数理解析研究所講究録 1315 巻 2003 年 24-31



# SPECTRAL PROPERTIES OF DIRAC SYSTEMS WITH COEFFICIENTS INFINITE AT INFINITY

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## 1 Introduction.

It is well known that a one-dimensional Schrödinger operator

$$-\frac{d^2}{dx^2} + q(x)$$

with potential $q$ satisfying $\lim_{x\to\pm\infty} q(x) = \infty$ has a purely discrete spectrum. If, on the other hand, $\lim_{x\to\infty} q(x) = -\infty$, the situation is entirely different. By a classical result, obtained independently by Hartman [6] and Shnol' [17], the spectrum is then purely absolutely continuous, filling the whole real line, if $|q(x)| = o(x^2)$ $(x \to \infty)$. In the limiting case $|q(x)| = O(x^2)$ $(x \to \infty)$ this is no longer true, as shown by Halvorsen [5] in a counterexample for which the essential spectrum has gaps. For potentials tending to $-\infty$ faster than $O(x^2)$, the singular end-point $\infty$, in the limit-point case in the above situations, changes its behaviour to an (oscillatory) limit-circle case, giving rise to a purely discrete spectrum again.

The relativistic counterpart of the Schrödinger operator is the Dirac operator

$$h = -i\sigma_2\frac{d}{dx} + m(x)\sigma_3 + q(x)$$

with Pauli matrices

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and locally integrable coefficients $m$, $q$. The coefficient $m$, corresponding to the mass of the particle, is often taken to be constant.
In many situations the Dirac operator has qualitatively similar spectral properties to the Schrödinger operator, but it generally differs in essential aspects. Thus it is always

unbounded below, and in the limit-point case at $\infty$. For constant or at least essentially bounded $m$, its spectrum is never purely discrete (see appendix of [12]). Furthermore, its main part is unitarily equivalent to its negative,

$$-i\sigma\frac{d}{dx} + m\sigma_3 - q \cong -(-i\sigma_2\frac{d}{dx} + m\sigma_3 + q),$$

and therefore the potentials $q$ and $-q$ give rise to spectra of the same qualitative structure. The usual interpretation of this fundamental difference to the Schrödinger operator is that the Dirac operator describes a particle-antiparticle pair rather than a single particle. In other words, the confinement of particles between high potential walls, familiar from nonrelativistic quantum mechanics, is absent from the Dirac theory, as the Dirac particle can penetrate any potential barrier by turning into an antiparticle.

The Dirac system with a divergent potential $\lim_{x\to\infty} q(x) = \infty$ (or, equivalently, $-\infty$) was first studied by Plesset [8] in the case of polynomial $q$, showing that the spectrum is purely (absolutely) continuous filling the whole real line. Rose and Newton [10] extended this observation to general eventually non-decreasing potentials; as shown below, this is correct although their proof contains a fatal error, as it incorrectly assumes that the presence of the mass term $m\sigma_3$ does not significantly change the asymptotics of the solutions of the eigenvalue equation for $h$.
Roos and Sangren [9] classified the qualitative spectral properties of one-dimensional Dirac operators in various situations, stating 'continous spectrum $-\infty < \lambda < \infty$' if $\lim_{x\to\infty} q(x) = \infty$. This would indeed appear plausible in view of the fact that the somewhat analogous Schrödinger operator with $\lim_{x\to\infty} q(x) = -\infty$ has this spectral structure except for potentials growing extremely fast, almost to the point of the loss of the limit-point property at $\infty$, and such a natural growth limit does not exist for the Dirac system.
Nevertheless, $\lim_{x\to\infty} q(x) = \infty$ by itself is consistent both with the existence of eigenvalues and of gaps in the essential spectrum, as demonstrated by examples in [13] and [14].
A closer look at the proof of Roos and Sangren reveals that they essentially assume a further condition on the potential of the type

$$q \in C^2(\,\cdot\,,\infty), \qquad \int^\infty \left(\frac{q'^2}{q^3} + \frac{|q''|}{q^2}\right) < \infty.$$

Thus their criterion for purely (absolutely) continuous spectrum covering the whole real line is effectively identical with that given by Titchmarsh [18]. It was subsequently simplified by Erdélyi [2] to

$$q \in AC_{\mathrm{loc}}(\,\cdot\,,\infty), \qquad \int^\infty \frac{|q'|}{q^2} < \infty.$$

The regularity condition on $q$ can mildly be weakened to the requirement that $q$ be locally of bounded variation, i.e. that

$$\sup \sum_{j=1}^{n} |q(x_j) - q(x_{j-1})| < \infty$$

where the supremum is taken over all finite collections $x_0 < x_1 < \cdots < x_n$ in the domain of $q$, $n \in \mathbf{N}$. Indeed, denoting by

$$Pf(x) := \sup \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))_+$$

the *positive variation* of a function $f : [c, \infty) \to \mathbf{R}$ of locally bounded variation (where the supremum is taken over all partitions with $x_0 = c$, $x_n = x$), we have (cf. [13])

**Proposition 1.** *Let $m = 1$, $q = w + r$, $w \in BV_{\mathrm{loc}}[c, \infty)$, $\lim_{x\to\infty} w(x) = \infty$, $r/w \in L^1[c, \infty)$. Then $h$ has purely absolutely continuous spectrum filling the real line if $1/w$ has bounded positive variation.*

If $q \in AC_{\mathrm{loc}}$, then $P(1/q) = \int_c^{\cdot} \frac{(q')_-}{q^2}$, recovering Erdélyi's result.
If $q$, not necessarily continuous, is eventually non-decreasing, its positive variation is eventually constant, which vindicates the Rose-Newton conjecture.

The above criterion can be made quantitative to yield a sufficient condition for the absence of eigenvalues (while permitting the possibility of gaps in the essential spectrum) — cf. [3], [15].

**Proposition 2.** *Let $q = w + r$, $w \in BV_{\mathrm{loc}}[c, \infty)$, $\lim_{x\to\infty} w(x) = \infty$, $r \in L^1_{\mathrm{loc}}[c, \infty)$, such that*

$$\limsup_{x\to\infty} \frac{1}{\log x} \left( P(1/w)(x) + \int_c^x \frac{|r|}{w} \right) < \frac{1}{2}.$$

*Then the eigenvalue equation $(-i\sigma_2 \frac{d}{dx} + \sigma_3 + q)\, u = \lambda u$ has no non-trivial $L^2(\,\cdot\,, \infty)$-solution for any $\lambda \in \mathbf{R}$.*

In the following, we shall present an approach which yields a transparent proof under minimal hypotheses for results of the above type.

## 2 A Central Theorem.

**Theorem.** *Let $M, M_1, Q, Q_1 \in L^1_{\mathrm{loc}}$ be real-valued functions such that $\dot{M} \geq 0$, $\lim_{x\to\infty} Q(x) = \infty$,*

$$\limsup_{x\to\infty} \frac{M(x)}{Q(x)} < 1, \qquad \text{and} \qquad \frac{M}{Q-M} \in BV_{\mathrm{loc}}[c, \infty).$$

*Let $\alpha$ be the non-decreasing function*

$$\alpha(x) := P\left(\frac{M}{Q-M}\right)(x) + \int_c^x \frac{|QM_1 - MQ_1|}{Q-M} \qquad (x \in [c, \infty)).$$

*Consider the equation*

$$(-i\sigma_2 \frac{d}{dx} + (M + M_1)\sigma_3 + Q - Q_1)\, u = 0. \tag{$*$}$$

*a) $(*)$ has no non-trivial solution $u \in L^2(\,\cdot\,, \infty)$ if $\int_c^x e^{-2\alpha} = \infty$.*

*b) All non-trivial solutions $u$ of $(*)$ have $\log|u|$ bounded if $\alpha(\infty) < \infty$.*

*Remark.* Taking $M = 1$, $M_1 = 0$, $Q = w - \lambda$ and $Q_1 = r$, we find that $\log|u|$ is bounded for all non-trivial solutions of

$$(-i\sigma_2 \frac{d}{dx} + \sigma_3 + (w + r))\, u = \lambda u$$

for any $\lambda \in \mathbb{R}$ if

$$P\left(\frac{1}{w - \lambda - 1}\right) + \int_c^{\cdot} \frac{|r|}{w - \lambda - 1},$$

or equivalently,

$$P\left(\frac{1}{w}\right) + \int_c^{\cdot} \frac{|r|}{w},$$

is bounded. In particular, there are no subordinate solutions in the sense of Gilbert-Pearson theory ([4], [1] for Dirac systems; a simple proof for the special case needed here can be found in [13]), and it follows that purely absolutely continuous spectrum covers all of $\mathbb{R}$, thus proving Proposition 1 above, which in turn entails all previous criteria.

The same choice of $M$, $M_1$, $Q$ and $Q_1$ proves Proposition 2. Indeed, under the hypotheses of Proposition 2, there is $x_0 \geq c$ such that $\alpha(x) \leq \frac{1}{2} \log x$ $(x \geq x_0)$, and hence

$$\int_c^\infty e^{-2\alpha} \geq \int_{x_0}^\infty \frac{dx}{x} = \infty.$$

The proof of our central Theorem uses the following Gronwall-type lemma for Stieltjes integrals, which can be proved mimicking the proof of [7] Theorem 1.4.

**Lemma 1.** *Let* $\alpha : [c, \infty)$ *be non-decreasing,* $\alpha(c) = 0$*, and* $f : [c, \infty) \to [0, \infty)$ *continuous such that*

$$f(x) \le C + \int_c^x f(t)\, d\alpha(t) \qquad (x \ge c)$$

*for some* $C > 0$*. Then* $f(x) \le Ce^{\alpha(x)}$ $(x \ge c)$*.*

*Proof* of the central Theorem.
Let $u$, $v$ be linearly independent, $\mathbf{R}^2$-valued solutions of $(*)$; then

$$(v_1^2)' = 2v_1v_2(M + M_1 - Q - Q_1), \quad (v_2^2)' = 2v_1v_2(M + M_1 + Q + Q_1).$$

The key to the problem is the function

$$R := |v|^2 + 2v_1^2 \frac{M}{Q - M} \in BV_{\mathrm{loc}}[c, \infty)$$

which can be interpreted geometrically as the square of the major radius of the ellipse in the $(v_1, v_2)$-plane on which the solution would move if the coefficients of the equation $(*)$ were frozen to their momentary values.
By the formula for integration by parts for Stieltjes integrals, we find

$$\begin{aligned}
R(x) - R(c) &= \int_c^x (|v|^2)' + \int_c^x 2\frac{M}{Q - M}(v_1^2)' + \int_c^x 2v_1^2\, d\left(\frac{M}{Q - M}\right) \\
&\le \int_c^x 4v_1v_2\left(M + M_1 + \frac{M}{Q - M}(M + M_1 - Q - Q_1)\right) \\
&\quad + \int_c^x 2v_1^2\, dP\left(\frac{M}{Q - M}\right) \\
&\le \int_c^x 2|v|^2 \frac{|QM_1 - MQ_1|}{Q - M} + \int_c^x 2|v|^2\, dP\left(\frac{M}{Q - M}\right);
\end{aligned}$$

and hence $|v(x)|^2 \le R(x) \le R(c) + \int_c^x 2|v|^2\, d\alpha$.
By Lemma 1, this implies $|v(x)|^2 \le R(c)e^{2\alpha(x)}$ $(x \ge c)$.
Now if $W$ is the Wronskian of the fundamental system $(u, v)$, then $|u|^2|v|^2 = W^2 + (u_1v_1 + u_2v_2)^2 \ge W^2$, and we conclude

$$|u(x)|^2 \ge \frac{W^2}{R(c)} e^{-2\alpha(x)} \qquad (x \ge c). \qquad \square$$

### 3 Angular Momentum.

The above results can be extended to three-dimensional spherically symmetric Dirac operators

$$H = -i\alpha \cdot \nabla = m(|\cdot|)\beta + q(|\cdot|),$$

where $\alpha_1, \alpha_2, \alpha_3$ and $\beta = \alpha_0$ are symmetric $4 \times 4$ Dirac matrices satisfying the anti-commutation relations $\alpha_i\alpha_j + \alpha_j\alpha_i = 0$.
By separation of variables in spherical polar coordinates, $H$ is unitarily equivalent to the direct sum of one-dimensional Dirac operators

$$h_k = -i\sigma_2\frac{d}{dx} + \sqrt{1+\frac{k^2}{x^2}}\,\sigma_3 + q + \frac{k}{2(x^2+k^2)} \qquad (x \in (0,\infty)),$$

$k \in \mathbb{Z} \setminus \{0\}$.
In the literature, the radial Dirac operator traditionally appears in the form

$$-i\sigma_2\frac{d}{dx} + \sigma_3 + \frac{k}{x}\sigma_1 + q,$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, but this operator is unitarily equivalent to the above $h_k$ (which has better behaviour at $\infty$) by virtue of the following observation (cf. [11] Lemma 3).

**Lemma 2.** *Let $I \subset \mathbb{R}$ be an interval, $q \in L^1_{\mathrm{loc}}(I)$, $m, l \in AC_{\mathrm{loc}}(I)$, $m > 0$. Then, with $\vartheta := \arctan l/m$ and*

$$A := \begin{pmatrix} \cos\vartheta/2 & -\sin\vartheta/2 \\ \sin\vartheta/2 & \cos\vartheta/2 \end{pmatrix},$$

*we have*

$$A^*\left(-i\sigma_2\frac{d}{dx} + m\sigma_3 + l\sigma_1 + q\right)A \cong -i\sigma_2\frac{d}{dx} + \sqrt{m^2+l^2}\,\sigma_3 + q + \frac{lm' - l'm}{2(m^2+l^2)}.$$

Under the hypotheses of Proposition 1 or Proposition 2, we can apply our central Theorem to $h_k$, choosing $M = 1$, $M_1 = \sqrt{1+(k/x)^2} - 1 \le k^2/x^2$, $Q = w - \lambda$ and $Q_1 = r + k/2(x^2+k^2)$; then

$$\begin{aligned}\alpha(x) &= P\left(\frac{1}{q-\lambda-1}\right)(x) + \int_c^x \frac{\left|(q(t)-\lambda)\left(\sqrt{1+\frac{k^2}{x^2}}-1\right) - r(t) - \frac{k}{2(t^2+k^2)}\right|}{q(t)-\lambda-1}\,dt \\ &\le P\left(\frac{1}{q-\lambda-1}\right)(x) + \int_c^x \frac{|r|}{q-\lambda-1} + \int_c^x \left(\frac{q(t)-\lambda}{q(t)-\lambda-1}\,k^2 + \frac{|k|}{2}\right)\frac{dt}{t^2}.\end{aligned}$$

The last integral remains bounded as $x \to \infty$. We thus obtain the following analogues of Propositions 1 and 2, with constant $m = 1$.

**Proposition 1$^a$.** *Under the hypotheses of Proposition 1, $h_k$ has purely absolutely continuous spectrum filling the real line, for all $k \neq 0$. As a consequence, $H$ has the same spectral structure.*

**Proposition 2$^a$.** *Under the hypotheses of Proposition 2, the eigenvalue equation*

$$(-i\sigma_2 \frac{d}{dx} + \sigma_3 + \frac{k}{x}\sigma_3 + q(x))\, u = \lambda u$$

*has no non-trivial $L^2(\,\cdot\,, \infty)$ solution for any lambda $\in \mathbf{R}$, $k \neq 0$. Consequently, $H$ has no eigenvalues.*

*Remarks.*

1. For more general perturbations $l\sigma_1$ instead of $\frac{k}{x}\sigma_1$, the above choice of $M$, $M_1$, $Q$, $Q_1$ does not always yield the best possible result for analogues of Proposition 2$^a$; see [15] Corollary 1.4., where a generalisation of the Evans-Harris criterion is obtained by choosing $M = \sqrt{1 + l^2}$, $M_1 = 0$.

2. In [16] analogues of Proposition 1$^a$ were obtained for spherically symmetric Dirac operators with a variable mass term $m$ which is either assumed to be dominated by $q$ near infinity, or equal to $q$. Such variable-mass Dirac systems have been proposed in the physical literature as models of quark confinement.
It turned out that for best results in this case it is advisable not to consider the above $h_k$, but to treat the angular momentum term in the usual representation by generalising the central Theorem to equations of the type

$$(-i\sigma_2 \frac{d}{dx} + M\sigma_3 + L\sigma_1 + Q)\, u = 0$$

(see Proposition 2 of [16]).

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