

## A Vista of Mean Zeta-Values

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**1. Introduction.** We shall try to view mean values of zeta-functions in a perspective brought out recently by R.W. Bruggeman and the present author [4]. They found a way to grasp the mean value

$$\mathcal{M}(\zeta^2, g) = \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 g(t) dt \tag{1.1}$$

in the spectral structure of  $L^2(\Gamma \backslash G)$ , with  $\Gamma = \text{PSL}_2(\mathbb{Z})$  and  $G = \text{PSL}_2(\mathbb{R})$ . It is shown that there exists a  $\Gamma$ -automorphic function on  $G$ , whose value at the unit element is closely related to  $\mathcal{M}(\zeta^2, g)$ , and whose spectral decomposition in  $L^2(\Gamma \backslash G)$  gives rise to that of  $\mathcal{M}(\zeta^2, g)$ . This amounts to an alternative and direct proof of the explicit formula for  $\mathcal{M}(\zeta^2, g)$  that was established as Theorem 4.2 in [15]. It is direct, because it entirely dispenses with the spectral theory of sums of Kloosterman sums that played a fundamental rôle in [15].

Each term in the cuspidal part of the explicit formula for  $\mathcal{M}(\zeta^2, g)$  is a product of two factors, arithmetic and geometric. They are expressed, respectively, in terms of Hecke series and an integral transform of the weight  $g$ . In the present article we are mainly concerned with this integral transform. An important advantage of the argument of [4] over that of [15] is in that [4] shows explicitly the way how the integral transform comes from basic geometric facts of the Lie group  $G$ , while [15] does not seem to yield readily such an explanation. We shall describe the mechanism thus revealed and proceed to an informal discussion to surmise possible extensions.

The article [4] is perhaps the first instance that any classical subject in Analytic Number Theory is dealt with wholly in the framework of the theory of linear Lie groups. The structural argument as this will find further applications in ANT; it is certainly not an intrusion from without.

**CONVENTION.** Notations are introduced where they are needed first time, and will remain effective thereafter unless otherwise stated. The weight function  $g$  is assumed, for the sake of simplicity, to be even, entire, real on  $\mathbb{R}$ , and of rapid decay in any fixed horizontal strip.

**2. Poincaré series.** The work [4] is a realization of a programme given in Section 4.2 of [15] (see also [13]). There the non-diagonal part of the integral of

$$\int_{-\infty}^{\infty} \zeta(z_1 + it)\zeta(z_2 - it)\zeta(z_3 + it)\zeta(z_4 - it)g(t)dt \tag{2.1}$$

in the region of absolute convergence is regarded as a sum over non-singular  $2 \times 2$  integral matrices. We obviously need to consider either section with matrices of positive or negative determinant. Hecke operators reduce it to a sum over the elements of  $\Gamma$ . Putting it formally, the programme is as follows: We relate the last sum with a certain Poincaré series

$$\mathcal{P}f(g) = \sum_{\gamma \in \Gamma} f(\gamma g), \quad g \in G. \tag{2.2}$$

We decompose this spectrally, and apply the operator

$$\mathcal{T} = \sum_{n=1}^{\infty} T_n n^{-w}, \quad \operatorname{Re} w > 1, \quad (2.3)$$

where  $T_n$  are Hecke operators (see (3.9) below). Then we specialize the result with  $g = 1$ . In [4] a sequence of  $f$  is chosen, so that the limit of  $\mathcal{TP}f(1)$  in  $f$  is equal to one of the sections of the non-diagonal part in question, e.g.,

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sigma_{z_1-z_4}(m) \sigma_{z_2-z_3}(m+n)}{m^{z_1}(m+n)^{z_2}} \hat{g} \left( \log \left( 1 + \frac{n}{m} \right) \right), \quad (2.4)$$

where  $\hat{g}$  is the Fourier transform of  $g$ . The choice is delicate. The  $f$  should be such that  $\mathcal{P}f$  is smooth enough to yield the point-wise convergence of the spectral decomposition. We note that an automorphic regularization too has to be taken into account, since  $\mathcal{P}f$  is not in  $L^2(\Gamma \backslash G)$  in general. In [4] this is done with a subtraction of an infinite sum of Eisenstein series over  $\Gamma \backslash G$  and thus has no relevance to the projections to cuspidal subspaces, however.

Our task is but analogous to a much simpler object: the projection of the Poincaré series

$$\sum_{n \in \mathbb{Z}} h(n+x) \quad (2.5)$$

to irreducible subspaces of  $L^2(\mathbb{Z} \backslash \mathbb{R})$ , where  $h$  is assumed to be smooth and compactly supported on  $(0, \infty)$ . The specialization to the unit element of the decomposition thus obtained is the sum formula

$$\sum_{n=1}^{\infty} h(n) = \int_0^{\infty} h(x) dx + 2 \sum_{n=1}^{\infty} \int_0^{\infty} h(x) \cos(2\pi n x) dx. \quad (2.6)$$

The kernel function  $\cos(2\pi x)$  is the Bessel function of representations of  $\mathbb{R}$ , in the sense of [8]. We shall indicate that in the expansion of  $\mathcal{P}f(1)$  the Bessel function of representations of  $G$  plays a rôle that is certainly more involved but similar in principle.

The Poisson sum formula (2.6) was employed by F.V. Atkinson [1] in his proof of an explicit formula for the mean square  $\mathcal{M}(\zeta, g)$  (see also Section 4.2 of [15]). In other words, his formula is a way to view  $\mathcal{M}(\zeta, g)$  in the spectral structure of  $L^2(\mathbb{Z} \backslash \mathbb{R})$ . In fact, the non-diagonal part in the Atkinson dissection, with which he started his argument, has an abelian construction, and thus it can be effectively analyzed with (2.6). By the same token, the non-diagonal part mentioned at the beginning of this section bears the group structure of  $G$ , and we are to exploit the fact accordingly. In passing, we remark that (2.6) is equivalent to the functional equation for  $\zeta(s)$ .

**3. Basic notion.** We need elements of the theory of  $\Gamma$ -automorphic representations of  $G$ . Thus, write

$$n[x] = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, \quad a[y] = \begin{bmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{bmatrix}, \quad k[\theta] = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad (3.1)$$

with matrices in the projective sense. Let  $N = \{n[x] : x \in \mathbb{R}\}$ ,  $A = \{a[y] : y > 0\}$ , and  $K = \{k[\theta] : \theta \in \mathbb{R}/\pi\mathbb{Z}\}$  so that  $G = NAK$  or  $G \ni g = nak$  be the Iwasawa decomposition

of  $G$ . The Haar measures on the groups  $N, A, K, G$ , are defined, respectively, by  $dn = dx$ ,  $da = dy/y$ ,  $dk = d\theta/\pi$ ,  $dg = dndadk/y$ , with Lebesgue measures  $dx, dy, d\theta$ .

The space  $L^2(\Gamma \backslash G)$  is composed of all left  $\Gamma$ -automorphic functions on  $G$ , vectors for short, which are square integrable against the measure  $dg$  over a fundamental domain of  $\Gamma$ . Elements of  $G$  act unitarily on functions in  $L^2(\Gamma \backslash G)$  from the right, and we have the orthogonal decomposition into invariant subspaces:

$$L^2(\Gamma \backslash G) = \mathbb{C} \cdot 1 \oplus {}^0L^2(\Gamma \backslash G) \oplus {}^eL^2(\Gamma \backslash G). \quad (3.2)$$

Here  ${}^0L^2$  is the cuspidal subspace spanned by functions whose Fourier expansions with respect to the left action of  $N$  have vanishing constant terms. The subspace  ${}^eL^2$  is spanned by integrals of Eisenstein series. The cuspidal subspace is decomposed into irreducible subspaces:

$${}^0L^2(\Gamma \backslash G) = \overline{\bigoplus V}. \quad (3.3)$$

The  $V$  is called also an irreducible cuspidal  $\Gamma$ -automorphic representation. The Casimir operator  $\Omega = y^2 (\partial_x^2 + \partial_y^2) - iy\partial_x\partial_\theta$  becomes a constant multiplication in each  $V$ ; that is,

$$\Omega|_{V^\infty} = \left( \nu_V^2 - \frac{1}{4} \right) \cdot 1, \quad (3.4)$$

where  $V^\infty$  is the set of all smooth vectors in  $V$ . Since  $\Gamma = \text{PSL}_2(\mathbb{Z})$ , there are no complementary series representations; hence we may assume either that  $i\nu_V < 0$  or that  $\nu_V$  is equal to half a positive odd integer. According to the right action of  $K$ , the space  $V$  is decomposed into  $K$ -irreducible subspaces

$$V = \overline{\bigoplus_p V_p}, \quad \dim V_p \leq 1, \quad (3.5)$$

where  $p$  runs over all integers. If it is not trivial,  $V_p$  is spanned by a  $\Gamma$ -automorphic function on which the right translation by  $k[\theta]$  becomes the multiplication by the factor  $\exp(2ip\theta)$ . It is called a  $\Gamma$ -automorphic form of spectral parameter  $\nu_V$  and weight  $2p$ .

Let us assume temporarily that  $V$  belongs to the unitary principal series, i.e.,  $i\nu_V < 0$  under our present setting. Then one can show that  $\dim V_p = 1$  for all  $p \in \mathbb{Z}$  and that there exists a complete orthonormal system  $\{\varphi_p \in V_p : p \in \mathbb{Z}\}$  of  $V$  such that

$$\varphi_p(\mathfrak{g}) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\varrho_V(n)}{\sqrt{|n|}} \mathcal{A}^{\text{sgn}(n)} \phi_p(\mathfrak{a}[|n|]\mathfrak{g}; \nu_V), \quad (3.6)$$

where  $\phi_p(\mathfrak{g}; \nu) = y^{\frac{1}{2}+\nu} \exp(2ip\theta)$  and

$$\mathcal{A}^\delta \phi_p(\mathfrak{g}; \nu) = \int_{-\infty}^{\infty} \exp(-2\pi i \delta x) \phi_p(\mathfrak{w}n[x]\mathfrak{g}; \nu) dx, \quad \delta = \pm, \quad \mathfrak{w} = k(\pi/2). \quad (3.7)$$

The  $\mathcal{A}^\delta$  is a specialization of the Jacquet operator. It should be observed that the coefficients  $\varrho_V(n)$  in (3.6) do not depend on the weight, a fact that can be shown by using the Maass operators. We may assume that each  $V$  is Hecke invariant; that is, for all positive integer  $n$ ,

$$T_n|_V = t_V(n) \cdot 1, \quad T_n = \frac{1}{\sqrt{n}} \sum_{d|n} \sum_{b \bmod d} L_{n[b/d]\mathfrak{a}[n/d^2]} \quad (3.8)$$

with a  $t_V(n) \in \mathbb{R}$ . Here  $L$  is the left translation. Also, the invariance  $\varphi_0(n^{-1}\mathfrak{a}) = \epsilon_V \varphi_0(n\mathfrak{a})$ ,  $\epsilon_V = \pm 1$ , can be assumed; thus we have  $\varrho_V(n) = \varrho_V(1) \epsilon_V^{\frac{1}{2}(1-\text{sgn}(n))} t_V(|n|)$ . With this, we associate to each  $V$  the Hecke series

$$H_V(s) = \sum_{n=1}^{\infty} t_V(n) n^{-s}, \quad (3.9)$$

which converges absolutely for  $\text{Re } s > 1$ , and continues to an entire function.

These concepts are readily extended to representations in the discrete series, i.e., those with  $\nu_V = \ell - \frac{1}{2}$ ,  $1 \leq \ell \in \mathbb{Z}$ . We have

$$\text{either } V = \bigoplus_{p \geq \ell} \overline{V_p} \quad \text{or} \quad V = \bigoplus_{p \leq -\ell} \overline{V_p}, \quad (3.10)$$

with  $\dim V_p = 1$ . The involution  $\mathfrak{g} = \mathfrak{n}ak \mapsto \mathfrak{n}^{-1}ak^{-1}$  interchanges the rôle of these two. As a counterpart of (3.6), we have, in the first case, a complete orthonormal system  $\{\varphi_p : p \geq \ell\}$  in  $V$ , such that

$$\varphi_p(\mathfrak{g}) = \pi^{\frac{1}{2}-\ell} \left( \frac{\Gamma(p+\ell)}{\Gamma(p-\ell+1)} \right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\varrho_V(n)}{\sqrt{n}} \mathcal{A}^+ \phi_p \left( \mathfrak{a}[n]\mathfrak{g}; \ell - \frac{1}{2} \right). \quad (3.11)$$

The same as (3.8) can be assumed. Thus  $\varrho_V(n) = \varrho_V(1)t_V(n)$ , and  $H_V(s)$  is defined as before.

**4. Explicit formula.** In terms of the above notion, Theorem 4.2 of [15] can be reformulated as

$$\begin{aligned} \mathcal{M}(\zeta^2, g) &= M(\zeta^2, g) + 2 \sum_V |\varrho_V(1)|^2 H_V \left( \frac{1}{2} \right)^3 \Theta(g, \nu_V) \\ &\quad + \int_{(0)} \frac{\left( \zeta \left( \frac{1}{2} + \nu \right) \zeta \left( \frac{1}{2} - \nu \right) \right)^3}{\zeta(1+2\nu) \zeta(1-2\nu)} \Theta(g, \nu) \frac{d\nu}{2\pi i}, \end{aligned} \quad (4.1)$$

where the path is the imaginary axis. The  $V$  is as in (3.3). The  $M$  and  $\Theta$  are integral transforms. The kernel of  $M$  is given explicitly in terms of logarithmic derivatives of the Gamma function. The construction of  $\Theta$  is our main concern, as has been stressed above.

The proof in [15] of the explicit formula (4.1) is via the spectral theory of sums of Kloosterman sums, which has inevitably made the argument far less structural. Although this fact does not matter in the quantitative study of the moment, it hinders us from discussing (4.1) with generalities in mind. Nevertheless, if one studies closely the proof, it will be seen that the kernel of  $\Theta(g, \nu)$  is a specialization of the Mellin or the multiplicative convolution of two Bessel kernels, i.e.,  $j_0$  and  $j_\nu$  below – the reason why the hypergeometric function turns up there. The  $j_0$  comes from the Voronoï formula or equivalently from the functional equation of the Estermann zeta-function; or more precisely, it can be traced back to the  $\Gamma$ -automorphic property of the Eisenstein series of weight zero, which is the automorphic function corresponding to the product of two values of the Riemann zeta-function (however, see the last paragraph of Section 6). The other Bessel kernel comes from the integral transform involved in the spectral expansion of sums of Kloosterman sums. In [7] it is observed that the latter is the Bessel function of representations of  $G$ , i.e., the

realization of the action of the Weyl element  $w = k(\pi/2)$  in terms of the Whittaker model over  $G$  (see (5.4) below).

With this, a structural description of  $\Theta$  is set out in [17]. To state it, let us put

$$j_\nu(u) = \pi \frac{\sqrt{|u|}}{\sin \pi\nu} \left( J_{-2\nu}^{\text{sgn}(u)}(4\pi\sqrt{|u|}) - J_{2\nu}^{\text{sgn}(u)}(4\pi\sqrt{|u|}) \right) \tag{4.2}$$

with  $J_\nu^+ = J_\nu$  and  $J_\nu^- = I_\nu$  in the ordinary notation for Bessel functions. This is the Bessel function of representations of  $G$ . Also put

$$\Xi(r, \nu) = \int_{\mathbb{R}^\times} j_0(-u) j_\nu\left(\frac{u}{r}\right) \frac{d^\times u}{\sqrt{|u|}}, \quad d^\times u = \frac{du}{|u|}. \tag{4.3}$$

Then we have, in (4.1),

$$\Theta(g, \nu) = \int_0^\infty g_c\left(\log\left(1 + \frac{1}{r}\right)\right) \frac{\Xi(r, \nu)}{\sqrt{r(r+1)}} dr, \tag{4.4}$$

with  $g_c$  the cosine transform of  $g$ . Note that the normalization in (3.6) has entailed differences in numerical factors in (4.1) and (4.4) from those corresponding in [17].

Here it should be stressed that the explicit formula (4.1) has also practical implications, not only revealing a structural relation between the zeta-function and automorphic forms: Let us write

$$\int_{-T}^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt = TP_4(\log T) + E_2(T), \quad T \geq 2, \tag{4.5}$$

where  $TP_4(\log T)$  is the main term with a polynomial  $P_4$  of degree 4, and  $E_2(T)$  stands for the error term. Ivić and the present author [10] (see also Theorem 5.3 of [15]) proved, via (4.1),

$$\int_0^V |E_2(T)|^2 dT \ll V^2 \log^{20} V, \quad V \geq 2, \tag{4.6}$$

which implies, by the way, the bound  $E_2(T) \ll T^{\frac{3}{2}} \log^8 T$ . The bound (4.6) is essentially the best possible, since the assertion  $E_2(T) = \Omega_\pm(\sqrt{T})$  is known to hold (Theorem 5.5 of [15]), and Ivić [9] demonstrated, by far more significantly, that the integral admits the lower bound of the order  $V^2$ . Both results are again via (4.1).

**5. Kirillov scheme.** The geometric information of each  $V$  is obviously contained in  $j_{\nu\nu}$  as far as (4.1) is concerned. A merit of the work [4] is in that it exhibits, in a structural mode, how this Bessel kernel enters into the scene. That is in effect an instance of applications of the harmonic analysis over the big cell of  $G$ . The procedure is termed as the Kirillov scheme in [4] because of its essential dependency on the Kirillov map defined below.

Thus let us give the fundamentals in this context. We extend (3.7) by

$$\mathcal{A}^\delta \phi(g) = \sum_p c_p \mathcal{A}^\delta \phi_p, \quad \phi = \sum_p c_p \phi_p, \tag{5.1}$$

where  $\phi$  is smooth, i.e.,  $|c_p| \ll (|p| + 1)^{-B}$  for each fixed  $B > 0$ . Note that the parameter  $\nu$  is actually involved here. It can be shown that  $\mathcal{A}^\delta \phi$  exists for any  $\nu$ , and

$$\mathcal{A}^\delta \phi(g) = \int_{\mathbb{R}} \exp(-2\pi\delta ix) \phi(\text{wn}[x]g) dx, \tag{5.2}$$

for those  $\nu$  in the domain where the integral converges uniformly. Then the Kirillov map  $\mathcal{K}$  is defined by

$$\mathcal{K}\phi(u) = \mathcal{A}^{\text{sgn}(u)}\phi(\mathfrak{a}[|u|]), \quad u \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}. \quad (5.3)$$

**Lemma 1.** *We have, for  $|\text{Re } \nu| < \frac{1}{2}$ ,*

$$\mathcal{K}R_w\phi(u) = \int_{\mathbb{R}^\times} j_\nu(u\lambda)\mathcal{K}\phi(\lambda)d^\times\lambda, \quad (5.4)$$

*with the right translation  $R$ .*

**Lemma 2.** *Let  $\nu \in i\mathbb{R}$ , and introduce the Hilbert space*

$$U_\nu = \overline{\bigoplus_p \mathbb{C}\phi_p}, \quad \phi_p(\mathfrak{g}) = \phi_p(\mathfrak{g}; \nu), \quad (5.5)$$

*equipped with the norm*

$$\|\phi\|_{U_\nu} = \sqrt{\sum_p |c_p|^2}, \quad \phi(\mathfrak{g}) = \sum_p c_p \phi_p(\mathfrak{g}). \quad (5.6)$$

*Then  $\mathcal{K}$  is a unitary map from  $U_\nu$  onto  $L^2(\mathbb{R}^\times, \pi^{-1}d^\times)$ .*

For the proof as well as the historical aspects of these assertions, see Section 4 of [4] and [17]. There extensions are made to the discrete and the complementary series, though the latter is irrelevant to our present situation.

**6. Projections.** Now, let  $\omega_V$  be the orthogonal projection to a  $V$  in the unitary principal series. We shall show very briefly how to fix  $\omega_V \mathcal{P}f$  with the Kirillov scheme. We may ignore the convergence issue.

The projection to  $V_p$  is, by the unfolding argument,

$$\begin{aligned} \langle \mathcal{P}f, \varphi_p \rangle_{\Gamma \backslash G} &= \int_G f(\mathfrak{g}) \overline{\varphi_p(\mathfrak{g})} d\mathfrak{g} \\ &= |\varrho_V(1)| \sum_{m=1}^{\infty} \frac{t_V(m)}{\sqrt{m}} (\Phi_p^+ + \epsilon_V \Phi_p^-) f_m(\nu_V), \end{aligned} \quad (6.1)$$

where  $f_m(\mathfrak{g}) = f(\mathfrak{a}[m]^{-1}\mathfrak{g})$  and

$$\Phi_p^\delta f(\nu) = \int_G f(\mathfrak{g}) \overline{\mathcal{A}^\delta \phi_p(\mathfrak{g})} d\mathfrak{g}. \quad (6.2)$$

Thus

$$\begin{aligned} \omega_V \mathcal{P}f(\mathfrak{g}) &= \sum_p \langle \mathcal{P}f, \varphi_p \rangle_{\Gamma \backslash G} \varphi_p(\mathfrak{g}) \\ &= |\varrho_V(1)|^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{t_V(m)t_V(n)}{\sqrt{mn}} \\ &\quad \times \left( \mathcal{B}^{(+,+)} + \mathcal{B}^{(-,-)} + \epsilon_V \mathcal{B}^{(+,-)} + \epsilon_V \mathcal{B}^{(-,+)} \right) f_m(\mathfrak{a}[n]\mathfrak{g}; \nu_V), \end{aligned} \quad (6.3)$$

$$\begin{aligned}\mathcal{B}^{(\delta_1, \delta_2)} f(g; \nu) &= \sum_p \Phi_p^{\delta_1} f(\nu) \mathcal{A}^{\delta_2} \phi_p(g; \nu) \\ &= \exp(2\pi i \delta_2 x) \sum_p \Phi_p^{\delta_1} f(\nu) \mathcal{A}^{\delta_2} \phi_p(a[y]) \exp(2ip\theta).\end{aligned}\quad (6.4)$$

Since our interest is in the value  $\varpi_V \mathcal{P}f(1)$ , we may restrict ourselves to the subgroup  $A$ . We have

$$\mathcal{B}^{(\delta_1, \delta_2)} f(a[y]; \nu) = \mathcal{K} \mathcal{L}^{\delta_1} f(\delta_2 y), \quad \mathcal{L}^{\delta} f = \sum_p \Phi_p^{\delta} f(\nu) \phi_p. \quad (6.5)$$

Assuming that  $\mathcal{L}^{\delta} f$  is a smooth vector in  $U_\nu$ , we have, by the unitarity assertion in Lemma 2,

$$\Phi_p^{\delta} f(\nu) = \langle \mathcal{L}^{\delta} f, \phi_p \rangle_{U_\nu} = \frac{1}{\pi} \int_{\mathbb{R}^{\times}} \mathcal{K} \mathcal{L}^{\delta} f(u) \overline{\mathcal{K} \phi_p(u)} d^{\times} u. \quad (6.6)$$

This means that if one can transform (6.2) into

$$\Phi_p^{\delta} f(\nu) = \frac{1}{\pi} \int_{\mathbb{R}^{\times}} Y^{\delta}(u) \overline{\mathcal{K} \phi_p(u)} d^{\times} u, \quad (6.7)$$

then it should follow that

$$\mathcal{B}^{(\delta_1, \delta_2)} f(a[y]; \nu) = Y^{\delta_1}(\delta_2 y), \quad (6.8)$$

because of the surjectivity assertion in the same lemma.

Since the integral in (6.2) is in fact over the big cell, we perform the change of variables accordingly. We have instead

$$\Phi_p^{\delta} f(\nu) = \int_0^{\infty} \int_{N_{wN}} f(a[u]g) \overline{R_g \mathcal{A}^{\delta} \phi_p(a[u])} d\dot{g} \frac{du}{u}. \quad (6.9)$$

Here  $g = n[x_1]wn[x_2]$  and  $d\dot{g} = dx_1 dx_2 / \pi$ . We observe that

$$R_g \mathcal{A}^{\delta} \phi_p(a[u]) = \exp(2\pi i \delta x_1 u) \mathcal{A}^{\delta} R_w R_{n[x_2]} \phi_p(a[u]), \quad (6.10)$$

and by Lemma 1

$$\begin{aligned}\mathcal{A}^{\delta} R_w R_{n[x_2]} \phi_p(a[u]) &= \mathcal{K} R_w R_{n[x_2]} \phi_p(\delta u) = \int_{\mathbb{R}^{\times}} j_{\nu}(\delta u \lambda) \mathcal{K} R_{n[x_2]} \phi_p(\lambda) d^{\times} \lambda \\ &= \int_{\mathbb{R}^{\times}} \exp(2\pi i x_2 \lambda) j_{\nu}(\delta u \lambda) \overline{\mathcal{K} \phi_p(\lambda)} d^{\times} \lambda.\end{aligned}\quad (6.11)$$

Inserting this into (6.9) we get

$$\begin{aligned}\Phi_p^{\delta} f(\nu) &= \frac{1}{\pi} \int_0^{\infty} \int_{\mathbb{R}^2} f(a[u]n[x_1]wn[x_2]) \exp(-2\pi i \delta x_1 u) \\ &\quad \times \int_{\mathbb{R}^{\times}} \exp(-2\pi i x_2 \lambda) j_{\nu}(\delta u \lambda) \overline{\mathcal{K} \phi_p(\lambda)} d^{\times} \lambda dx_1 dx_2 \frac{du}{u}.\end{aligned}\quad (6.12)$$

Hence we find via (6.8) that

$$\begin{aligned}\mathcal{B}^{(\delta_1, \delta_2)} f(a[y]; \nu) &= \int_0^{\infty} j_{\nu}(\delta_1 \delta_2 y u) \\ &\quad \times \int_{\mathbb{R}^2} f(a[u]n[x_1]wn[x_2]) \exp(-2\pi i \delta_1 u x_1 - 2\pi i \delta_2 y x_2) dx_1 dx_2 \frac{du}{u},\end{aligned}\quad (6.13)$$

which ends the application of the Kirillov scheme.

This is admittedly highly formal. For instance, the last step requires an exchange of the order of integration in (6.12), which is non-trivial. Nevertheless, the procedure exhibits how the Bessel kernel  $j_\nu$  comes into  $\Theta$ . With the choice of the sequence of  $f$  made in [4], the above is all validated. There each  $f$  is such that  $f(a[y]g) = y^z f(g)$  with a fixed  $z$ ,  $\operatorname{Re} z > \frac{1}{2}$ . Thus, in (6.1) the sum yields  $H_V(z + \frac{1}{2})$ , while  $f_m$  is replaced by the plain  $f$ , which simplifies the discussion considerably. The operator  $\mathcal{T}$  given in (2.2) is responsible for another Hecke series. In view of the factor  $H_V(\frac{1}{2})^3$  in (4.1), we need to have one more Hecke series as a factor. That comes out of the sum over  $n$  in (6.3) when we take the limit in  $f$ . The contribution of the discrete series representations and the projection to  ${}^eL^2(\Gamma \backslash G)$  are treated similarly. In this way we reach an expression equivalent to Lemmas 4.5 and 4.6 of [15] combined, without recourse to the spectral theory of Kloosterman sums. The rest of the argument to establish (4.1) is the same as in Sections 4.6–4.7 of [15], which is a procedure of analytic continuation. Another feature of [4] to be mentioned is that it gives also a structural understanding of the non-spectral term (4.3.16) of [15] that is called a residual contribution there.

One might see somewhat remotely in the last integral over the entire plane a reason why we have the Bessel factor  $j_0$  in (4.3). This is, however, different from our brief explanation made in the paragraph following (4.1). The formula (6.13) has been deduced without touching any arithmetic objects such as Eisenstein series. Thus, the factor  $j_0$  should rather be regarded as a geometric characteristic of the big cell surfacing in conjunction with the peculiarity of the moment  $\mathcal{M}(\zeta^2, g)$ .

**7. Extrapolation.** Here we shall discuss possible extensions of the above in order to have a glimpse of a unified theory of mean values of automorphic  $L$ -functions that has long been sought for and is still to be discovered.

*1:* An immediate extension of the explicit formula (4.1) is to the mean squares  $\mathcal{M}(\zeta_F, g)$  of Dedekind zeta-functions  $\zeta_F$  of quadratic number fields  $F$ . The underlying Lie group is the same as  $G$  but Hecke congruence subgroups replace  $\Gamma$ . Less immediate is the extension to the fourth moment  $\mathcal{M}(\zeta_F^2, g)$  with real quadratic number fields  $F$  of class number one. The same for imaginary quadratic number fields of class number one is far more difficult but has nonetheless been included in our extensions. In the real quadratic case among these two the Lie group is the product of two copies of  $\operatorname{PSL}_2(\mathbb{R})$ , and the discrete subgroups are the Hilbert modular groups. In the imaginary case we have instead  $\operatorname{PSL}_2(\mathbb{C})$  and Bianchi groups. The explicit formulas for these mean values of Dedekind zeta-functions are established in [16], [3], and [5] (see also [2, Part X]), respectively. Note that [5] treats the Gaussian field only for the sake of simplicity. Those works depend on spectral expansions of sums of corresponding Kloosterman sums in much the same way as [15] does.

To dispense with this dependency, we need to construct the Poincaré series like that in [4], but it should not raise any inherent difficulties of new type. The condition that  $F$  is of class number one is imposed to have  $\zeta_F$  defined as a sum over integers of  $F$  rather than over integral ideals, and thus the relation between  $\mathcal{M}(\zeta_F^2, g)$  and the discrete groups over  $F$  becomes as visible as the case of  $\mathcal{M}(\zeta^2, g)$ . Hence the condition appears to be superficial or rather a technical matter, although we have not dealt with the details for the general case yet. In any event here is a problem that will be settled probably without much efforts; but an additional complexity will be caused by the plurality of inequivalent cusps. It should be added that the real quadratic case, even with the class number being equal to one, contains a distinctive problem induced by the existence of infinitely many units. In [3] this is overcome with an instance of partition of one; otherwise the situation is fairly analogous to that of



the Riemann zeta-function.

These three mean values and  $\mathcal{M}(\zeta^2, g)$  are much alike each other in the culminating explicit formulas. However, the technical difficulty varies among their proofs, and the most conspicuous is in the case of  $\mathcal{M}(\zeta_F^2, g)$  with imaginary  $F$ , as indicated above. A reason for this is in that the maximal compact subgroup  $SU(2)$  of  $PSL_2(\mathbb{C})$  is non-commutative. Nevertheless, the structure (4.1)–(4.4) extends gracefully to imaginary  $F$ , although the continuous spectral part involves now a sum over all Grössencharakteren, an aspect shared by the real quadratic case as well. Interesting is the rôle played by the Bessel function of representations of  $PSL_2(\mathbb{C})$ . It is much similar to that of  $j_\nu$  above. Moreover, the counterpart of  $j_0$  appears in an essentially identical configuration. See [2, Part XIII] for the details.

2: So far we have been concerned with the situations in which the mean value in question can be embedded, in a sense, into a Poincaré series. They are analogous to each other at least ostensibly, because of their general dependency upon the harmonic analysis over  $GL_2$ . However, our view has to be altered, when we move to the mean square  $\mathcal{M}(H_V, g)$  of a particular Hecke  $L$ -function  $H_V$ . Because of the fact that the functional equation for  $H_V$  is virtually the same as that for the product of values of the Riemann zeta-function at two shifted arguments, one may presume that  $\mathcal{M}(H_V, g)$  should admit a spectral decomposition resembling that of  $\mathcal{M}(\zeta^2, g)$ . This appears to be a natural conjecture; but it has been confirmed so far only in the case of  $V$  in the discrete series, and the unitary principal series case has not been resolved as yet.

We shall make precise the situation with the discrete series, quoting the main result of [14], but with a new outlook. Thus, let  $D$  be such an irreducible representation among those  $V$  defined by (3.3); we may assume that the first decomposition in (3.10) takes place with  $D$ . Let  $\Omega|_D = (\ell_D - \frac{1}{2})^2 - \frac{1}{4}$  with a positive integer  $\ell_D$ , and write

$$\psi_D(g) = \exp(2i\ell_D\theta)y^{\ell_D} \sum_{n=1}^{\infty} t_D(n)n^{\ell_D-\frac{1}{2}} \exp(2\pi i(x+iy)n), \quad (7.1)$$

in place of (3.11) with  $V = D$ ,  $p = \ell_D$ . Also put

$$\psi_V(g) = \sqrt{y} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} t_V(n)K_{\nu_V}(2\pi|n|y) \exp(2\pi inx), \quad (7.2)$$

in place of (3.6) with  $p = 0$ , where  $K_\nu$  is the  $K$ -Bessel function. Via multiple applications of Maass operators, these automorphic forms generate the spaces  $D$  and  $V$ , respectively.

With this, the cuspidal part of  $\mathcal{M}(H_D, g)$  can be put as

$$(-1)^{\ell_D} 2^{6\ell_D} \pi^{4\ell_D-1} \sum_V \frac{|\varrho_V(1)|^2 \langle \psi_V, |\psi_D|^2 \rangle_{\Gamma \backslash G/K}}{\Gamma(2\ell_D - \frac{1}{2} + \nu_V) \Gamma(2\ell_D - \frac{1}{2} - \nu_V)} H_V \left( \frac{1}{2} \right) \Theta_{\ell_D}(g, \nu_V). \quad (7.3)$$

Here

$$\Theta_\ell(g, \nu) = \int_0^\infty \left(1 + \frac{1}{r}\right)^{\ell-\frac{1}{2}} g_c \left( \log \left(1 + \frac{1}{r}\right) \right) \frac{\Xi_\ell(r, \nu)}{\sqrt{r(r+1)}} dr, \quad (7.4)$$

with

$$\Xi_\ell(r, \nu) = \int_{\mathbb{R}^\times} |u|^{\ell-\frac{1}{2}} j_{\ell-\frac{1}{2}}(-u) j_\nu \left( \frac{u}{r} \right) \frac{d^\times u}{\sqrt{|u|}}. \quad (7.5)$$

Observe that  $j_{\ell_D-\frac{1}{2}}(-u) j_{\nu_V}(u/r) \equiv 0$  for any  $V$  in the discrete series; thus the sum (7.3) is actually over  $V$  in the unitary principal series. The non-cuspidal part of  $\mathcal{M}(H_D, g)$  involves

the Rankin  $L$ -function attached to  $D$  but is omitted here because (7.3) is sufficient for our present purpose.

Thus there is a remarkable similarity between  $\mathcal{M}(\zeta^2, g)$  and  $\mathcal{M}(H_D, g)$  in their spectral expansions. Specializing (7.4)–(7.5) with  $\ell = \frac{1}{2}$ , we recover (4.3)–(4.4). However, the proof of (7.3) is different from either of the two proofs of (4.1), and it rests instead on an inner-product argument. That is, the discussion of [14] starts with an inner-product of  $|\psi_D|^2$  and a Poincaré series of Selberg's type, a device that generates the Dirichlet series

$$\sum_{m=1}^{\infty} \frac{t_D(m)t_D(m+n)}{(m+n)^s}, \quad (7.6)$$

which is analogous to the inner sum of (2.4). Since the inner product decomposes spectrally, so does this function too. The rest of the argument is to integrate the expansion. One should note that [14] is free from any use of Kloosterman sums and has the appeal of being functional. The step for (7.6) is crucial, for Hecke eigenvalues do not have the structure analogous to that of the divisor function  $\sigma_\alpha$  with which our deduction of (2.4) is made. We remark that conversely (2.4) has not been generated via the inner product argument.

We add that the counterpart of (4.6) for the mean square of  $H_V$  is given in [14]. As to the  $\Omega$ -result, it should follow if we have

$$\langle \psi_V, |\psi_D|^2 \rangle_{\Gamma \backslash G/K} \neq 0 \quad (7.7)$$

for at least one  $V$ . This remains in the state of a conjecture as in [14].

§: Here emerges three fundamental problems:

- (a) Does the Poincaré series approach to  $\mathcal{M}(\zeta^2, g)$  extend to  $\mathcal{M}(H_D, g)$ ?
- (b) Does the inner-product argument for  $\mathcal{M}(H_D, g)$  extend to  $\mathcal{M}(\zeta^2, g)$ ?
- (c) Prove an explicit formula for  $\mathcal{M}(H_V, g)$  with  $V$  in the unitary principal series.

Problems (b) and (c) are discussed in the important work [11] of M. Jutila. He forged, via an inner-product approach, a unified treatment of the mean values  $\mathcal{M}(\zeta^2, g)$ ,  $\mathcal{M}(H_D, g)$  and  $\mathcal{M}(H_V, g)$  with the above specifications. His results are asymptotic formulas for these mean values, which closely resemble (4.1). Being asymptotic, they are not exact as (4.1); but the approximation is good enough for principal applications such as discussing the mean square of the error terms in the corresponding unweighted mean values. Thus the analogue of (4.6) for the mean square of  $H_V$  is obtained in [11], which is quite an achievement.

Let us be uncompromising, however: Problem (c) has to be solved genuinely. It appears highly likely to us that (a) has an affirmative answer. We are yet to construct the Poincaré series in question, but there should not be a need to recast substantially the Kirillov scheme for this aim, as is pointed to by the appearance of  $j_\nu$  in (7.5). If this is indeed the case, then it should be realistic to presume that Problem (c) will be resolved in a similar fashion. That is to say, we conjecture that there exists a unified way via the Poincaré series approach to deal with mean squares of automorphic  $L$ -functions. Our belief stems from another aspect as well, i.e., the contribution of the discrete series to  $\mathcal{M}(\zeta^2, g)$ . Although this has turned out to be negligible in applications, the identity (4.1) would fail to hold unless we include it. It seems proper for us to claim that the function  $H_V$  in (c) is closer to  $\zeta^2$  than  $H_D$  in (b). Thus  $\mathcal{M}(H_V, g)$  with such a  $V$  should accommodate contributions of all  $\Gamma$ -automorphic representations. This plausible inference strongly suggests that the mean value problem of automorphic  $L$ -functions in general should be a subject attached to linear Lie groups but not to their quotients like the upper half plane  $G/K$ , excepting  $\mathcal{M}(H_D, g)$  as is seen above.

Yet we cannot deny the possibility that (b) will turn out to be the right way to proceed along, although the inner-product should anyway be taken fully over  $\Gamma \backslash G$ . Here relevant is a certain result of the type of addition theorem for the Whittaker function: In Jutila's discussion on  $\mathcal{M}(\zeta^2, g)$  and  $\mathcal{M}(H_V, g)$ , a difficulty occurs when a separation of variables is tried on the product of two values of the Whittaker function; and that is indeed the reason why he obtained approximative results instead of explicit spectral expansions. He worked with automorphic forms over the upper half plane; and their weights are fixed. We think it likely that the difficulty could be resolved if we take into account all the weights, i.e., an addition theorem. This is but close to what is developed in Section 6; see (6.4) in particular.

4: As to higher power moments of the Riemann zeta-function, the present author muses occasionally that a hoard could be hidden in [6].

*Small things stir up great* — [12]

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