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Some Historical note on random fields

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1 Introduction

We are interested in a generalization of the study of stochastic processes and a mathematical formulation of random complex systems. Such motivations lead us to the investigation of random fields which have a higher dimensional parameter or which are parametrized by a manifold with certain geometric structures. In the course of such investigations, we are naturally led to revisit the theory of classical functional analysis.

First of all we would like to discuss the motivations. One of the motivations of our work on the variational calculus for random fields came from the observation of Lévy's Brownian motion. While we studied Lévy's Brownian motion, we have tried many attempts how to express the complex way of dependence of the Brownian motion. The dependence becomes extremely complex as soon as we come to the higher dimensional parameter case. First, we considered a conditional expectation of the variable at some point in the parameter space, given its values at the points on some manifold C. Let it vary, and we found that the conditional expectation of the value at the point changes drastically in a complex manner. Such a change happens even the manifold C is a segment.

In other words, a conditional expectation mentioned above can be considered as a random field $X(C)$, and it can be viewed as a functional of C. It would be a good method to clarify the complex structure of a random field $X(C)$ by taking its variation.

Some other motivations are

i) C. E. Shannon's method of characterising probability distributions by using the information quantity (entropy),

ii) Lévy's infinitesimal equation for a stochastic process,

$$\delta X(t) = \Phi(X(s), s \leq t, Y(t), t, dt),$$

where $Y(t)$ is the innovation, tells us the impotance of innovation,

iii) Wiener's theory of prediction

(1) Time series 1949.

(2) Nonlinear problems in random theory. 1958. Coding-decoding.

Our original purpose is the investigation of random fields parameterised by a manifold C. One of the powerful methods is the so-called *innovation approach*, where the given fields are expressed in terms of a generalized stochastic process with independent values at every instant.

To fix the idea, such a generalized stochastic process may be taken to be either(Gaussian) white noise or a Poisson noise. The main part of our study, therefore, involves the analysis of functionals, in general nonlinear functionals, of those noises. Thus, we are led to the analysis of such functionals.

There are two typical cases; namely

(a) functionals are assumed to have finite variance, hence L^2 theoretic approach can be applied,

(b) sample function properties of the given system are to be clarified at each step of the analysis.

What we are going to discuss in this report will mainly be concerned with the case (a). Again to fix the idea we assume the collection of functionals of Gaussian white noise. We are therefore given a Hilbert space $(L^2) = L^2(E^*, \mu)$ of square integrable functionals of a white noise, where μ is the probability distribution of white noise. There is an established theory of representation of (L^2)-functionals of x, a sample function of white noise which is a generalized function, say in the dual space E^* of some nuclear space E. Namely, introduce the *S-transform* of those functionals. Let $\varphi(x)$ be in (L^2), then its S-transform is defined by

$$(S\varphi)(\xi) = \exp(-\frac{1}{2}\|\xi\|^2) \int_{E^*} \exp[\langle x, \xi\rangle]\varphi(x)d\mu(x), \ \xi \in E. \tag{1}$$

The collection of S-transforms of (L^2)-functional forms a Reproducing Kernel Hilbert space $\mathbf{F}$ which is isomorphic to (L^2). In short, φ has a representation in terms of a functional (indeed, non-random functional). We can therefore appeal to the classical theory of functional analysis.

If, in particular, the functional is parameterized by a manifold C, expressible as $\varphi(C, x)$, then the S-transform is of the form $U(C, \xi)$. We note that the variation $\delta\varphi(C, x)$, x being fixed, then the S-transform is the variation $\delta U(C, \xi)$, ξ now being fixed. Also, the partial derivative $\frac{\partial}{\partial x(t)}$ acting on φ corresponds to the Fréchet derivative of U with respect to the variable ξ.

Thus, we see that, important part of the analysis of random fields can be reduced to the calculus of classical (non-random) fields through the S-transform.

Needless to say, the stochastic calculus for random fields requires important additional study which is proper for random functions. This report, apart from this problem, reviews some classical results on functional analysis which are useful in the study of stochastic variational calculus.

2 Classical fields and random fields

In this section we shall breifly review the classical fields and random fields. The random fields can be transformed to the classical random fields and thus we can appeal the theory of classical fields

1. Classical field

The variation calculus for classical (non-random) fields is developed in the first half of the 20th century in line with the Classical theory of Functional analysis. We will briefly discuss some important development on this subject in the following.

1) Hadamard(1908) : Green's function $g(a, b; C)$

At the end of 19th century Hadamard first encountered the calculus of variations when working on Wave theory, Elasticity, and Geometrical Problems; Geodesics.

He discussed the functional operation in 1903 in his paper *On the functional operations*. Comptes Rendus 136, 351-354.

In the preface of his book *Leçon's sur le calcul des variations,* Paris, published in 1910, we can see his idea as follows.

The calculus of variation is nothing else than the first chapter of the theory which is nowadays called the calculus of functionals, and whose development will undoubtment will undoubtedly be one of the first tasks of the future. It is this idea which inspired me above all, in the course of lectures

I gave this topic at the Collége de France as well as in the preparation of this work.

The Green function $G(a, b; C)$ can be considered as a functional of a curve C.

2) Tonelli(1922) : Line integral

We may consider the line integrals, discussed in the book by Tonelli, as a classical field.

3) Lévy(1951)

Systematic approach to the variations of the integrals over a curve has been done by P. Lévy in 1951 and we can apply it to random field.

2. Fields in Physics

Various directions of quantum dynamics had beautiful interplay with stochastic analysis, like mathmatical physics in 18th century. We make a short visit in the following.

1) Quantum Fields : P.A.M. Dirac.

His 1933 paper "The Lagrangian in quantum mechanics" seems to be a generalization of the action principle to quantum fields. The paper contained a profound suggestion.

We may imagine, if understood correctly, that he expected the action is computed as in the case of Markov process. We modified slightly. Also, our idea is that the Markov property, even multiple Markov case, we do not want to assume the regularity of paths. We have therefore defined the Markov property in analogy with the case of Gaussian processes.

2) $P(\phi)$ theory

3) String theory (Y. Nambu and others)

4) Tomonaga-Schwinger equation (S. Tomonaga)

3. Random fields

Random fields parametrized by a manifold C are studied by Lévy, McKean, Hida, etc. Namely,

P. Lévy (1948) : Multi-dimensional Brownian motion restricted on hyperplane.

Lévy's Brownian motion $B(a), a \in R^d$ is a natural generalization of the ordinary Brownian motion $B(t), t \in R$.

Let $\mathbf{B} = \{B(a), a \in R^d\}$ be a Gaussian system. If the conditions

i) $E(X(a)) = 0$ for every $a \in R^d$,

ii) $X(o) = 0$, where o is the origin of R^d,

iii) $E(X(a) - X(b))^2 = |b - a|$, where $|\cdot|$ is the norm in R^d,

are satisfied, then $\mathbf{B}$ is called the *Lévy Browniann motion.*
The covariance function is given by

$$E(X(a)X(b)) = \frac{1}{2}(|a| + |b| - |a - b|).$$

Many interesting results have been obtained starting from P. Lévy's book, 1948 [8]. The representations have been obtained by McKean and Chentsov. Mckean's expansion of $B(a)$ is significant for our study. It should be noted that the Lévy Brownian motion enjoys profound, complex way of dependence from the view point of probability. Indeed $B(a)$ contains more complexity compared to the case of the ordinary one-dimensional parameter Brownian motion. From the information theoretical viewpoint, we can say that more information is carried in the case of $B(a)$ when the paremetr varies.

Lévy's Brownian motion is indeed a good representative of random fields with multi-dimensional parameter in many respect. Not only the Brownian motion itself but also derived fields and processes are also interesting. The results in 1990 by the author illustrate one of the properties of complex dependence.

We need a system of innovations in many directions; either along the radial directions, or along the smooth curves, or along the surface, and so forth. Hence, the $X(a)$ has infinite multiplicity along the direction of development of a. This is another example of significant characteristics of the Brownian motion $X(a)$ with multi-dimensional parameter.

The Lévy flight with multi-dimensional parameter is also another good example of a random field parametrized by a point in multi-dimensional space [13].

2) H. P. McKean (1951) : Markov property : splitting field.

   McKean gave the Markov property by using spliiting field.

3) T. Hida and Si Si : Random field with a parameter C, which is taken as a smooth manifold in R^d.

   We define random fields $X(C)$ to be functionals (in fact, generalized white noise functionals) of a white noise $x(u)$. Namely,

$$\{X(C) = X(C, x); C \in \mathbf{C}\}$$

   where

$$\begin{aligned} \mathbf{C} &= \{C; C \in C^2, \text{diffeomorphic to } S^1, (C) \text{ is convex}\}, \qquad (2) \\ (C) &: \text{ the domain enclosed by } C. \end{aligned}$$

   Note. For the definition of $\mathbf{C}$, convexity of C is required. The reason for this requirement is that when the variation of $X(C)$ is computed, the convexity assumption is necessary.

   We further assume that

$$X(C) \in (L^2).$$

   Hence, the S-transform can be applied to $X(C)$. Let the S-transform be denoted by $U(C, \xi)$:

$$U(C, \xi) = S(X(C))(\xi). \qquad (3)$$

   We are therefore ready to appeal to the classical theory of functional analysis.

   In the case of linear functional of x, the $X(C)$ is expressed in the form

$$X(C) = \int_{R^2} F(C, u)x(u)du^2.$$

   An interesting class involves such integrals as

$$X(C) = \int_{(C)} F(C, u)x(u)du^2,$$

   where (C) denotes the domain enclosed by C. The expression above is called a *causal* representation of $X(C)$ in terms of white noise. Such a representation is called causal since the white noise $x(u), u$ being inside

Topics on such random fields $X(C)$ have been discussed in our papers. Various interesting properties are found by analogy with the canonical representation of a Gaussian process $X(t)$.

Throughout these cases, the idea of the innovation approach is sitting behind.

3 Variational Calculus

Here we wish to list some mathematicians who contributed in the developement of variational calculus for functions of a *manifold.*

1. G. Darboux, Théorie des surfaces tom III. Livre 6ème. 1894

   First variation of arc length is used to have geodesic line.

2. J. Hadamard, Leçons sur le calcul dess variations. 1910. Herman et Fils.

3. E. Goursat, Cours d'analyse mathématique. Tome III, Gauthier-Villars. Chapt.XXXIV Calcul des variations (1911).

   This is elementary, however it was done in a systematic approach to calculus of variations.

4. V. Volterra, Leçons sur les fonctions des lihnes. 1913, Chapt. IX, Section 16. Opere Matematiche. vol. V.

5. P. Lévy

   1) Sur quelques équations définissant des fonctions de ligne. C.R. 151 (1910) 373-375.

      It can be seen as the study of the Hadamard equation.

   2) Les équations intégro-différentielles définissant des fonctions de lignes. Thèse.1911.

      General variational equations, in particular the Hadamard equation, are discussed. Integrability condition is discussed, too.

   3) Sur la variation de la distribution de lélectricité sur un conducteur dont la surface se déforme. Bull. Soc, math. de France 46 (1918) 35-68. (not in the Collected Works.)

      This paper can be viewed as an application of the Hadamard equation.

4) Lecons d'analyse fonctionnelle. Gauthier-Villars, 1922.
   Lévy has discussed functions of line and surface. Lévy Laplacian is discussed in connection with the second variation.
5) Cours de mécanique. Gauthier-Villars, 1928. Chapt. XII Systèmes flexibles.
   In this book he considered the functionals defined as integrasl over a curve and take the variations.
   Equilibrium of a string restricted to a surface is also discussed and has come to the variational equation of the form:
$$\delta I = -\int_{arcAB} k\delta n ds.$$
6) Problèmes concrets d'analyse fonctionnelle. Gauthier-Villars, 1951.
   Part I and Part II can be seen as the generalization of the Hadamard equation. We see many significant examples of functionals of C and their variations.
   In Part III, functionals with certain singularity are taken and the singular part of their second order variation determines the Lévy Laplacian. The analysis employed there gives strong influence to white noise analysis. Significant connections with white noise theory is concerned with the fact that essentially infinite dimensional character can be illustrated through the discussions.

6. L. Tonelli

   Tonelli's work on line integral can be seen in his book *Fondamenti di Calcolo delle Variazioni*, 1922-1924 (2 volumes), Bologna.

7. C.E. Shannon,

   In his book, The mathematical theory of communications (1949),haracterization of probability distributions is done by using the entropy. Optimality is discussed by using the quantity of information.

Addendum Stochastic variational equations.

As is mentioned in Section 1, we are interested in stochastic variational equations. The Lévy infinitesimal equation
$$\delta X(t) = \Phi(X(s), s \leq t, Y(t), t, dt),$$
where $Y(t)$ is the innovation, can be generalized to the infinitesimal equation
$$\delta X(C) = \Phi(X(C'), C' < C, Y(s), s \in C, C, \delta C),$$

for random fields, in which $\{Y(s), s \in C\}$ is innovation of $X(C)$. This infinitesimal equation can be considered as a stochastic variational equation for a random field $X(C)$.

If $X(C)$ is assumed to be a functional of white noise, then apply the S-transform to have $U(C,\xi)$. Then, the variational equation for $U(C) \doteq U(C,\xi)$ corresponds to the stochastic variational equation for $X(C)$ and thus we can aply the classical variational calculus. There $\xi(s)$ corresponds to the innovation $Y(s)$.

We now come to the another case where $X(C)$ is a functional of Poisson noise, then innovation is transformed to $\exp[i\xi(t)] - 1$.

Existence and uniqueness of the stochastic variational equation can be discussed in this case, too.

A rigorous definition of the innovation has been given by the paper [1].

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