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HERMITE'S CONSTANT AND RATIONAL POINTS OF FLAG VARIETIES

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This note is a continuation of our survey article [W6]. We give an account of further development of generalized Hermite constants after [W6]. First half of this note is almost the same as the article [W7] written in Japanese, but in the last half, we will state new results (Theorems 5, 6 and 7) .

1. Hermite's constant and some generalizations. As I already wrote a survey of Hermite's constant in [W6], I skip the details of the history of Hermite's constant. I only mention a definition of Hermite–Rankin constant and some important results in this section.

Let $\mathcal{L}^n$ be the set of all lattices of rank n in the Euclidean space $\mathbb{R}^n$. For $L \in \mathcal{L}^n$, we denote by $d(L)$ the volume of the fundamental parallelepiped of L and by $m_1(L)$ the square of the length of minimal vectors in L, i.e., $m_1(L) = \min_{0\neq x\in L} \|x\|^2$. Then the constant

$$\gamma_n = \max_{L\in\mathcal{L}^n} \frac{m_1(L)}{d(L)^{2/n}}$$

is called Hermite's constant.

For $1 \leq d \leq n-1$, define the lattice invariant $m_d(L)$ by

$$m_d(L) = \min_{\substack{x_1,\cdots,x_d\in L\\ x_1\wedge\cdots\wedge x_d\neq 0}} \det({}^t x_i x_j)_{1\leq i,j\leq d}.$$

Then Rankin [R] defined the following constant:

$$\gamma_{n,d} = \max_{L\in\mathcal{L}^n} \frac{m_d(L)}{d(L)^{2d/n}},$$

As a generalization of Hermite–Rankin constant, Thunder [T2] defined the constant $\gamma_{n,d}(k)$ for any algebraic number field k. Rankin's constant $\gamma_{n,d}$ coincides with $\gamma_{n,d}(\mathbb{Q})$. We will recall Thunder's definition of $\gamma_{n,d}(k)$ in the next section (see Example 1) and express $\gamma_{n,d}(k)$ in terms of fundamental Hermite constants of GL_n. Thunder proved the following:

(1) $\gamma_{n,d}(k) = \gamma_{n,n-d}(k)$ for $1 \leq d \leq n-1$
(2) $\gamma_{n,d}(k) \leq \gamma_{m,d}(k)(\gamma_{n,m}(k))^{d/m}$ for $1 \leq d < m \leq n-1$

$$(3)\quad \left(\frac{n|D_k|^{d(n-d)/2}}{\mathrm{Res}_{s=1}\zeta_k(s)} \frac{\prod_{j=n-d+1}^{n} Z_k(j)}{\prod_{j=2}^{d} Z_k(j)} \right)^{2/(n[k:\mathbb{Q}])} \leq \gamma_{n,d}(k) \leq \left(\frac{2^{r_1+r_2}|D_k|^{1/2}}{V(n)^{r_1/n}V(2n)^{r_2/n}} \right)^{2d/[k:\mathbb{Q}]}.$$

Here $Z_k(s) = (\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{1-s}\Gamma(s))^{r_2}\zeta_k(s)$ denotes the zeta function of k, $V(n) = \pi^{n/2}/\Gamma(1+n/2)$ the volume of the unit ball in $\mathbb{R}^n$, D_k the discriminant of k and r_1 (resp. r_2) the number of real (resp. imaginary) places of k. Originally, in the case of $k = \mathbb{Q}$, (1) and (2) are due to Rankin, and (3) for $d = 1$ is due to Minkowski and Hlawka.

We particularly write $\gamma_n(k)$ for $\gamma_{n,1}(k)$. Newman ([N, XI]) and Icaza ([I]) also considered $\gamma_n(k)$ based on Humbert's reduction theory. Tables below show the known explicit values of $\gamma_n(k)$ (cf. [BCIO], [G-L], [N], [R]).

n	2	3	4	5	6	7	8
γ_n	$2/\sqrt{3}$	$\sqrt[3]{2}$	$\sqrt{2}$	$\sqrt[5]{8}$	$\sqrt[6]{64/3}$	$\sqrt[7]{64}$	2

$\gamma_{4,2} = 3/2$

d	-1	-2	-3	-7	-11	2	3	5
$\gamma_2(\mathbb{Q}(\sqrt{d}))$	$\sqrt{2}$	2	$\sqrt{6}/2$	$\sqrt{21}/3$	$\sqrt{22}/2$	$2/\sqrt{2\sqrt{6}-3}$	2	$2/\sqrt[4]{5}$

2. Fundamental Hermite constants. Thunder's definition shows that $\gamma_{n,d}(k)$ is a quantity attached to the Grassmann variety of d-dimensional subspaces in k^n. This suggests that there exists an analogue of Hermite's constant for any generalized flag variety G/Q, where G denotes a connected reductive algebraic group defined over k and Q a k-parabolic subgroup of G. We introduced such a constant in terms of a strongly k-rational representation π of G in [W1]. This constant, say γ_π^G, was named a generalized Hermite constant attached to π, because $\gamma_{n,d}(k)$ is equal to $\gamma_{\pi_d}^{GL_n}$ of the d-th exterior representation π_d of GL_n. A strongly k-rational representation is used for embedding k-rationally G/Q into a projective space. We note that there are infinitely many strongly k-rational representations of G if G is isotropic. In a subsequent paper [W5], we gave a more natural definition of the generalized Hermite constant of G/Q provided that Q is maxmal. This new definition depends only on G, Q and does not need a strongly k-rational representaton π. We write $\gamma(G,Q,k)$, or simply γ_Q, for this new constant. Two constants γ_π^G and γ_Q have a relation of the form $\gamma_\pi^G = (\gamma_Q)^{c_\pi}$, where c_π is a positive rational number depending on π. In other words, γ_Q is considered as an essential part of γ_π^G in the sense that it is independent of any embedding of G/Q into a projective space. In this section, we first recall the definition of γ_Q, and then we state some properties of γ_Q.

In the following, k denotes a global field, i.e., an algebraic number field or a function field of one variable over a finite field. We fix a connected reductive algebraic group G defined over k, a minimal k-parabolic subgroup P of G and a maximal standard k-parabolic subgroup Q of G. By "standard", we means Q contains P. To define notations, we take a connected k-subgroup R of G. Let $R(k)$ denote the group of k-rational points of R, $R(\mathbb{A})$ the adele group of R and $\mathbf{X}_k^*(R)$ the module of k-rational characters of R. For $a \in R(\mathbb{A})$, define the homomorphism $\vartheta_R(a)$ from $\mathbf{X}_k^*(R)$ into the group $\mathbb{R}_+$ of positive real numbers by $\vartheta_R(a)(\chi) = |\chi(a)|_{\mathbb{A}}$ for $\chi \in \mathbf{X}_k^*(R)$, where $|\cdot|_{\mathbb{A}}$ stands for the idele norm of the idele group of k. Then ϑ_R gives rise to a homomorphism from $R(\mathbb{A})$ into $\mathrm{Hom}(\mathbf{X}_k^*(R), \mathbb{R}_+)$. The kernel of ϑ_R is denoted by $R(\mathbb{A})^1$. If R is a standard k-parabolic subgroup, U_R and M_R stand for the unipotent radical and a Levi subgroup of R, respectively. If R is a minimal k-parabolic subgroup P, we can take M_P as the centralizer of a maximal k-split torus S of G. In general, we take M_R such that $M_P \subset M_R$. The maximal central k-split torus of M_R is denoted by Z_R. We fix a good maximal compact subgroup K of $G(\mathbb{A})$.

We define the height function H_Q on $G(\mathbb{A})$. Since Q is maximal, $\mathbf{X}_k^*(M_Q/Z_G)$ is of rank one and has a generator $\widehat{\alpha}_Q$ such that $\widehat{\alpha}_Q|_S$ is contained in the closed cone generated by the simple roots with respect to (P, S). Define the map $z_Q \colon G(\mathbb{A}) \longrightarrow Z_G(\mathbb{A})M_Q(\mathbb{A})^1\backslash M_Q(\mathbb{A})$ by $z_Q(g) = Z_G(\mathbb{A})M_Q(\mathbb{A})^1 m$ if $g = umh$, $u \in U_Q(\mathbb{A}), m \in M_Q(\mathbb{A})$ and $h \in K$. This is well defined and a left $Z_G(\mathbb{A})Q(\mathbb{A})^1$-invariant. Then the function $H_Q \colon G(\mathbb{A}) \longrightarrow \mathbb{R}_+$ is defined by $H_Q(g) = |\widehat{\alpha}_Q(z_Q(g))|_{\mathbb{A}}^{-1}$ for $g \in G(\mathbb{A})$.

We set $Y_Q = Q(\mathbb{A})^1\backslash G(\mathbb{A})^1$ and $X_Q = Q(k)\backslash G(k)$. Then X_Q is regarded as a subset of Y_Q. Since $Z_G(\mathbb{A})^1 = Z_G(\mathbb{A}) \cap G(\mathbb{A})^1 \subset M_Q(\mathbb{A})^1$, z_Q maps $Y_Q = Q(\mathbb{A})^1\backslash G(\mathbb{A})^1$ to $M_Q(\mathbb{A})^1\backslash(M_Q(\mathbb{A}) \cap G(\mathbb{A})^1)$. Namely, we have the following commutative diagram:

$$\begin{array}{ccc} Y_Q & \xrightarrow{z_Q} & M_Q(\mathbb{A})^1\backslash(M_Q(\mathbb{A}) \cap G(\mathbb{A})^1) \\ \downarrow & & \downarrow \\ Z_G(\mathbb{A})Q(\mathbb{A})^1\backslash G(\mathbb{A}) & \xrightarrow{z_Q} & Z_G(\mathbb{A})M_Q(\mathbb{A})^1\backslash M_Q(\mathbb{A}) \end{array}$$

Since both vertical arrows are injective, H_Q is restricted to Y_Q. Let $B_T = \{y \in Y_Q \colon H_Q(y) \leq T\}$ for $T > 0$. We can prove the following.

Proposition. *For $T > 0$ and any $g \in G(\mathbb{A})^1$, $B_T \cap X_Q g$ is a finite subset of Y_Q. Hence, one can define the function*

$$\Gamma_Q(g) = \min\{T > 0 \colon B_T \cap X_Q g \neq \emptyset\} = \min_{y \in X_Q g} H_Q(y)$$

on $G(\mathbb{A})^1$. Then the maximum

$$\gamma(G, Q, k) = \max_{g \in G(\mathbb{A})^1} \Gamma_Q(g)$$

exists.

The constant $\gamma_Q = \gamma(G, Q, k)$ is called the fundamental Hermite constant of (G, Q) over k. An interesting thing is a similarity between the definitions of γ_n and γ_Q. Namely, γ_n is represented as

$$\gamma_n = \max_{\substack{g \in GL_n(\mathbb{R}) \\ |\det g| = 1}} \min\{T > 0 \mid B_T^n \cap g\mathbb{Z}^n \neq \{0\}\},$$

where B_T^n denotes the ball of radius T with center 0 in $\mathbb{R}^n$. On the other hand, by definition,

$$\gamma_Q = \max_{g \in G(\mathbb{A})^1} \min\{T > 0 \colon B_T \cap X_Q g \neq \emptyset\}.$$

Thus X_Q plays a role of the lattice $\mathbb{Z}^n$ and B_T is an analogue of the ball B_T^n. In some cases, it is more convenient to consider the constant

$$\widetilde{\gamma}(G, Q, k) = \max_{g \in G(\mathbb{A})} \min_{y \in X_Q g} H_Q(g).$$

If k is an algebraic number field, then $\widetilde{\gamma}(G, Q, k)$ is always equal to $\gamma(G, Q, k)$. The next example shows a relation between $\gamma(GL_n, Q, k)$ and $\gamma_{n,d}(k)$.

Example 1. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be a standard k-basis of k^n. We identify the group of linear automorphisms of k^n with $GL_n(k)$. For $1 \leq d \leq n-1$, $Q_d(k)$ denotes the stabilizer of the subspace spanned by $\mathbf{e}_1, \cdots, \mathbf{e}_d$ in $GL_n(k)$. A k-basis of the d-th exterior product $\bigwedge^d k^n$ is formed by the elements $\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_d}$ with $I = \{1 \leq i_1 < i_2 < \cdots < i_d \leq n\}$. The global height H_d on $\bigwedge^d k^n$ is defined to be

$$H_d\Big(\sum_I a_I \mathbf{e}_I\Big) = \prod_{\substack{w \\ \text{infinite}}} \left(\sum_I |a_I|_w^{2/[k_w:\mathbb{R}]} \right)^{[k_w:\mathbb{R}]/2} \prod_{\substack{v \\ \text{finite}}} \sup_I(|a_I|_v),$$

where $|\cdot|_v$ denotes the usual normalized absolute value of the completion field k_v at a place v of k. We can define the constant

$$\widehat{\gamma}_{n,d}(k) = \max_{g \in GL_n(\mathbb{A})} \min_{\substack{x_1, \cdots, x_d \in k^n \\ x_1 \wedge \cdots \wedge x_d \neq 0}} \frac{H_d(gx_1 \wedge \cdots \wedge gx_d)}{|\det g|_{\mathbb{A}}^{d/n}}.$$

If k is an algebraic number field, $\widehat{\gamma}_{n,d}(k)^{2/[k:\mathbb{Q}]}$ is none other than Thunder's definition of $\gamma_{n,d}(k)$. It is immediate to see that

$$\frac{H_d(g^{-1}\mathbf{e}_1 \wedge \cdots \wedge g^{-1}\mathbf{e}_d)}{|\det g^{-1}|_{\mathbb{A}}^{d/n}} = H_{Q_d}(g)^{\gcd(d,n-d)/n}$$

for $g \in GL_n(\mathbb{A})$, and hence

$$\widehat{\gamma}_{n,d}(k) = \widetilde{\gamma}(GL_n, Q_d, k)^{\gcd(d,n-d)/n}.$$

In general, $Z_{GL_n}(\mathbb{A})GL_n(\mathbb{A})^1$ is an index finite normal subgroup of $GL_n(\mathbb{A})$, but it is not necessarily equal to $GL_n(\mathbb{A})$ if k is a function field. Let Ξ be a complete set of representatives for the cosets of $Z_{GL_n}(\mathbb{A})GL_n(\mathbb{A})^1 \backslash GL_n(\mathbb{A})$. If we put

$$\begin{aligned}
\widehat{\gamma}_{n,j}(k)_\xi &= \max_{g \in Z_{GL_n}(\mathbb{A})GL_n(\mathbb{A})^1 \xi} \min_{\substack{x_1, \cdots, x_d \in k^n \\ x_1 \wedge \cdots \wedge x_d \neq 0}} \frac{H_d(gx_1 \wedge \cdots \wedge gx_d)}{|\det g|_{\mathbb{A}}^{d/n}} \\
&= \frac{1}{|\det \xi|_{\mathbb{A}}^{d/n}} \max_{g \in GL_n(\mathbb{A})^1 \xi} \min_{\substack{x_1, \cdots, x_d \in k^n \\ x_1 \wedge \cdots \wedge x_d \neq 0}} H_d(gx_1 \wedge \cdots \wedge gx_d)
\end{aligned}$$

for $\xi \in \Xi$, then

$$\widehat{\gamma}_{n,d}(k) = \max_{\xi \in \Xi} \gamma_{n,d}(k)_{\xi},$$

and in particular, for the unit element $\xi = 1$,

$$\widehat{\gamma}_{n,d}(k)_1 = \gamma(GL_n, Q_d, k)^{\gcd(d,n-d)/n}.$$

If k is a number field, $Z_{GL_n}(\mathbb{A})GL_n(\mathbb{A})^1 = GL_n(\mathbb{A})$ holds, and hence one has

$$\widetilde{\gamma}(GL_n, Q_d, k) = \gamma(GL_n, Q_d, k) = \gamma_{n,d}(k)^{n[k:\mathbb{Q}]/(2\gcd(d,n-d))}.$$

We summarize the properties of $\gamma(G, Q, k)$.

Theorem 1. *Assume the exact sequence*

$$1 \longrightarrow Z \longrightarrow G \xrightarrow{\ \beta\ } G' \longrightarrow 1$$

of connected reductive groups defined over k satisfies the following two conditions:

- *Z is central in G.*
- *Z is isomorphic to a product of tori of the form $R_{k'/k}(GL_1)$, where each k'/k is a finite separable extension and $R_{k'/k}$ denotes the functor of restriction of scalars from k' to k.*

Then $\gamma(G, Q, k)$ is equal to $\gamma(G', \beta(Q), k)$.

Theorem 2. *If k/ℓ is a finite separable extension, then $\gamma(R_{k/\ell}(G), R_{k/\ell}(Q), \ell)$ is equal to $\gamma(G, Q, k)$.*

Theorem 3. *Let R and Q be two different maximal standard k-parabolic subgroups of G, $Q^R = M_R \cap Q$ a maximal standard parabolic subgroup of M_R and $M_Q^R = M_R \cap M_Q$ a Levi subgroup of Q^R. We write $\widehat{\alpha}_Q^R$ for the $\mathbb{Z}$-basis $\widehat{\alpha}_{Q^R}$ of $\mathbf{X}_k^*(M_Q^R/Z_R)$. Then $\mathbb{Q}$-vector space $\mathbf{X}_k^*(M_Q^R/Z_G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is spanned by $\widehat{\alpha}_Q^R$ and $\widehat{\alpha}_R|_{M_Q^R}$. If we take $\omega_1, \omega_2 \in \mathbb{Q}$ such that*

$$\widehat{\alpha}_Q|_{M_Q^R} = \omega_1 \widehat{\alpha}_Q^R + \omega_2 \widehat{\alpha}_R|_{M_Q^R},$$

then one has an inequality of the form

$$\gamma(G, Q, k) \leq \widetilde{\gamma}(M_R, Q^R, k)^{\omega_1} \gamma(G, R, k)^{\omega_2}.$$

Example 2. We illustrate that Theorem 1 and Theorem 3 are generalizations of the duality relation (1) and Rankin's inequality (2) in §1, respectively. We use the same notations as in Example 1. First, we consider the automorphism $\beta\colon GL_n \longrightarrow GL_n$ defined by $\beta(g) = w_0({}^t g^{-1})w_0^{-1}$, where

$$w_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in GL_n(k).$$

Since $\beta(Q_d) = Q_{n-d}$, Theorem 1 deduces

$$\gamma(GL_n, Q_d, k) = \gamma(GL_n, Q_{n-d}, k).$$

If k is a number field, this implies the duality relation (1). Next, for $i, j \in \mathbb{Z}$ with $1 \leq i < j \leq n-1$, we take two maximal standard k-parabolic subgroups $R = Q_j$ and $Q = Q_i$ of GL_n. Then, $M_R = GL_j \times GL_{n-j}$, $M_Q = GL_i \times GL_{n-i}$ and $M_Q^R = GL_i \times GL_{j-i} \times GL_{n-j}$. It is easy to see

$$\omega_1 = \frac{n}{j}\frac{\gcd(i, j-i)}{\gcd(i, n-i)}, \qquad \omega_2 = \frac{i}{j}\frac{\gcd(j, n-j)}{\gcd(i, n-i)}.$$

Theorem 3 deduces

$$\gamma(GL_n, Q_i, k) \leq \widetilde{\gamma}(M_{Q_j}, Q_i^{Q_j}, k)^{\frac{n}{j}\frac{\gcd(i,j-i)}{\gcd(i,n-i)}} \gamma(GL_n, Q_j, k)^{\frac{i}{j}\frac{\gcd(j,n-j)}{\gcd(i,n-i)}}.$$

If k is a number field, this and Example 1 imply Rankin's inequality (2).

Let $\tau(G)$ (resp. $\tau(Q)$) be the Tamagawa number of G (resp. Q) and $\omega_{\mathbb{A}}^G$ (resp. $\omega_{\mathbb{A}}^{U_Q}$ and $\omega_{\mathbb{A}}^{M_Q}$) the Tamagawa measure of $G(\mathbb{A})$ (resp. $U_Q(\mathbb{A})$ and $M_Q(\mathbb{A})$). The modular character δ_Q^{-1} of $Q(\mathbb{A})$ is defined by the relation $d\omega_{\mathbb{A}}^{U_Q}(m^{-1}um) = \delta_Q(m)^{-1} d\omega_{\mathbb{A}}^{U_Q}(u)$ for $u \in U_Q(\mathbb{A})$ and $m \in M_Q(\mathbb{A})$. We define constants $\widehat{e}_Q$ and $C_{G,Q}$ as follows:
- $\delta_Q(m) = |\widehat{\alpha}_Q(m)|_{\mathbb{A}}^{\widehat{e}_Q}$ for all $m \in M(\mathbb{A})$.
- $d\omega_{\mathbb{A}}^G(g) = C_{G,Q}^{-1}\delta_Q(m)^{-1} d\omega_{\mathbb{A}}^{U_Q}(u) d\omega_{\mathbb{A}}^{M_Q}(m) d\nu_K(h)$ for all $g = umh$, $u \in U_Q(\mathbb{A})$, $m \in M_Q(\mathbb{A})$ and $h \in K$.

Here ν_K denotes the Haar measure of K normalized so that $\nu_K(K) = 1$. By an argument of the mean value theorem, we can show the following theorem.

Theorem 4. *One has an estimate of the form*

$$\left(C_{G,Q} \cdot D_{G,Q} \cdot E_Q \cdot \frac{\tau(G)}{\tau(Q)}\right)^{1/\widehat{e}_Q} \leq \gamma(G, Q, k),$$

where $D_{G,Q}$ and E_Q are given as follows:

$$D_{G,Q} = \begin{cases} \dfrac{[\mathbf{X}_k^*(Z_G) : \mathbf{X}_k^*(G)]}{[\mathbf{X}_k^*(Z_Q) : \mathbf{X}_k^*(M_Q)]} & (\mathrm{ch}(k) = 0), \\[2ex] \dfrac{(\log q)^{\mathrm{rank}\, \mathbf{X}_k^*(G)}[\mathrm{Hom}(\mathbf{X}_k^*(G), q^{\mathbb{Z}}) : \mathrm{Im}\, \vartheta_G]}{(\log q)^{\mathrm{rank}\, \mathbf{X}_k^*(M_Q)}[\mathrm{Hom}(\mathbf{X}_k^*(M_Q), q^{\mathbb{Z}}) : \mathrm{Im}\, \vartheta_{M_Q}]} & (\mathrm{ch}(k) > 0), \end{cases}$$

$$E_Q = \begin{cases} \widehat{e}_Q[\mathbf{X}_k^*(Z_Q/Z_G) : \mathbf{X}_k^*(M_Q/Z_G)] & (\mathrm{ch}(k) = 0), \\ (1 - q_0^{-\widehat{e}_Q}) & (\mathrm{ch}(k) > 0). \end{cases}$$

Here, if $\mathrm{ch}(k) > 0$, then q denotes the cardinality of the constant field of k and $q_0 > 1$ the generator of the subgroup $|\widehat{\alpha}_Q(M_Q(\mathbb{A}) \cap G(\mathbb{A})^1)|_{\mathbb{A}}$ of the cyclic group $q^{\mathbb{Z}}$ generated by q. Moreover, this inequality is strict if $\mathrm{ch}(k) > 0$.

We note that $\gamma(G, Q, k) \in q_0^{\mathbb{Z}}$ if $\mathrm{ch}(k) > 0$.

Example 3. Let $G = GL_n$ and $Q = Q_d$. If $\mathrm{ch}(k) = 0$, Theorem 4 is essentially the same as the lower bound of (3) in §1. If $\mathrm{ch}(k) > 0$, we obtain $q_0 = q^{n/\gcd(d,n-d)}$ and

$$\left(\frac{q^{(g(k)-1)(d(n-d)+1)}(q-1)(1-q^{-n})}{h_k} \frac{\prod_{i=n-d+1}^{n} \zeta_k(i)}{\prod_{i=2}^{d} \zeta_k(i)} \right)^{1/\gcd(d,n-d)} < \gamma(GL_n, Q_d, k),$$

where $g(k)$ denotes the genus of k, h_k the divisor class number of k and $\zeta_k(s)$ the congruence zeta function of k. On the other hand, from the definition of $\widehat{\gamma}_{n,d}(k)$ and Thunder's theorem on an analogue of Minkowski's second convex bodies theorem ([T1]), it follows that

$$1 \le \gamma(GL_n, Q_d, k) \le \widetilde{\gamma}(GL_n, Q_d, k) \le q^{ndg(k)/\gcd(d,n-d)} = q_0^{dg(k)}.$$

If $g(k) = 0$, i.e., k is a rational function field over $\mathbb{F}_q$, this implies $\gamma(GL_n, Q_d, k) = \widetilde{\gamma}(GL_n, Q_d, k) = 1$. If $g(k) = 1$ and $d = 1$, the first inequality and the upper bound of the second inequality give

$$q^{n-1} \cdot \frac{(q-1)(q^{2n} + a_1 q^n + q)}{(q + a_1 + 1)(q^{2n} - q^{n+1})} < \gamma(GL_n, Q_1, k) \le \widetilde{\gamma}(GL_n, Q_1, k) \le q^n,$$

where $h_k = a_1 + q + 1$. Combining this with the Hasse–Weil bound $|a_1| \le 2\sqrt{q}$, we have $\gamma(GL_n, Q_1, k) = \widetilde{\gamma}(GL_n, Q_1, k) = q^n$ provided that $h_k \le q - 1$.

Except for the case where G is either an inner form of a general linear group or an orthogonal group defined over an algebraic number field ([W2], [W3]), we have no any result on an upper bound of $\gamma(G, Q, k)$.

Theorems 1 – 4 and Example 3 were proved in [W5]. Furthermore, we can add a small result on $\widetilde{\gamma}(GL_n, Q_1, k)$.

Theorem 5. *We define the constant* Δ_k *as follows:*

$$\Delta_k = \begin{cases} |D_k| & (k \text{ is an algebraic number field of absolute discriminant } D_k). \\ q^{2g(k)-2} & (k \text{ is a function field of genus } g(k) \text{ and constant field } \mathbb{F}_q). \end{cases}$$

If ℓ is a separable extension of k with degree r, then

$$\frac{\widetilde{\gamma}(GL_n, Q_1, \ell)}{\Delta_\ell^{n/2}} \le r^{-nrs_k/2} \cdot \frac{\widetilde{\gamma}(GL_{nr}, Q_1, k)}{\Delta_k^{nr/2}},$$

where s_k denotes the number of infinite places of k.

This theorem was first proved in [O-W] in the case of $k = \mathbb{Q}$. See [W8] for a genral case.

3. Behavior of fundamental Hermite constants under isogenies. Theorem 1 asserts that the fundamental Hermite constants is invariant under some kind of central extensions. It is natural to ask how the fundamental Hermite constatns behaves under central isogenies. We can show the following general result.

Theorem 6. *Let*

$$1 \longrightarrow F \longrightarrow \widehat{G} \xrightarrow{\beta} G \longrightarrow 1$$

be a separable central k-isogeny of a connected reductive k-group G and Q a maximal k-parabolic subgroup of G. Then

$$\gamma(\widehat{G}, \beta^{-1}(Q), k)^{d_\beta} \le \gamma(G, Q, k),$$

where $d_\beta = [\mathbf{X}_k^*(M_{\beta^{-1}(Q)}/Z_{\widehat{G}}) : \mathbf{X}_k^*(M_Q/Z_G)]$.

If k is an algebraic number field, we have a more precise result. We assume G is an almost simple isotropic group and

$$1 \longrightarrow F \longrightarrow \widetilde{G} \xrightarrow{\beta} G \longrightarrow 1$$

is the simply connected covering of G defined over k. Let $\mathfrak{o}$ (resp. $\mathfrak{o}_v$) be the ring of integers in k (resp. k_v for a finite place v of k). We fix an $\mathfrak{o}$-model of G and take the group $G(\mathfrak{o}_v)$ of $\mathfrak{o}_v$-rational points of G. We set

$$G(\mathbb{A}_\infty) = \prod_{\substack{w \\ \text{infinite}}} G(k_v) \times \prod_{\substack{v \\ \text{finite}}} G(\mathfrak{o}_v).$$

It is known that $G(k)G(\mathbb{A}_\infty)$ is a normal subgroup of $G(\mathbb{A})$ and $G(k)G(\mathbb{A}_\infty)\backslash G(\mathbb{A}) = G(k)\backslash G(\mathbb{A})/G(\mathbb{A}_\infty)$ is a finite set ([P-R, Proposition 8.8]). Let Ξ_G be a complete set of representatives of $G(k)G(\mathbb{A}_\infty)\backslash G(\mathbb{A})$. For each $\xi \in \Xi_G$, we set

$$\gamma(G, Q, k)_\xi = \max_{g \in G(k)G(\mathbb{A}_\infty)\xi} \min_{x \in X_Q g} H_Q(x),$$

and especially

$$\gamma(G, Q, k)_1 = \max_{g \in G(k)G(\mathbb{A}_\infty)} \min_{x \in X_Q g} H_Q(x).$$

It is obvious that

$$\gamma(G, Q, k) = \max_{\xi \in \Xi_G} \gamma(G, Q, k)_\xi .$$

Theorem 7. *Being the notations and assumptions as above, we have*

$$\gamma(\widetilde{G}, \beta^{-1}(Q), k)^{d_\beta} = \gamma(G, Q, k)_1 .$$

Theorems 6 and 7 will be proved in [W9]. As a corollary of Theorems 1 and 7, we obtain the following.

Corollary. *If k is an algebraic number field, then*

$$\gamma(SL_n, Q_d \cap SL_n, k)^{n/\gcd(d,n-d)} = \gamma(PGL_n, Z_{GL_n}\backslash Q_d, k)_1$$

In particular, if the ideal class group $I_k = k^\times \mathbb{A}_\infty^\times \backslash \mathbb{A}^\times$ *of k satisfies* $I_k = I_k^n$, *then*

$$\gamma(SL_n, Q_d \cap SL_n, k)^{n/\gcd(d,n-d)} = \gamma(GL_n, Q_d, k) = \gamma(PGL_n, Z_{GL_n}\backslash Q_d, k).$$

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