数理解析研究所講究録 1316 巻 2003 年 13-18



# On the Number of Poles of the First Painlevé Transcendents and Higher Order Analogues II

Shun Shimomura

Department of Mathematics

下村 俊 (慶應大・理工)

## 1. Introduction

Let $w(z)$ be an arbitrary solution of the first Painlevé equation

(PI) $$w'' = 6w^2 + z$$

($' = d/dz$). Then, $w(z)$ is a transcendental meromorphic function, and every pole is double. The counting function for poles is defined by

$$N(r,w) = \int_0^r \big(n(\rho,w) - n(0,w)\big)\frac{d\rho}{\rho} + n(0,w)\log r,$$

where $n(r,w)$ denotes the number of poles inside the disk $|z| \le r$, each counted according to its multiplicity. By a well-known argument in the Nevanlinna theory ([4, §2.4]), we have

(1.1) $$\liminf_{r\to\infty} \frac{m(r,w)}{T(r,w)} = 0, \quad \text{namely,} \quad \limsup_{r\to\infty} \frac{N(r,w)}{T(r,w)} = 1,$$

which implies $N(r,w) \to \infty$ as $r \to \infty$. Here, $m(r,w)$ and $T(r,w)$ are, respectively, the proximity and the characteristic functions defined by

$$m(r,w) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |w(re^{i\phi})|d\phi, \qquad \log^+ x = \max\{0, \log x\},$$
$$T(r,w) = m(r,w) + N(r,w)$$

(for the standard notation and basic facts in the Nevanlinna theory, see [2], [4]). For the magnitude of $N(r,w)$, the following is known ([1], [5], [6], [9]):

(1.2) $$r^{5/2}\log r \ll N(r,w) \ll r^{5/2},$$

which implies that the growth order of $w(z)$

$$\sigma(w) = \limsup_{r\to\infty} \frac{\log T(r,w)}{\log r}$$

is equal to 5/2. (We write $f(r) \ll g(r)$ (or $g(r) \gg f(r)$) if $f(r) = O(g(r))$ as $r \to \infty$.)

A sequence of higher order analogues of (PI) is given by the following:

$$(\mathrm{PI}_{2\nu}) \qquad d_{\nu+1}[w] + 4z = 0, \qquad \nu \in \mathbf{N}$$

(cf. [1, §16]; [3]). Here, $d_\nu[w]$ ($\nu = 0, 1, 2, ...$) are differential polynomials in $w$ determined by

$$(1.3) \quad d_0[w] = 1,$$

$$(1.4) \quad Dd_{\nu+1}[w] = (D^3 - 8wD - 4w')d_\nu[w], \qquad D = d/dz, \qquad \nu \in \mathbf{N} \cup \{0\}.$$

Since

$$d_2[w]/4 = -w'' + 6w^2 + C_1 w + C_0,$$

where $C_j \in \mathbf{C}$ ($j = 0, 1$) are arbitrary, equation ($\mathrm{PI}_2$) essentially coincides with (PI). In general, ($\mathrm{PI}_{2\nu}$) is a $2\nu$-th order nonlinear equation; e.g. for $\nu = 2, 3$,

$$(\mathrm{PI}_4)_0 \quad w^{(4)} = 20ww'' + 10(w')^2 - 40w^3 + z,$$

$$(\mathrm{PI}_6)_0 \quad w^{(6)} = 28ww^{(4)} + 56w'w^{(3)} + 42(w'')^2 - 280(w^2w'' + w(w')^2 - w^4) + z,$$

where the arbitrary constants corresponding to $C_j$ of ($\mathrm{PI}_2$) are taken to be 0. Let $w_\nu(z)$ be an arbitrary meromorphic solution of ($\mathrm{PI}_{2\nu}$). It is interesting to evaluate the growth order of $w_\nu(z)$. The following result gives a lower estimate of it:

**Theorem 1.1.** *For every* $\nu \in \mathbf{N}$,

$$(1.5) \qquad \limsup_{r\to\infty} \frac{\log N(r, w_\nu)}{\log r} \geq \frac{2\nu+3}{\nu+1},$$

*namely the growth order of* $w_\nu(z)$ *is not less than* $(2\nu+3)/(\nu+1)$.

As an immediate consequence, we have

**Corollary 1.2.** *Equation* ($\mathrm{PI}_{2\nu}$) *admits no rational solutions.*

Viewing Theorem 1.1 combined with (1.2), we pose the following:

**Conjecture.** *The growth order of* $w_\nu(z)$ *is equal to* $(2\nu+3)/(\nu+1)$.

We sketch the proof of Theorem 1.1, illustrating the particular case $\nu = 2$. The full proof is found in [8].

## 2. Sketch of the proof of Theorem 1.1 for ($\mathrm{PI}_4$)

The basic idea is the same as in the proof for (PI) (cf. [7]). Suppose the contrary:

$$(2.1) \qquad \limsup_{r\to\infty} \frac{\log N(r, w_2)}{\log r} < \frac{7}{3},$$

namely, for some $\varepsilon > 0$, $N(r, w_2) \ll r^{7/3-\varepsilon}$, from which it follows that

$$n(r) = n(r, w_2) \ll r^{7/3-\varepsilon}, \tag{2.2}$$

because

$$N(2r, w_2) \geq \int_r^{2r} (n(\rho, w_2) - n(0, w_2))\frac{d\rho}{\rho} \geq (n(r, w_2) + O(1)) \log 2.$$

Starting from (2.1), we will derive a contradiction. Let $\{a_j\}_{j=1}^{\infty}$ (or $\{a_j\}_{j=1}^{q}$, $q \in \mathbf{N}$) be the sequence of all distinct poles of $w_2(z)$ arranged as $|a_1| \leq \cdots \leq |a_j| \leq \cdots$. It is easy to check that, around each pole $a_j$,

$$w_2(z) = c(j)(z - a_j)^{-2} + O(1),$$

where $c(j) = 1$ or $3$. By this fact combined with (2.2), we write $w_2(z)$ in the form

$$w_2(z) = \Phi(z) + \varphi(z), \tag{2.3}$$

$$\Phi(z) = \sum_{j=1}^{\infty} c(j)((z - a_j)^{-2} - a_j^{-2}), \tag{2.4}$$

where $\varphi(z)$ is an entire function; and in (2.4), if $a_1 = 0$ the term $(z - a_1)^{-2} - a_1^{-2}$ should be replaced by $z^{-2}$. Under (2.2), we have the following lemmas whose proofs are similar to those of [7, Lemmas 1.1 and 1.2].

**Lemma 2.1.** *For every $r > 1$, there exists $z_r$ satisfying*

$$0.7r \leq |z_r| \leq r, \quad \sum_{|a_j|<2r} |z_r - a_j|^{-2} \ll r^{1/3-\varepsilon/2}, \quad \sum_{|a_j|<2r} |z_r - a_j|^{-3} \ll r^{1/2-\varepsilon}.$$

**Lemma 2.2.** *Let $r$ be an arbitrary number satisfying $r > 1$. Then,*

$$\sum_{|a_j|\geq 2r} |(z - a_j)^{-2} - a_j^{-2}| \ll r^{1/3-\varepsilon}, \quad \sum_{|a_j|\geq 2r} |z - a_j|^{-3} \ll 1$$

*for $|z| \leq r$, and*

$$\sum_{0<|a_j|<2r} |a_j^{-2}| \ll r^{1/3-\varepsilon}.$$

By a well-known argument of the Nevanlinna theory, it is shown that $\varphi(z)$ is a polynomial. Note that $|\Phi(z)| \leq |\sum_{|a_j|<2r}| + |\sum_{|a_j|\geq 2r}|$. By Lemmas 2.1 and 2.2, for every $r > 1$, there exists $z_r$, $0.7r \leq |z_r| \leq r$ satisfying

$$\begin{aligned} &|\Phi(z_r)| \ll r^{1/3-\varepsilon/2}, \qquad |\Phi'(z_r)| \ll r^{1/2-\varepsilon}, \\ &|\Phi''(z_r)| \ll r^{2/3-\varepsilon}, \qquad |\Phi^{(4)}(z_r)| \ll r^{1-3\varepsilon/2}. \end{aligned} \tag{2.5}$$

(2.6) $$w_2(z_r) \ll \left(|w_2^{(4)}(z_r)| + |w_2(z_r)||w_2''(z_r)| + |w_2'(z_r)|^2 + |z_r|\right)^{1/3}$$
$$\ll |w_2^{(4)}(z_r)|^{1/3} + |w_2(z_r)|^{1/3}|w_2''(z_r)|^{1/3} + |w_2'(z_r)|^{2/3} + |z_r|^{1/3}.$$

Substituting $w_2^{(k)}(z_r) = \varphi^{(k)}(z_r) + \Phi^{(k)}(z_r)$ $(k = 0, 1, 2, 4)$ into (2.6) and using $|\Phi^{(k)}(z_r)| \ll r^{1/3+k/6}$ (cf. (2.5)), we have

(1) $$|\varphi(z_r)| \ll r^{1/3} + |\varphi^{(4)}(z_r)|^{1/3}$$
$$+ (r^{1/9} + |\varphi(z_r)|^{1/3})(r^{2/9} + |\varphi''(z_r)|^{1/3}) + r^{1/3} + |\varphi'(z_r)|^{2/3},$$

which implies that $\varphi(z) \equiv C \in \mathbf{C}$. Then, by $(\mathrm{PI}_4)$,

$$0.7r \le |z_r| \ll |w_2^{(4)}(z_r)| + |w_2(z_r)||w_2''(z_r)| + |w_2'(z_r)|^2 + |w_2(z_r)|^3 \ll r^{1-\varepsilon},$$

which is a contradiction. Thus Theorem 1.1 with $\nu = 2$ follows.

## 3. Genaral case

To treat the general case, we need to know some facts related to the terms of the differential polynomial $d_{\nu+1}[w]$. Let $[w, w', ..., w^{(p)}]^{\boldsymbol{\iota}}$ denote the monomial $w^{\iota_0}(w')^{\iota_1} \cdots (w^{(p)})^{\iota_p}$, where $\boldsymbol{\iota} = (\iota_0, \iota_1, ..., \iota_p) \in (\mathbf{N} \cup \{0\})^{p+1}$. For this monomial with $\boldsymbol{\iota} = (\iota_0, \iota_1, ..., \iota_p)$, we define the weight of it by

$$||\boldsymbol{\iota}|| := \sum_{\kappa=0}^{p} (2+\kappa)\iota_\kappa.$$

Then, $d_{\nu+1}[w]$ is written in the form:

**Lemma 3.1.** *For every $\nu \in \mathbf{N} \cup \{0\}$,*

$$d_{\nu+1}[w] = \gamma_{\nu+1} w^{\nu+1} + \sum_{||\boldsymbol{\iota}|| \le 2(\nu+1),\ \iota_0 \le \nu} c_{\boldsymbol{\iota}} [w, w', ..., w^{(2\nu)}]^{\boldsymbol{\iota}}, \qquad \boldsymbol{\iota} = (\iota_0, \iota_1, ..., \iota_{2\nu}),$$

*where $c_{\boldsymbol{\iota}} \in \mathbf{C}$, $\gamma_{\nu+1} \in \mathbf{C} \setminus \{0\}$.*

To show Theorem 1.1 for the general case, we start from the supposition that

$$N(r, w_\nu) \ll r^{(2\nu+3)/(\nu+1)-\varepsilon},$$

which implies that

(3.1) $$n(r, w_\nu) \ll r^{(2\nu+3)/(\nu+1)-\varepsilon}$$

for some $\varepsilon > 0$. Let $\{a_j\}_{j=1}^{\infty}$ (or $\{a_j\}_{j=1}^{q}$) be a sequence of distinct poles of $w_\nu(z)$. Around $a_j$, we have

$$w_\nu(z) = c(a_j)(z - a_j)^{-2} + O(1),$$

where $c(a_j) = k(a_j)(k(a_j)+1)/2$ for some $k(a_j) \in \{1, ..., \nu\}$. By (3.1), $w_\nu(z)$ is written in the form

$$w_\nu(z) = \sum_{a_j} c(a_j)\big((z-a_j)^{-2} - a_j^{-2}\big) + \varphi(z),$$

where $\varphi(z)$ is an entire function. Instead of Lemmas 2.1 and 2.2, we have the following under supposition (3.1):

**Lemma 3.2.** *For every $r > 1$, there exists $z_r$ satisfying*

$$0.7r \le |z_r| \le r, \quad \sum_{|a_j|<2r} |z_r - a_j|^{-2} \ll r^{1/(\nu+1)-\varepsilon/2}, \quad \sum_{|a_j|<2r} |z_r - a_j|^{-3} \ll r^{(3/2)/(\nu+1)-\varepsilon}.$$

**Lemma 3.3.** *Let $r$ be an arbitrary number such that $r > 1$. Then*

$$\sum_{|a_j|\ge 2r} \big|(z-a_j)^{-2} - a_j^{-2}\big| \ll r^{1/(\nu+1)-\varepsilon}, \qquad \sum_{|a_j|\ge 2r} |z-a_j|^{-3} \ll 1$$

*for $|z| \le r$, and*

$$\sum_{0<|a_j|<2r} |a_j^{-2}| \ll r^{1/(\nu+1)-\varepsilon}.$$

Using Lemmas 3.2 and 3.3 combined with Lemma 3.1, we prove Theorem 1.1 for the general case.

# References


[1] Gromak, V. I., Laine, I. and Shimomura, S., *Painlevé Differential Equations in the Complex Plane* Walter de Gruyter, Berlin, New York 2002.

[2] Hayman, W. K., *Meromorphic Functions* Clarendon Press, Oxford, 1964.

[3] Kudryashov, N. A., The first and second Painlevé equations of higher order and some relations between them, Phys. Lett. A **224**, 353–360 (1997).

[4] Laine, I., *Nevanlinna Theory and Complex Differential Equations* Walter de Gruyter, Berlin, New York, 1993.

[5] Shimomura, S., Growth of the first, the second and the fourth Painlevé transcendents, to appear in Math. Proc. Cambridge Philos. Soc.

[6] Shimomura, S., Lower estimates for the growth of Painlevé transcendents, to appear in Funkcial. Ekvac.

[7] Shimomura, S., On the number of poles of the first Painlevé transcendents and higher order analogues, SūrikaisekikenkyūshoKōkyūroku No. 1296, 124–127 (2002).

[8] Shimomura, S., Poles and $\alpha$-points of meromorphic solutions of the first Painlevé hierarchy, Preprint (2003).

[9] Steinmetz, N., Value distribution of Painlevé transcendents, Israel J. Math. **128**, 29–52 (2002).