)$, $\mathbf{a}_P(\xi) = \xi = {}^t(\xi', \xi_3)$, $\mathbf{a}_{SV}(\xi) = {}^t(-\frac{\xi_3}{|\xi'|}\xi', |\xi'|)$, $\mathbf{a}_{SH}(\xi) = \frac{1}{|\xi'|}{}^t(-\xi_2, \xi_1, 0)$,

$$\begin{aligned}
\Delta^{SVO}_{\pm}(\sigma,\omega) &= (\frac{c_P}{c_S})^2\left(1-2|\omega'|^2\right)^2 \pm 4\frac{i\sigma}{|\sigma|}\frac{c_P}{c_S}|\omega'|^2\omega_3\eta(\omega),\ \eta(\omega) = \sqrt{(\frac{c_P}{c_S})^2|\omega'|^2-1},\\
\Delta^{SVO}(\omega) &= |\Delta^{SVO}_{\pm}(\sigma,\omega)| = \sqrt{(\frac{c_P}{c_S})^4(1-2|\omega'|^2)^4 + 16(\frac{c_P}{c_S})^2|\omega'|^4\omega_3^2\eta(\omega)^2}.
\end{aligned}$$

The functions $\phi_0^{\alpha,r} = \phi_0^{\alpha,r}(\mathbf{x};\sigma,\omega)$ are defined of the forms:

$$\begin{aligned}
\phi_0^{P,r}(\mathbf{x};\sigma,\omega) &= -\frac{\Delta^P_-(\omega)}{\Delta^P_+(\omega)}e^{i\sigma c_P^{-1}\check{\omega}\cdot\mathbf{x}}\mathbf{a}_P(\check{\omega}) - \frac{\tilde{\Delta}^P(\omega)}{\Delta^P_+(\omega)}e^{i\sigma c_S^{-1}\xi^P(\omega)\cdot\mathbf{x}}\mathbf{a}_{SV}(\xi^P(\omega)),\\
\phi_0^{SV,r}(\mathbf{x};\sigma,\omega) &= -\frac{\tilde{\Delta}^{SV}(\omega)}{\Delta^{SV}_+(\omega)}e^{i\sigma c_P^{-1}\xi^{SV}(\omega)\cdot\mathbf{x}}\mathbf{a}_P(\xi^{SV}(\omega)) - \frac{\Delta^{SV}_-(\omega)}{\Delta^{SV}_+(\omega)}e^{i\sigma c_S^{-1}\check{\omega}\cdot\mathbf{x}}\mathbf{a}_{SV}(\check{\omega}),\\
\phi_0^{SH,r}(\mathbf{x};\sigma,\omega) &= e^{i\sigma c_S^{-1}\check{\omega}\cdot\mathbf{x}}\mathbf{a}_{SH}(\check{\omega}),\\
\phi_0^{SVO,r}(\mathbf{x};\sigma,\omega) &= -\frac{\tilde{\Delta}^{SVO}(\omega)}{\Delta^{SVO}(\omega)}e^{i\sigma c_S^{-1}\omega'\cdot\mathbf{x}'}e^{-|\sigma|c_P^{-1}\eta(\omega)x_3}\mathbf{a}_P(\xi^{SVO}(\sigma,\omega))\\
&\qquad -\frac{\Delta^{SVO}_-(\sigma,\omega)}{\Delta^{SVO}(\omega)}e^{i\sigma c_S^{-1}\check{\omega}\cdot\mathbf{x}}\mathbf{a}_{SV}(\check{\omega}),
\end{aligned}$$

where $\mathbf{x}' = {}^t(x_1, x_2)$, $\xi^P(\omega) = {}^t\left(\frac{c_S}{c_P}\omega', \xi_3^P(\omega)\right)$, $\xi^{SV}(\omega) = {}^t\left(\frac{c_P}{c_S}\omega', \xi_3^{SV}(\omega)\right)$, $\xi^{SVO}(\sigma,\omega) = {}^t\left(\frac{c_P}{c_S}\omega', \frac{i\sigma}{|\sigma|}\eta(\omega)\right)$, $\xi_3^P(\omega) = \sqrt{1-(\frac{c_S}{c_P})^2|\omega'|^2}$, $\xi_3^{SV}(\omega) = \sqrt{1-(\frac{c_P}{c_S})^2|\omega'|^2}$,

$$\begin{aligned}
\Delta^P_{\pm}(\omega) &= (\frac{c_P}{c_S})^2\left(1-2(\frac{c_S}{c_P})^2|\omega'|^2\right)^2 \pm 4\frac{c_S}{c_P}|\omega'|^2\omega_3\xi_3^P(\omega),\\
\Delta^{SV}_{\pm}(\omega) &= (\frac{c_P}{c_S})^2\left(1-2|\omega'|^2\right)^2 \pm 4\frac{c_P}{c_S}|\omega'|^2\omega_3\xi_3^{SV}(\omega),\\
\tilde{\Delta}^P(\omega) &= \frac{4c_S}{c_P}\omega_3|\omega'|((\frac{c_P}{c_S})^2 - 2|\omega'|^2),\\
\tilde{\Delta}^{SV}(\omega) &= \tilde{\Delta}^{SVO}(\omega) = -\frac{4c_P}{c_S}\omega_3|\omega'|(1-2|\omega'|^2).
\end{aligned}$$

We introduce the following generalized eigenfunctions of A_0:

$$\begin{aligned}
&\phi_0^{\alpha}(\mathbf{x};\sigma,\omega) = \phi_0^{\alpha,i}(\mathbf{x};\sigma,\omega) + \phi_0^{\alpha,r}(\mathbf{x};\sigma,\omega) \qquad\qquad (\alpha\in\Lambda'),\\
&\phi_0^{R}(\mathbf{x};\sigma,\zeta) = \sqrt{2\pi\rho_0}C_0^R e^{i\sigma c_R^{-1}\zeta\cdot\mathbf{x}'}\sum_{j=1}^{2} C_j^R e^{-|\sigma|c_R^{-1}\xi_R^{(j)}x_3}\mathbf{a}_R^{(j)}(\sigma,\zeta).
\end{aligned}$$

Here $\xi_R^{(1)} = \sqrt{1-(c_R/c_P)^2}$, $\xi_R^{(2)} = \sqrt{1-(c_R/c_S)^2}$, $C_1^R = 2-(c_R/c_S)^2$, $C_2^R = -2\xi_R^{(1)}$, $\mathbf{a}_R^{(1)}(\sigma,\zeta) = {}^t(\zeta, \frac{i\sigma}{|\sigma|}\xi_R^{(1)})$, $\mathbf{a}_R^{(2)}(\sigma,\zeta) = {}^t(\xi_R^{(2)}\zeta, \frac{i\sigma}{|\sigma|})$, and the positive constant C_0^R is taken satisfying $|\sigma|(2\pi\rho_0 c_R)^{-1}\int_0^\infty |\phi_0^R(\mathbf{x};\sigma,\zeta)|^2 dx_3 = 1$. Note that the constant C_0^R depends only on c_P, c_S and c_R. The function ϕ_0^R is the eigenfunction corresponding to the Rayleigh waves.

Using the above generalized eigenfunctions, we can make a spectral representation of A_0 (i.e. the generalized Fourier transform) in the sense of the scattering theory of Wilcox type. We set $\langle\cdot\rangle^{-s_0}\mathcal{H}_0 = \{\mathbf{f} \in L^2_{loc}(\mathbb{R}^3_+)\,;\, \langle\mathbf{x}\rangle^{s_0}\mathbf{f}(\mathbf{x}) \in \mathcal{H}_0\}$, where $\langle\mathbf{x}\rangle = (1+|\mathbf{x}|^2)^{1/2}$. For any function $\mathbf{f} \in \langle\cdot\rangle^{-s_0}\mathcal{H}_0$ we put

$$\begin{aligned}
(\mathcal{F}^0_\alpha(\sigma)\mathbf{f})(\omega) &= \rho_0^{-1/2} c_\alpha^{-3/2}(-i\sigma)(\mathbf{f}, \phi_0^\alpha(\cdot\,;\sigma,\omega))_{\mathcal{H}_0} \qquad (\alpha \in \Lambda'),\\
(\mathcal{F}^0_R(\sigma)\mathbf{f})(\zeta) &= \rho_0^{-1/2} c_R^{-3/2}(-i\sigma)(\mathbf{f}, \phi_0^R(\cdot\,;\sigma,\zeta))_{\mathcal{H}_0}.
\end{aligned}$$

Note that $\mathcal{F}^0_\alpha(\sigma)$ are bounded operators from $\langle\cdot\rangle^{-s_0}\mathcal{H}_0$ to $L^2(S^2_\alpha)$ for any $\sigma \in \mathbb{R}$ if we take $s_0 > 1$. Set $X = \langle\cdot\rangle^{-s_0}\mathcal{H}_0$, $N = \oplus_{\alpha\in\Lambda} L^2(S^2_\alpha)$ $(\Lambda = \Lambda' \cup \{R\})$ and $\mathcal{F}^0(\sigma) = {}^t(\mathcal{F}^0_P(\sigma), \mathcal{F}^0_{SV}(\sigma), \mathcal{F}^0_{SVO}(\sigma), \mathcal{F}^0_{SH}(\sigma), \mathcal{F}^0_R(\sigma))$. Then, we can show that $\mathcal{F}^0(\sigma)$ is a $B(X, N)$-valued measurable function and the spectral representation in the sense of the scattering theory of Wilcox type. From the operator $\mathcal{F}^0(\sigma)$, using Theorem 4.1 in **[4]**, we can construct the translation representation T_0 and the spectral representation $\mathcal{T}_0$ in the sense of the scattering theory of Lax-Phillips type.

In the scattering theory of the Wilcox type, there is no essential difference among the differences caused by multiplications of unitary transforms to generalized eigenfunctions used to define the generalized Fourier transform. But in the theory of Lax and Phillips type, these unitary transforms make differences in the properties of the outgoing and incoming subspaces $D^0_\pm$. This is why we do not use the generalized eigenfuncion obtained by **[1]** straightforwardly.

3 Translation representation for the free case

In this section we consider the translation representation of free space problem and give the concrete form of it. The corresponding outgoing (resp. incoming) subspace does not satisfy the property similar to (1.1). Rather than that we can show that there does not exist any translation representation having (1.1) (for the detail, see **[5]**).

For the self-adjoint operator A_0, the space $\mathcal{H}(A_0^{1/2})$ denotes the completion of the domain $D(A_0^{1/2})$ of $A_0^{1/2}$ with the norm $\|A_0^{1/2}\mathbf{f}\|$ for $\mathbf{f} \in D(A_0^{1/2})$. We introduce the Hilbert space $H_0 = \mathcal{H}(A_0^{1/2}) \times \mathcal{H}_0$, which is an abstruct formulation of the energy space for the usual wave equation. We also define the one parameter family of unitary operators $\{U_0(t)\}$ corresponding to problem (2.1) by $U_0(t)\vec{\mathbf{f}} = {}^t(\mathbf{u}(t), \partial_t\mathbf{u}(t))$.

For any $\vec{\mathbf{f}} \in \mathcal{Y}^0_{s_0} = \{\vec{\mathbf{f}} = {}^t(\mathbf{f}_1, \mathbf{f}_2)\,;\, \langle \mathbf{x}\rangle^{s_0}\mathbf{f}_1(\mathbf{x}) \in H^1(\mathbb{R}^3_+),\ \mathbf{f}_2 \in \langle\cdot\rangle^{-s_0}\mathcal{H}_0\}$ $(s_0 \in \mathbb{R})$, we set

$$
\begin{aligned}
&(3.1) \qquad T_{0,\alpha}\vec{\mathbf{f}}\,(s,\cdot) &=& \int_{\mathbb{R}} e^{-is\sigma}\mathcal{T}_{0,\alpha}\vec{\mathbf{f}}\,(\sigma,\cdot)\,d\sigma \quad (\alpha\in\Lambda),\\
&(3.2) \qquad \mathcal{T}_{0,\alpha}\vec{\mathbf{f}}\,(\sigma,\cdot) &=& (2\pi)^{-1}\left\{i\sigma(\mathcal{F}^0_\alpha(-\sigma)\mathbf{f}_1)(\cdot) + (\mathcal{F}^0_\alpha(-\sigma)\mathbf{f}_2)(\cdot)\right\} \quad (\alpha\in\Lambda).
\end{aligned}
$$

Since $\mathcal{Y}^0_{s_0}(\subset D(A_0^{1/2})\times\mathcal{H}_0)$ is dense in H_0, from Theorem 4.1 in [4], the mapping $T_0 = {}^t(T_{0,P}, T_{0,SV}, T_{0,SVO}, T_{0,SH}, T_{0,R})$ (resp. $\mathcal{T}_0 = {}^t(\mathcal{T}_{0,P}, \mathcal{T}_{0,SV}, \mathcal{T}_{0,SVO}, \mathcal{T}_{0,SH}, \mathcal{T}_{0,R})$) becomes the translation (resp. the spectral) representation of $\{U_0(t)\}$. Now we obtain the following theorem.

Theorem 3.1 *For $\{U_0(t)\}$, we can construct a bounded linear operator $T_0 : H_0 \to L^2(\mathbb{R}; N)$ such that*

$$
\begin{cases}
\left\|T_0\vec{\mathbf{f}}\right\|^2_{L^2(\mathbb{R};N)} = 4(2\pi)^2\|\vec{\mathbf{f}}\|^2_{H_0} \quad \text{for any } \vec{\mathbf{f}}\in H_0,\\
T^\pm \text{ is surjective,}
\end{cases}
$$
$$
T_0U_0(t) = \tau_tT_0 \text{ for any } t\in\mathbb{R}, \text{ where } \tau_t : k(s)\mapsto k(s-t).
$$

Each element of T_0 corresponds to the reflection phenomenon or surface waves. For example, $T_{0,SVO}$ corresponds to the total reflection phenomenon and $T_{0,R}$ corresponds to the Rayleigh surface wave, etc. We give the concrete representations of the translation representation T_0. We set $\widetilde{\mathcal{S}}_0 = \{\vec{\mathbf{f}} = {}^t(\mathbf{f}_1, \mathbf{f}_2)\,;\, \mathbf{f}_j \in \mathcal{S}(\overline{\mathbb{R}^3_+})\}$, where $\mathcal{S}(\overline{\mathbb{R}^3_+}) = \{\vec{\mathbf{f}}\,|_{\mathbb{R}^3_+}\,;\, \vec{\mathbf{f}} \in \mathcal{S}(\mathbb{R}^3)\}$ and $\mathcal{S}(\mathbb{R}^3)$ is the usual Schwartz's function space consisting of rapidly decreasing functions in $\mathbb{R}^3$. From (3.1) and (3.2), it follows that

$$
T_{0,\alpha}\vec{\mathbf{f}}\,(s,\omega) = \rho_0^{-1/2}c_\alpha^{-3/2}\sum_{j=0}^{1}(-\partial_s)^{2-j}\left\{F^{-1}\left[(\mathbf{f}_{1+j}\,,\,\phi^\alpha_0(\cdot;\sigma,\omega))_{H_0}\right](s)\right\} \quad (\alpha\in\Lambda).
$$

Hence we have the concrete forms of $T_{0,\alpha}$. The operators $T^\pm_{0,\alpha}$ $(\alpha = \Lambda)$ are of the forms:

$$
\begin{aligned}
T_{0,P}\vec{\mathbf{f}}\,(s,\omega) &= (c_P\rho_0)^{1/2}\Bigg[\mathbf{a}_P(\check{\omega})\cdot(\mathcal{R}_P\vec{\mathbf{f}})(c_Ps,\check{\omega}) - \frac{\Delta^P_-(\omega)}{\Delta^P_+(\omega)}\mathbf{a}_P(\omega)\cdot(\mathcal{R}_P\vec{\mathbf{f}})(c_Ps,\omega)\\
&\qquad - \frac{c_S^2}{c_P^2}\frac{\tilde{\Delta}^P(\omega)}{\Delta^P_+(\omega)}\mathbf{a}_{SV}(\xi^P(\omega))\cdot(\mathcal{R}_S\vec{\mathbf{f}})(c_Ss,\xi^P(\omega))\Bigg],\\
T_{0,SV}\vec{\mathbf{f}}\,(s,\omega) &= (c_S\rho_0)^{1/2}\Bigg[\mathbf{a}_{SV}(\check{\omega})\cdot(\mathcal{R}_S\vec{\mathbf{f}})(c_Ss,\check{\omega}) - \frac{\Delta^{SV}_-(\omega)}{\Delta^{SV}_+(\omega)}\mathbf{a}_{SV}(\omega)\cdot(\mathcal{R}_S\vec{\mathbf{f}})(c_Ss,\omega)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{c_P^2}{c_S^2}\frac{\tilde{\Delta}^{SV}(\omega)}{\Delta_+^{SV}(\omega)}\mathbf{a}_P(\xi^{SV}(\omega))\cdot(\mathcal{R}_P\vec{\mathbf{f}})(c_Ps,\xi^{SV}(\omega))\Big],\\
T_{0,SH}\vec{\mathbf{f}}\,(s,\omega) &= (c_S\rho_0)^{1/2}\Big\{\mathbf{a}_{SH}(\check{\omega})\cdot(\mathcal{R}_S\vec{\mathbf{f}})(c_Ss,\check{\omega}) + \mathbf{a}_{SH}(\omega)\cdot(\mathcal{R}_S\vec{\mathbf{f}})(c_Ss,\omega)\Big\},\\
T_{0,SVO}\vec{\mathbf{f}}\,(s,\omega) &= (c_S\rho_0)^{1/2}\Big[\frac{(c_P/c_S)^2(1-2|\omega'|^2)^2}{\Delta^{SVO}(\omega)}\Big\{\mathbf{a}_{SV}(\check{\omega})\cdot(\mathcal{R}_S\vec{\mathbf{f}})(c_Ss,\check{\omega})\\
&\qquad - \mathbf{a}_{SV}(\omega)\cdot(\mathcal{R}_S\vec{\mathbf{f}})(c_Ss,\omega)\Big\}\\
&\quad + \frac{4(c_P/c_S)|\omega'|^2\omega_3\eta(\omega)}{\Delta^{SVO}(\omega)}\Big\{\mathbf{a}_{SV}(\check{\omega})\cdot(\kappa(D_s)\mathcal{R}_S\vec{\mathbf{f}})(c_Ss,\check{\omega})\\
&\qquad + \mathbf{a}_{SV}(\omega)\cdot(\kappa(D_s)\mathcal{R}_S\vec{\mathbf{f}})(c_Ss,\omega)\Big\}\\
&\quad - \frac{c_P^2}{c_S^2}\frac{\tilde{\Delta}^{SVO}(\omega)}{\Delta^{SVO}(\omega)}\Big\{\begin{pmatrix}\frac{c_P}{c_S}\omega'\\ 0\end{pmatrix}\cdot(\tilde{\mathcal{R}}_P^+\vec{\mathbf{f}})(c_Ps,\frac{c_P}{c_S}\omega',\eta(\omega))\\
&\qquad + \begin{pmatrix}0\\ \eta(\omega)\end{pmatrix}\cdot(\tilde{\mathcal{R}}_P^-\vec{\mathbf{f}})(c_Ps,\frac{c_P}{c_S}\omega',\eta(\omega))\Big\}\Big],\\
T_{0,R}\vec{\mathbf{f}}\,(s,\zeta) &= (c_R\rho_0)^{1/2}\sqrt{2\pi\rho_0}C_R^0\sum_{j=1}^{2}\Big\{C_{j,R}^{(1)}\begin{pmatrix}\zeta\\ 0\end{pmatrix}\cdot(\tilde{\mathcal{R}}_R^+\vec{\mathbf{f}})(c_Rs,\zeta,\xi_R^{(j)})\\
&\qquad + C_{j,R}^{(2)}\begin{pmatrix}0\\ 1\end{pmatrix}\cdot(\tilde{\mathcal{R}}_R^-\vec{\mathbf{f}})(c_Rs,\zeta,\xi_R^{(j)})\Big\}
\end{aligned}
$$

for any $\vec{\mathbf{f}} = {}^t(\mathbf{f}_1,\mathbf{f}_2) \in \tilde{\mathcal{S}}_0$, where " $\cdot$ " means the inner product of $\mathbb{C}^3$, and the operators $\mathcal{R}_\alpha$, $\tilde{\mathcal{R}}_\alpha^\pm$ are defined by

$$
\begin{aligned}
&\mathcal{R}_\alpha\vec{\mathbf{g}}\,(s,\xi) = c_\alpha\partial_s^2\mathcal{R}^0\mathbf{g}_1(s,\xi) - \partial_s\mathcal{R}^0\mathbf{g}_2(s,\xi) \quad (\alpha = P,S),\\
&\tilde{\mathcal{R}}_\alpha^\pm\vec{\mathbf{g}}\,(s,\xi',\xi_3) = c_\alpha\partial_s^2\tilde{\mathcal{R}}_\pm^0\mathbf{g}_1(s,\xi',\xi_3) - \partial_s\tilde{\mathcal{R}}_\pm^0\mathbf{g}_2(s,\xi',\xi_3) \quad (\alpha = S,R).
\end{aligned}
$$

Here, $\mathcal{R}^0\mathbf{h}(s,\xi) = \int_{\{\mathbf{x}\in\mathbb{R}^3_+;\,\mathbf{x}\cdot\xi=s\}}\mathbf{h}(\mathbf{x})\,dS_\mathbf{x}$ is the usual Radon transform and $\tilde{\mathcal{R}}_\pm^0$ are the operators defined by

$$
\begin{aligned}
\tilde{\mathcal{R}}_+^0\mathbf{h}(s,\xi',\xi_3) &= \frac{1}{\pi}\int_{\mathbb{R}^3_+}\frac{\xi_3x_3}{(\xi_3x_3)^2+(\xi'\cdot\mathbf{x}'-s)^2}\mathbf{h}(\mathbf{x})\,d\mathbf{x},\\
\tilde{\mathcal{R}}_-^0\mathbf{h}(s,\xi',\xi_3) &= \frac{1}{\pi}\int_{\mathbb{R}^3_+}\frac{s-\xi'\cdot\mathbf{x}'}{(\xi_3x_3)^2+(\xi'\cdot\mathbf{x}'-s)^2}\mathbf{h}(\mathbf{x})\,d\mathbf{x}.
\end{aligned}
$$

In the formulae of $T_{0,\alpha}$, the Radon transform $\mathcal{R}^0$ appears only in the terms represented by $\mathcal{R}_\alpha$. This is the same as those in the cases of the problems for the whole space (cf. [7], [14]). In our case, however, we must add the new terms with $\tilde{\mathcal{R}}_\alpha^\pm$. These terms come from the evanescent waves and the Rayleigh surface waves.

By using the translation representation stated above, we can introduce the components of the waves. Set $U_{0,P}(t) = U_0(t)(T_0)^{-1}\,{}^t(T_{0,P},0,0,0,0)$, then $U_{0,P}(t)$ represents the element concerning the reflection phenomena of P-waves, set $U_{0,SVO}(t) =$

$U_0(t)(T_0)^{-1}\,{}^t(0,0,T_{0,SVO},0,0)$, then $U_{0,SVO}(t)$ represents the element concerning the total reflection phenomena and set $U_{0,R}(t) = U_0(t)(T_0)^{-1}\,{}^t(0,0,0,0,T_{0,R})$, then $U_{0,R}(t)$ represents the element concerning the Rayleigh surface waves. The others are also defined in the same way. As is in [5] we can obtain the concrete form of $U_{0,\alpha}(t)$ $(\alpha \in \Lambda)$ using the components of the translation representation $T^{0,\alpha}$ (cf. Theorem 6.2 in [5]). Using these concrete expressions we can characterize the outgoing and incoming subspaces associated with the translation representation T_0. Note that for T_0, $D^0_\pm$ are given by

$$T_0(D^0_\pm) = \{k \in L^2(\mathbb{R};N)\,;\, k(s) = 0 \text{ for } \pm s < 0\}.$$

Then we know that D^0_+ and D^0_- are the outgoing and incoming subspaces. We obtain the characterization of $D^0_\pm$ as follows:

Theorem 3.2 *For any $\vec{\mathbf{f}} \in H_0$, $\vec{\mathbf{f}}$ belongs to $D^0_\pm$ if and only if*

$$\text{(3.3)} \qquad \operatorname{supp} \mathcal{P}\left[(U_0(t) - U_{0,SR}(t))\vec{\mathbf{f}}\right]_1 \subset \{\mathbf{x} \in \mathbb{R}^3_+\,;\, \pm c_S t < |\mathbf{x}|\,\} \text{ and}$$

$$\text{(3.4)} \qquad \operatorname{supp} \mathcal{P}\left[U_0(t)\vec{\mathbf{f}}\right]_1\Big|_{x_3=0} \subset \{\mathbf{x}' \in \partial\mathbb{R}^3_+\,;\, \pm c_R t < |\mathbf{x}'|\,\}$$

hold for any $\pm t > 0$, where $U_{0,SR}(t) = U_{0,SVO}(t) + U_{0,R}(t)$ and $\mathcal{P}\,{}^t(u_1,u_2,u_3) = {}^t(u_1,u_2)$. For any $\vec{\mathbf{f}} \in D^0_\pm$, we also have the following properties:

(i) $\operatorname{supp} U_{0,\alpha}(t)\vec{\mathbf{f}} \subset \{\mathbf{x} \in \overline{\mathbb{R}^3_+}\,;\, \pm c_\alpha t < |\mathbf{x}|\,\}$ *for $\alpha = P, SV, SH$;*

(ii) *for $\alpha = SVO, R$, the functions $U_{0,\alpha}(t)\vec{\mathbf{f}}\,(\mathbf{x})$ are C^∞-function in $\pm c_\alpha t > |\mathbf{x}|$, and for any $\delta > 0$, $q \in \mathbb{N} \cup \{0\}$ and multi-indices $\gamma \in (\mathbb{N} \cup \{0\})^3$, there exists a constant $C_{\delta,q,\gamma} > 0$ independent of $\vec{\mathbf{f}} \in D^0_\pm$ such that*

$$\left|\partial_t^q \partial_{\mathbf{x}}^\gamma \left[U_{0,\alpha}(t)\vec{\mathbf{f}}\right]_l(\mathbf{x})\right| \le C_{\delta,q,\gamma}(1+|t|)^{-l+\frac{1}{2}-q-|\gamma|}\|T_{0,\alpha}\vec{\mathbf{f}}\,\|_{L^2(\mathbb{R};L^2(S^2_\alpha))}$$
$$\textit{for any } l = 1,2,\ (t,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^3_+,\ \pm c_\alpha(1-\delta)t \ge |\mathbf{x}|;$$

(iii) $(\mathcal{P}\,[U_{0,\alpha}(t)]_l\,\vec{\mathbf{f}})(\mathbf{x}',0) = 0$ *on $\pm\, c_\alpha t > |\mathbf{x}'|$ for $\alpha \in \Lambda$ and $l = 1,2$.*

Remark 3.3 Our generalized eigenfunctions $\phi_0^{SVO}(\mathbf{x};\sigma,\omega)$ and $\phi_0^R(\mathbf{x};\sigma,\zeta)$ are different from the ones in Dermenjian and Guillot [1]. We multiply $\frac{\Delta_+^{SVO}(\sigma,\omega)}{\Delta^{SVO}(\omega)}$ and $i\sigma/|\sigma|$ to Dermenjian and Guillot's ones respectively. This reason is that we wish to obtain Theorem 3.2. Namely if we choose these eigenfunctions in other way, we can no longer obtain the characterizations (3.3) and (3.4).

In the case of Cauchy problem, $D_{\pm}^{0}$ has the property

$$D_{\pm}^{0}=\{\vec{\mathbf{f}}={}^{t}(f_1,f_2)\in H\,;\,U(t)\vec{\mathbf{f}}=0 \text{ in } |x|\le \pm ct\} \tag{3.5}$$

for some constant $c>0$ standing for the latest speed of propagation speed of the waves (see e.g. [7], [14]). But in the case of (2.1), there do not exist translation representations such that corresponding outgoing and incoming subspaces $D_{\pm}^{0}$ have the same property as (3.5).

Proposition 3.4 *For any $\vec{\mathbf{f}}\in H_0$, the following two statements* (a) *and* (b) *are equivalent to each other:*

(a) $\operatorname{supp} U_0(t)\vec{\mathbf{f}}\subset\{\mathbf{x}\in\mathbb{R}_+^3\,;\,\pm c_R t<|\mathbf{x}|\}$ *for any* $\pm t>0$.

(b) $\vec{\mathbf{f}}\in D_{\pm}^{0},\ \ T_{0,SVO}\vec{\mathbf{f}}(s,\omega)=0$ *for any* $(s,\omega)\in\mathbb{R}\times S_{SVO}^2$, *and* $T_{0,R}\vec{\mathbf{f}}(s,\zeta)=0$ *for any* $(s,\zeta)\in\mathbb{R}\times S_R^2$.

In the cases of the Cauchy problems of the wave equation and the elastic wave equation, the condition (a) in Proposition 3.4 just characterizes the set $D_{\pm}^{0}$ (cf. [7] and [14]). Consequently, Proposition 3.4 shows that in the case of the elastic wave equation in the half-space, the condition (a) in Proposition 3.4 is too strong to characterize $D_{\pm}^{0}$. These differences come from existence of "the evanescent waves" $\phi_0^{SVO,r}(\mathbf{x};\sigma,\omega)$ corresponding to the total reflection phenomena and the Rayleigh surface waves $\phi_0^{R}(\mathbf{x};\sigma,\zeta)$, which is caused by existence of the boundary. In the case of the transmission problem of the scalar valued wave equation, we also see such phenomena since there also exist evanescent waves. (cf. [4]).

Lastly, we give other decay estimates for the solution to (2.1), which are required to prove existence and completeness of the wave operatores. For any $r\in\mathbb{R}$, we set $D_{\pm}^{r}=U_0(\pm(C_{min}^0)^{-1})D_{\pm}^{0}$, where $C_{min}^{0}=c_R$. Note that c_R is the slowest speed of the waves for the free problem (2.1).

Proposition 3.5 (i) *For any $\vec{\mathbf{g}}\in D_{+}^{r}$ $(r\in\mathbb{R})$, $U_0(t)\vec{\mathbf{g}}\in C^\infty$ in $t>(C_{min}^0)^{-1}(|\mathbf{x}|+r)$.*
(ii) *For any $m\in\mathbb{N}\cup\{0\}$ and $\delta>0$, there exists a constant $C_m>0$ such that*

$$\sum_{|\alpha|+j=m}\|\langle\cdot\rangle^{-(1+\delta)}\partial_t^j\partial_{\mathbf{x}}^{\alpha}[U_0(t)\vec{\mathbf{g}}]_1\|_{L^2(\mathbb{R}_+^3\cap B_R)}$$
$$\le C_m R^{-(1+\delta)+n/2}(t+(C_{min}^0)^{-1}(r-R))^{-n/2+1-m}\|\vec{\mathbf{g}}\|_{H_0}$$

for any $\vec{\mathbf{g}} \in D^r_+$, $R \geq 1$, $t > (C^0_{min})^{-1}(R-r)$.

(iii) *For any* $m \in \mathbb{N} \cup \{0\}$, *there exists a constant* $C_m > 0$ *such that*

$$\sum_{|\alpha|+j=m} \|\partial_t^j \partial_{\mathbf{x}}^\alpha [U_0(t)\vec{\mathbf{g}}]_1\|_{L^2(\mathbb{R}^3_+ \cap B_{R+2})} \leq C_m(1+t)^{-n/2+1-m} \|\vec{\mathbf{g}}\|_{D(L_0^{\max\{0,m-1\}})}$$

for any $\vec{\mathbf{g}} \in D(L_0^{m-1}) \cap D_+^{R_0+2}$ $(m \geq 2)$, $\vec{\mathbf{g}} \in H_0 \cap D_+^{R_0+2}$ $(m = 0, 1)$, $t \geq 0$.

(iv) *For any* $\ell \in \mathbb{N} \cup \{0\}$ *and* $\delta' > 0$ *there exists a constant* $C_{\ell,\delta'} > 0$ *such that*

$$\sum_{1 \leq |\alpha|+j \leq 1+\ell} \|\langle\cdot\rangle^{-(1+\delta')} \partial_t^j \partial_x^\alpha [U_0(t)\vec{\mathbf{f}}]_1\|_{L^2(\mathbb{R}^3_+)} \leq C_{\ell,\delta'}(1+t)^{-(1+\delta')} \|\vec{\mathbf{f}}\|_{H_\ell(\mathbb{R}^3_+),\delta'}$$

$$\text{for any } t \geq 0, \vec{\mathbf{f}} \in D(L_0^\ell) \text{ with } \|\vec{\mathbf{f}}\|_{H_\ell(\mathbb{R}^3_+),\delta'} < +\infty.$$

where $\|\vec{\mathbf{f}}\|_{H_\ell(\mathbb{R}^3_+),\delta'} = \sum_{|\alpha| \leq \ell} \{\|\langle\cdot\rangle^{(1+\delta')} \partial_x^\alpha \nabla_{\mathbf{x}} \mathbf{f}_1\|_{L^2(\mathbb{R}^3_+)} + \|\langle\cdot\rangle^{(1+\delta')} \partial_x^\alpha \mathbf{f}_2\|_{L^2(\mathbb{R}^3_+)}\}$.

4 Existance of the wave operators

From now on we proceed to the perturbed case. In this section, we consider existence of the wave operators. For the perturbed case, in the same manner as the free case we can formulate problem (1.2). Let $\mathcal{H} = L^2(\Omega; \mathbb{C}^3, \rho(\mathbf{x})d\mathbf{x})$ and A be the Hilbert space with the norm $\|\mathbf{f}\|^2_{\mathcal{H}} = \left\{\int_\Omega |\mathbf{f}(\mathbf{x})|^2 \rho(\mathbf{x})d\mathbf{x}\right\}^{1/2}$ and the self-adjoint operator an $\mathcal{H}$ defined by

$$A\mathbf{u} = -(\rho(\mathbf{x}))^{-1}\mathcal{A}(\mathbf{x}, \partial_{\mathbf{x}})\mathbf{u} \quad (\mathbf{u} \in D(A) = \{\mathbf{u} \in H^2(\Omega; \mathbb{C}^3) \mid \mathcal{N}(\mathbf{x}, \partial_{\mathbf{x}})\mathbf{u} = 0\}).$$

For the operator A, we also define the Hilbert spaces $\mathcal{H}(A^{1/2})$ and $H = \mathcal{H}(A^{1/2}) \times \mathcal{H}$, and the one parameter family of unitary operators $\{U(t)\}$ corresponding to problem (1.2).

We take a cut-off function $\psi \in C^\infty(\mathbb{R}^3)$ so that $0 \leq \psi \leq 1$, $\psi(\mathbf{x}) = 1$ in $|\mathbf{x}| > R_0 + 2$ and $\psi(\mathbf{x}) = 0$ in $|\mathbf{x}| < R_0 + 1$. We define the wave operator $W_\pm \in B(H_0, H)$ by

$$W_\pm = s - \lim_{t \to \pm\infty} U(-t) J_\psi U_0(t) \in B(H_0, H),$$

where $J_\psi \in B(H_0, H)$ is given by $J_\psi \vec{\mathbf{f}} = {}^t(\psi \mathbf{f}_1, \psi \mathbf{f}_2)$.

Proposition 4.1 *The wave operators are partially isometric and satisfy*

$$U(t)W_\pm = W_\pm U_0(t) \quad \text{for any } t \in \mathbb{R}.$$

In what follows, we consider only the outgoing (i.e. $t \to \infty$) case since the incoming (i.e. $t \to -\infty$) case is the same. To show Proposition 4.1, we need some existence theorem which is useful to choose decaying part of solutions. We set $\dot{H}^m(\Omega) = \{\mathbf{u} \in H^m_{loc}(\Omega)\,;\, \partial^\alpha_{\mathbf{x}}\mathbf{u} \in L^2(\Omega)$ for $1 \le |\alpha| \le m, \lim_{r\to\infty} r^{-2}\int_{r\le|\mathbf{x}|\le 2r} |\mathbf{u}(\mathbf{x})|^2 d\mathbf{x} = 0\}$. For $\mathbf{w} \in \bigcap_{j=0}^1 C^j(\mathbb{R}; \dot{H}^{2+\ell-j}(\Omega))$, $\partial_t\mathbf{w} \in \bigcap_{j=0}^1 C^j(\mathbb{R}; H^{1+\ell-j}(\Omega))$ $(\exists\ell \in \mathbb{N}\cup\{0\})$, we consider the following mixed problem:

$$\text{(4.1)} \qquad \begin{cases} (\partial_t^2 - (\rho(\mathbf{x}))^{-1}\mathcal{A}(\mathbf{x},\partial_{\mathbf{x}}))\mathbf{u}(t,\mathbf{x}) = \mathbf{q}_{\mathbf{w}}(t,\mathbf{x}) & \text{in } \mathbb{R}\times\Omega, \\ \mathcal{N}(\mathbf{x},\partial_{\mathbf{x}})\mathbf{u}(t,\mathbf{x}) = \mathbf{m}_{\mathbf{w}}(t,\mathbf{x}) & \text{on } \mathbb{R}\times\partial\Omega, \\ \lim_{t\to\infty}\|\mathbf{u}(t,\cdot)\|_{1,\Omega} = 0, \end{cases}$$

where $\|\mathbf{u}(t,\cdot)\|_{p,\Omega} = \sum_{1\le|\alpha|+j\le p}\|\partial_t^j\partial_{\mathbf{x}}^\alpha\mathbf{u}(t,\cdot)\|_{L^2(\Omega)}$ $(p = 1,2,\ldots)$ and

$$\begin{aligned} \mathbf{q}_{\mathbf{w}}(t,\mathbf{x}) &= (\rho(\mathbf{x}))^{-1}\mathcal{A}(\mathbf{x},\partial_{\mathbf{x}})(\psi(\mathbf{x})\mathbf{w}(t,\mathbf{x})) - \rho_0^{-1}\psi(\mathbf{x})\mathcal{A}_0(\partial_{\mathbf{x}})\mathbf{w}(t,\mathbf{x}) \\ &= ((\rho(\mathbf{x}))^{-1}\mathcal{A}(\mathbf{x},\partial_{\mathbf{x}}) - \rho_0^{-1}\mathcal{A}_0(\partial_{\mathbf{x}}))(\psi(\mathbf{x})\mathbf{w}(t,\mathbf{x})) + \rho_0^{-1}[\mathcal{A}_0(\partial_{\mathbf{x}}),\psi]\mathbf{w}(t,\mathbf{x}). \\ \mathbf{m}_{\mathbf{w}}(t,\mathbf{x}) &= \psi(\mathbf{x})\mathcal{N}_0(\partial_{\mathbf{x}})\mathbf{w}(t,\mathbf{x}) - \mathcal{N}(\mathbf{x},\partial_{\mathbf{x}})(\psi(\mathbf{x})\mathbf{w}(t,\mathbf{x})) \\ &= (\mathcal{N}_0(\partial_{\mathbf{x}}) - \mathcal{N}(\mathbf{x},\partial_{\mathbf{x}}))(\psi(\mathbf{x})\mathbf{w}(t,\mathbf{x})) - (\mathcal{N}_0(\partial_{\mathbf{x}})\psi)(\mathbf{x})\cdot\mathbf{w}(t,\mathbf{x}). \end{aligned}$$

We also set $F_p(t) = \sum_{1\le j+|\alpha|\le p}\|\partial_t^j\partial_{\mathbf{x}}^\alpha\mathbf{w}(t,\cdot)\|_{L^2(\Omega)}$ $(p = 1,2,\ldots)$.

Proposition 4.2 *Set $\ell = 0$, then we have the following:*

(i) *if we assume that $F_{1+p} \in L^1([0,\infty))$ and $\lim_{t\to\infty} F_0(t) = 0$ ($p = 0$ or 1), then problem (4.1) admits a unique solution $\mathbf{u}$ such that $\mathbf{u} \in \bigcap_{j=0}^{1+p} C^j(\mathbb{R}; \dot{H}^{1+p-j}(\Omega))$ and $\partial_t\mathbf{u} \in \bigcap_{j=0}^{p} C^j(\mathbb{R}; H^{p-j}(\Omega))$.*

(ii) *The solution $\mathbf{u}$ stated in* (i) *has the following estimate:*

there exists a constant $C > 0$ depending only on Ω and $\mathcal{A}(\mathbf{x},\partial_{\mathbf{x}})$ such that

$$\|\mathbf{u}(t,\cdot)\|_{1+p,\Omega} \le C\left\{\sup_{t\le\tau}\left(\sum_{j=0}^{p} F_j(\tau)\right) + \int_t^\infty F_{1+p}(\tau)d\tau\right\} \qquad \textit{for any } t \in \mathbb{R}.$$

Especially, when $p = 1$ we have

$$\|\mathbf{u}(t,\cdot)\|_{2,\Omega} \le C\|{}^t(\mathbf{w}(0,\cdot),\partial_t\mathbf{w}(0,\cdot))\|_{D(L_0)} \qquad \textit{for any } t \in \mathbb{R}.$$

From Proposition 3.5, we can only obtain $F_0(t) = O(t^{-1/2}) \notin L^1([1,\infty))$. Hence, usual existence theorem dose not ensure Proposition 4.2. We have to take into account of the forms of $\mathbf{q}_{\mathbf{w}}$ and $\mathbf{m}_{\mathbf{w}}$ to show Proposition 4.2. Using Proposition 4.2, we obtain Proposition 4.1. For $\vec{\mathbf{f}} \in D_{\pm}^{R_0+2}$, we take the solution $\mathbf{w}(t,\mathbf{x}) = [U_0(t)\vec{\mathbf{f}}]_1$, where $[\vec{\mathbf{f}}]_1 = \mathbf{f}_1$. The uniqueness of the solution to (1.2) implies that

$$U(-t)J_\psi U_0(t)\vec{\mathbf{f}} = (J_\psi + V(0))\vec{\mathbf{f}} - U(-t)V(t)\vec{\mathbf{f}}$$

for any $t \geq 0$ and $\vec{\mathbf{f}} \in D(L_0) \cap D_{\pm}^{R_0+2}$, where $V(t)\vec{\mathbf{f}} = {}^t(\mathbf{v}(t,\cdot), \partial_t \mathbf{v}(t,\cdot))$ and $\mathbf{v}(t,\mathbf{x})$ is the solution to (4.1) obtained in Proposition 4.2 with respect to the solution $\mathbf{w}(t,\mathbf{x})$ of (2.2). Hence Proposition 3.5 and Proposition 4.2 implies that the limit $W_{\pm}\vec{\mathbf{f}}$ is defined for $\vec{\mathbf{f}} \in D(L_0) \cap D_{\pm}^{R_0+2}$ and is partially isomorphic. Since $\bigcup_{t\in\mathbb{R}} U_0(t)\{D(L_0) \cap D_{\pm}^{R_0+2}\} = (\bigcup_{t\in\mathbb{R}} U_0(t) D_{\pm}^{R_0+2}) \cap D(L_0)$ is dense in H_0, we obtain Proposition 4.1.

5 Completeness of the wave operators

We define closed subspace $D_{\pm}$ in H by $D_{\pm} = W_{\pm}(D_{\pm}^0)$. Our purpose is to show that D_+ (resp. D_-) is outgoing (resp. incoming) subspce of $\{U(t)\}$. Since $\bigcup_{t\in\mathbb{R}} U(t) D_{\pm} = W_{\pm}(\bigcup_{t\in\mathbb{R}} U_0(t) D_{\pm}^0)$, this is equivalent to show the completeness of $W_{\pm}$. In what follows, we concentrate on the outgoing case.

Definitely to show the completeness, we need " local decay property". From Theorem 1.2 in [6], we have $\sigma_p(A) = \emptyset$, which implies the following local decay property shown in the same way as in [7]:

Lemma 5.1 *There is a sequence $\{t_j\} \subset \mathbb{R}$ such that $\lim_{j\to\infty} t_j = \infty$ and for any $\vec{\mathbf{f}} \in H$, $w - \lim_{j\to\infty} U(t_j)\vec{\mathbf{f}} = \mathbf{0}$ in H in the weak topology.*

Using the local decay property, we show the completeness of W_+, i.e. D_+ is outgoing.

Theorem 5.2 *The wave operators $W_{\pm}$ are complete in the sense of $R(W_{\pm}) = H$, where $R(W_{\pm})$ is the range of $W_{\pm}$.*

Hence, in this sense, we can say that the scattering theory of Lax and Phillips can be also formulated.

To show Theorem 5.2, for all $\vec{\mathbf{f}} \in D(L)$, it sufficies to find $\vec{\mathbf{g}} \in H_0$ satisfying $\vec{\mathbf{f}} = W_{\pm}\vec{\mathbf{g}}$. This is simplest approach, however, it seems to be different to find $\vec{\mathbf{g}}$ directly. Usually Theorem 5.2 are shown by contradiction arguments (cf. [7], [8] etc). In [2], Professor Ikawa gives an intresting approach to obtain $\vec{\mathbf{g}}$. He makes an successible approximation of $\vec{\mathbf{g}}$ by using a weak version of the decomposition due to Morawetz [10] originally. Since Huygens principle dose not hold, however, we have to change the decomposition as follows:

Proposition 5.3 *There exist constants C_1 and $C_2 > 0$ depending only on Ω and $\mathcal{A}(\mathbf{x}, \partial_{\mathbf{x}})$ such that the following decomposition holds:*

for any $\vec{\mathbf{f}} \in D(L)$, there exist $T_0 = T_0(\vec{\mathbf{f}}) > 0$, $\vec{\mathbf{g}}_0 \in H_0$, $\vec{\mathbf{f}}_0 \in H$ and $\vec{\mathbf{z}}_0(t) \in C([T_0, \infty); H)$ such that

$$\|\vec{\mathbf{g}}_0\|_{H_0} \leq C_2 \|\vec{\mathbf{f}}\|_{D(L)}, \|\vec{\mathbf{f}}_0\|_{D(L)} \leq C_3 \|\vec{\mathbf{f}}\|_{D(L)}, \|\vec{\mathbf{f}}_0\|_H \leq 2^{-1}\|\vec{\mathbf{f}}\|_H,$$
$$\|\vec{\mathbf{z}}_0(t)\|_H \leq (1 + t - T_0)^{-1/4}\|\vec{\mathbf{f}}\|_H \qquad (\text{for any } t \geq T_0 \geq 0)$$

and $U(t)\vec{\mathbf{f}}$ is decomposed as

$$U(t)\vec{\mathbf{f}} = J_\psi U_0(t - T_0)\vec{\mathbf{g}}_0 + \vec{\mathbf{z}}_0(t) + U(t - T_0)\vec{\mathbf{f}}_0 \quad \text{in } H \ (t \geq T).$$

Note that $C_1, C_2 > 0$ do not depend on $\vec{\mathbf{f}} \in D(L)$, however, $T_0 = T_0(\vec{\mathbf{f}})$ may depend on $\vec{\mathbf{f}}$. This is the meaning of "a weak version".

Using Proposition 5.3 iteratively, we can follow the argument of Ikawa [**2**]. Thus we can obtain sequences $T_j > 0$, $\vec{\mathbf{g}}_j \in H_0$ $\vec{\mathbf{f}}_j \in D(L)$ and $\vec{\mathbf{z}}_j \in C([\tilde{T}_j, \infty), H)$ satisfying

$$\|\vec{\mathbf{g}}_j\|_{H_0} \leq C_1\|\vec{\mathbf{f}}_{j-1}\|_H \qquad \|\vec{\mathbf{f}}_j\|_{D(L)} \leq C_1\|\vec{\mathbf{f}}_{j-1}\|_{D(L)} \qquad \|\vec{\mathbf{f}}_j\|_H \leq \frac{1}{2}\|\vec{\mathbf{f}}_{j-1}\|_H$$
$$\|\vec{\mathbf{z}}_j(t)\|_H \leq (1 + t - \tilde{T}_j)^{-1/2}\|\vec{\mathbf{f}}_{j-1}\|_H$$
$$U(t - \tilde{T}_{j-1})\vec{\mathbf{f}}_{j-1} = J_\psi U_0(t - \tilde{T}_j)\vec{\mathbf{g}}_j + \vec{\mathbf{z}}_j(t) + U(t - \tilde{T}_j)\vec{\mathbf{f}}_j \quad \text{in } H$$

for any $t \geq \tilde{T}_j$ and $j \in \mathbb{N}$, where $\tilde{T}_j = \sum_{p=1}^{j} T_p$. From this we can conclude that the limit $\vec{\mathbf{g}} = \sum_{j=0}^{\infty} U_0(-\tilde{T}_j)\vec{\mathbf{g}}_j \in H_0$ exists and this $\vec{\mathbf{g}}$ is just the solution of $\vec{\mathbf{f}} = W_+\vec{\mathbf{g}}$. Hence we have Theorem 5.2.

Next we show Proposition 5.3. Let P_+^r ($r \in \mathbb{R}$) be the orthogonal projection to the space $(D_\pm^r)^\perp$. We introduce a regularization $\tilde{P}_+^\varphi$ of $P_+^{R_0+2}$ defined by $\tilde{P}_+^\varphi\vec{\mathbf{g}} = T_0^{-1}[\varphi(\cdot)T_0\vec{\mathbf{g}}(\cdot)]$, where $\varphi \in C^\infty(\mathbb{R})$ so that $0 \leq \varphi \leq 1$, $\varphi(s) = 1$ in $s < (C_{min}^0)^{-1}(R_0+2)$, $\varphi(s) = 0$ in $s > (C_{min}^0)^{-1}(R_0+3)$. These operators work as "cutoff" for energy escaping part of data in H_0 (cf. [**9**]).

We also need an extension operator $E_{\mathcal{A}_0} \in B(D(L), D(L_0))$ and $E_{\mathcal{A}} \in B(H_0, D(L))$ to keep regularities of the solutions after they are cut off. Basically this is made using usual Seeley extension.

Lemma 5.4 *There exists an operator $E_{\mathcal{A}_0} \in B(D(L), D(L_0))$ such that the following prpoerties hold:*

(i) $[E_{\mathcal{A}_0}\vec{\mathbf{f}}\,]_1(\mathbf{x}) = \mathbf{f}_1(\mathbf{x})$ *in* $\mathbb{R}^3_+ \cap \Omega$ *or* $|\mathbf{x}| \geq R_0 + 2$ *for some fixed* $b > 0$,

(ii) $[E_{\mathcal{A}_0}\vec{\mathbf{f}}\,]_2(\mathbf{x}) = \mathbf{f}_2(\mathbf{x})$ *in* $\mathbb{R}^3_+ \cap \Omega$,

(iii) *there exists a constant* $C_3 > 0$ *such that*

$$\|\partial^\alpha_{\mathbf{x}}\{[E_{\mathcal{A}_0}\vec{\mathbf{f}}\,]_1 - \mathbf{f}_1\}\|_{L^2(\Omega\cap B_{R_0+2})} \leq C_3\|\partial^\alpha_{\mathbf{x}}\mathbf{f}_1\|_{L^2(\Omega\cap B_{R_0+2})} \quad \textit{for any } \vec{\mathbf{f}} \in D(L).$$

Further, if we consider the case replaced $\mathcal{A}(\mathbf{x}, \partial_{\mathbf{x}})$ *to* $\mathcal{A}_0(\partial_{\mathbf{x}})$, Ω *to* $\mathbb{R}^3_+$ *and* L *to* L_0 *respectively, we also have an operator* $E_{\mathcal{A}} \in B(H_0, D(L))$ *satisfying* $E_{\mathcal{A}} J_\psi \in B(D(L_0), D(L))$ *and all properties (i) - (iii) correspongind to the case.*

Next we can state our decomposition of the solution $U(t)\vec{\mathbf{f}}$ which gives the basis on the proof of Proposition 5.3. Let $H_m(\Omega) = \dot{H}^{1+m}(\Omega) \times H^m(\Omega)$.

Lemma 5.5 *For any* $\vec{\mathbf{f}} \in D(L)$, $U(t)\vec{\mathbf{f}}$ *can be decomposed as*

$$U(t)\vec{\mathbf{f}} = U(t - T - \tilde{T})\vec{\mathbf{f}} + W_T(t)\vec{\mathbf{f}} + J_\psi U_0(t - T)(I - \tilde{P}^\varphi_+)E_{\mathcal{A}_0}U(T)\vec{\mathbf{f}}$$
$$\textit{for any } t, T, \tilde{T} \geq 0 \textit{ with } t \geq T + \tilde{T},$$

where $V_T(\tilde{T}) \in B(D(L))$ *and* $W_T(t) \in C([0,\infty) : B(D(L), H_1(\Omega)))$ *have the following estimates:*

there exist constants C_4, C_5, $C_6 > 0$ *depending only on* $\mathcal{A}(\mathbf{x}, \partial_{\mathbf{x}})$ *and* Ω *such that*

$$\sup_{T,\tilde{T}\geq 0} \|V_T(\tilde{T})\|_{B(D(L))} \leq C_4$$
$$\sup_{t\geq T\geq 0} \{(1 + t - T)^{1/2}\|W_T(t)\|_{B(D(L),H)} + \|W_T(t)\|_{B(D(L))}\} \leq C_5$$
$$\|V_T(\tilde{T})\|_H \leq C_6\Big\{(1 + \tilde{T})^{-1/2}\|\vec{\mathbf{f}}\,\|_{D(L)} + \|\tilde{P}^\varphi_+ E_{\mathcal{A}_0}U(T)\vec{\mathbf{f}}\,\|_{H_0}$$
$$+ |||U(T)\vec{\mathbf{f}}\,|||_{H_0(\Omega\cap B_{R_{\tilde{T}}})} + \|\left[U(T)\vec{\mathbf{f}}\right]_1\|_{L^2(\Omega\cap B_{R_{\tilde{T}}})}\Big\}$$

for any $\tilde{T}$, $T \geq 0$ *and* $\vec{\mathbf{f}} \in D(L)$. *In the above,* $R_{\tilde{T}} = 2(R_0 + 3 + C_{max}\tilde{T})$ *and* $C_{max} = \max\{C^1_{max}, C^0_{max}\}$, *where* C^1_{max} *and* C^0_{max} *are the finite propagation speed of (1.2) and (2.2) respectively.*

To show Lemma 5.5, for all $t \geq T + \tilde{T}$, T, $\tilde{T} \geq 0$, we separeate $U(t)$ as

$$U(t + T) = J_\psi U_0(t)(I - \tilde{P}^\varphi_+)E_{\mathcal{A}_0}U(T) + \tilde{W}_T(t) + U(t - \tilde{T})U(\tilde{T})V^{(1)}_T + V^{(2)}_T(\tilde{T}),$$

where $V_T^{(1)} = (I - E_{\mathcal{A}} J_\psi E_{\mathcal{A}_0})U(T) + E_{\mathcal{A}} \tilde{P}_+^\varphi E_{\mathcal{A}_0} U(T)$ and $V_T^{(2)}(t) = (U(t)E_{\mathcal{A}} J_\psi - J_\psi U_0(t))(I - \tilde{P}_+^\varphi)E_{\mathcal{A}_0} U(T) - \tilde{W}_T(t)$. In the above, $\tilde{W}_T(t)\vec{\mathbf{f}} = {}^t(\mathbf{v}(t,\cdot), \partial_t \mathbf{v}(t,\cdot))$ is determined by the solution $\mathbf{v}(t,\cdot)$ to (4.1) for choosing $\mathbf{w}(t,\cdot) = [U_0(t)(I - \tilde{P}_+^\varphi)E_{\mathcal{A}_0} U(T)\vec{\mathbf{f}}]_1$. From the uniqueness of the solution to (1.2), it follows that $V_T^{(2)}(t) = U(t - \tilde{T})V_T^{(2)}(\tilde{T})$. We define $V_T(\tilde{T})$ in Lemma 5.5 by $V_T(\tilde{T}) = U(\tilde{T})V_T^{(1)} + V_T^{(2)}(\tilde{T})$. Using local decay estimates for the solutions of the free problem (i.e. Proposition 3.5), Lemma 5.4 and the property of the finite propagation speed, we can obtain Lemma 5.5.

To obtain Proposition 5.3, we need various local decay property which are derived from Lemma 5.1.

Lemma 5.6 *For any $\vec{\mathbf{f}} \in D(L)$, it follows that*

(i) *for any $R > 0$, $\lim_{j\to\infty}\{\|U(t_j)\vec{\mathbf{f}}\|_{H_0(\Omega\cap B_R)} + \|[U(t_j)\vec{\mathbf{f}}]_1\|_{L^2(\Omega\cap B_R)}\} = 0$,*

(ii) $\lim_{j\to\infty} \|\tilde{P}_+^\varphi E_{\mathcal{A}_0} U(t_j)\vec{\mathbf{f}}\|_{H_0} = 0$,

where $\{t_j\}$ is the sequence obtained in Lemma 5.1.

Proof. (i) is obtained by well known argument using Rellich compactness Theorem (cf. e.g. [7] or [2]). To obtain (ii), we require the following Lemma which is shown by using the formula of T_0 (cf. §3).

Lemma 5.7 *There is a constant $C > 0$ such that for any $\delta > 0$, the following estimate holds:*

$$\int_{-\infty}^{-r} \|T_0 \vec{\mathbf{g}}\,(s)\|_H^2 ds \le C r^{-2(1+\delta)} \|\vec{\mathbf{g}}\,\|_{H_0(\mathbb{R}^3_+),\delta}$$

for any $\vec{\mathbf{g}} \in H_0$, $\|\vec{\mathbf{g}}\,\|_{H_0(\mathbb{R}^3_+),\delta} < \infty$, $r \ge 2(C^0_{min})^{-1} r$.

We set $G(t) = E_{\mathcal{A}_0} U(t) - U_0(t) E_{\mathcal{A}_0}$. The Duhamel principle and Proposition 4.2, we have $G(t)\vec{\mathbf{f}} = \int_0^t U_0(t-s)Q(s)\vec{\mathbf{f}}\,ds$ in H_0 for any $\vec{\mathbf{f}} \in D(L^2)$, where $Q(t)$ is a $B(D(L_0^2), H)$-valued continuous function having the estimate

$$\|Q(t)\vec{\mathbf{f}}\,\|_{H_0(\mathbb{R}^3_+),\delta} \le C_\delta \|\vec{\mathbf{f}}\,\|_{D(L^2)} \quad \text{for any } t \in \mathbb{R},\ \vec{\mathbf{f}} \in D(L^2)$$

for any $\delta > 0$ fixed.

Hence from Lemma 5.7 and (iv) in Proposition 3.5, we obtain

$$\|P_+^{-r} G(t)\vec{\mathbf{f}}\,\|_{H_0} \le C_\delta r^{-\delta} \|\vec{\mathbf{f}}\,\|_{D(L^2)} \quad \text{for any } r \ge 2(C^0_{min})^{-1} \text{ and } \vec{\mathbf{f}} \in D(L^2).$$

This estimate, Lemma 5.1 and the fact that for any $r \in \mathbb{R}$ $\tilde{P}_+^{\varphi} - P_+^{-r} \in B(D(L_0), H_0)$ is compact imply $\lim_{j\to\infty} \|G(t_j)\vec{\mathbf{f}}\|_{H_0} = 0$. By definition, it follows that

$$\|\tilde{P}_+^{\varphi} U_0(t) F_{\mathcal{A}_0} \vec{\mathbf{f}}\|_{H_0}^2 \le C \int_{-\infty}^{(C_{min}^0)^{-1}-t} \|T_0 F_{\mathcal{A}_0} \vec{\mathbf{f}}(s)\|_N^2 ds.$$

Hence we obtain (ii) of Lemma 5.6.

For $\vec{\mathbf{f}} \in D(L)$, we take $\tilde{T} = \tilde{T}(\vec{\mathbf{f}}) > 0$ so that $C_5(1+\tilde{T})^{-1/2}\|\vec{\mathbf{f}}\|_{D(L)} + 4C_6(1+\tilde{T})^{-1/2}\|\vec{\mathbf{f}}\|_{D(L)} \le \|\vec{\mathbf{f}}\|_H$. Then we have $\|W_T(t)\vec{\mathbf{f}}\|_H \le (1+t-T)^{-1/4}\|\vec{\mathbf{f}}\|_H$ for any $t, T \ge 0$ with $t \ge T + \tilde{T}$, where $W_T(t)$ is in Lemma 5.5.

For this $\tilde{T} > 0$, from Lemma 5.6, we can choose $N \in \mathbb{N}$ such that

$$\|V_{t_N}(\tilde{T})\vec{\mathbf{f}}\|_H \le \frac{1}{2}\|\vec{\mathbf{f}}\|_{H_0}.$$

Thus, if we take $\vec{\mathbf{f}}_0 = V_{t_N}(\tilde{T})\vec{\mathbf{f}} \in D(L)$, $\vec{\mathbf{g}} = U_0(\tilde{T})(I - \tilde{P}_+^{\varphi})E_{\mathcal{A}_0}U(t_N)\vec{\mathbf{f}} \in D(L_0)$, $\vec{\mathbf{z}}_0(t) = W_+(t)\vec{\mathbf{f}}$ and $T_0 = t_N + \tilde{T}$, Lemma 5.5 implies that they satisfy all properties in Proposition 5.3, which completes the proof of Proposition 5.3.

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KAWASHITA, MISHIO
DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HIROSHIMA UNIVERSITY,
HIGASHI-HIROSHIMA, 739-8526 JAPAN
KAWASHITA, WAKAKO
FACULTY OF INTEGRATED ARTS AND SCIENCES, HIROSHIMA UNIVERSITY (TEMPORARY)
HIGASHI-HIROSHIMA, 739-8521 JAPAN
SOGA, HIDEO
FACULTY OF EDUCATION, IBARAKI UNIVERSITY, MITO, IBARAKI, 310-8512, JAPAN