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Asymptotic expansions of the non-holomorphic Eisenstein series

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Abstract

In this report we describe one asymptotic expansion of the non-holomorphic Eisenstein series using Airy functions. We might expect that the property of the Eisenstein series on the complex parameter is simple because of the "good behavior" of its constant terms. But the fact is that it is not so, and this makes our study interesting. We will find an analogy between Eisenstein series and the square of the Riemann zeta-function in the point of view of the asymptotic expansion.

1 Eisenstein series

Let $k \geq 0$ be an even integer and H be the upper half plane. The non-holomorphic Eisenstein series for $SL_2(\mathbb{Z})$ is defined by

$$E_k(z,s) = y^s \sum_{\{c,d\}} (cz+d)^{-k}|cz+d|^{-2s}. \tag{1}$$

Here $z = x+\sqrt{-1}\,y \in H$, $s \in \mathbb{C}$ and the summation is taken over $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$, a complete system of representation of $\left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in SL_2(\mathbb{Z})\right\}\backslash SL_2(\mathbb{Z})$. The right-hand side of (1) converges absolutely and locally uniformly on $\left\{(z,s) \mid z \in H, \Re(s) > 1-\frac{k}{2}\right\}$, and $E_k(z,s)$ has a meromorphic continuation to the whole s-plane.

In this report we consider the Eisenstein series $E(z,s) = E_0(z,s)$. Let $i = \sqrt{-1}$, $s = \sigma+it \in \mathbb{C}$ and $\zeta(s)$ be the Riemann zeta-function. For $\Re(s) > 1$, $\zeta(2s)E(z,s)$ is expressed by

$$\zeta(2s)E(z,s) = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d) \neq (0,0)}} \frac{y^s}{|cz+d|^{2s}}.$$

The Fourier expansion is as follows;

$$\begin{aligned} \zeta(2s)E(z,s) &= \zeta(2s)y^s + \sqrt{\pi}\zeta(2s-1)\tfrac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s} \\ &\quad +4\tfrac{\pi^s}{\Gamma(s)}\sqrt{y}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)K_{s-\frac{1}{2}}(2\pi ny)\cos(2\pi nx), \end{aligned} \tag{2}$$

$$\sigma_s(l) = \sum_{d|l,\, d>0} d^s,$$

and $K_\nu(\tau)$ $(\nu,\, \tau \in \mathbb{C})$ is the modified Bessel function defined by the integral

$$K_\nu(\tau) = \frac{1}{2}\int_0^\infty u^{\nu-1} \exp\left(-\tfrac{1}{2}\tau(u+\tfrac{1}{u})\right) du.$$

It is well-known that the Fourier expansion (2) gives the holomorphic continuation of $\zeta(2s)E(z,s)$ to the whole s-plane except for the simple pole at $s=1$, and gives the functional equation

$$\pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = \pi^{-1+s}\Gamma(1-s)\zeta(2-2s)E(z,1-s).$$

2 Asymptotic expansion

Based on Olver's works [4]-[7], Balogh ([1], [2]) gave one uniform asymptotic expansion of K_{it} $(t \in \mathbb{R})$ for large values t using Airy functions. The Airy function is defined by

$$\mathrm{Ai}(\xi) = \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}u^3 + \xi u\right) du = \frac{1}{\sqrt{3}\pi}\xi^{\frac{1}{2}} K_{\frac{1}{3}}\left(\tfrac{2}{3}\xi^{\frac{3}{2}}\right).$$

The uniform asymptotic expansion due to Balogh is as follows;

$$K_{it}(\tau) = \frac{\sqrt{2}\pi}{t^{\frac{1}{3}}} \exp\left(-\frac{\pi}{2}t\right)\left(\frac{\rho}{r^2-1}\right)^{\frac{1}{4}} \left\{ \mathrm{Ai}(\xi)\left(1+\sum_{k=1}^{m} t^{-2k}A_k(\rho)\right) \right.$$
$$\left. + t^{-\frac{4}{3}}\mathrm{Ai}'(\xi)\sum_{k=0}^{m-1} t^{-2k}B_k(\rho) + \varepsilon_{2m+1} \right\}, \tag{3}$$

where $t \in \mathbb{R}$, $\tau \in \mathbb{C}$, and r, ρ, ξ are parameters such that

$$r = \tau/t, \quad \tfrac{2}{3}\rho^{\frac{3}{2}} = (r^2-1)^{\frac{1}{2}} - \operatorname{arcsec} r, \quad \xi = t^{\frac{2}{3}}\rho.$$

The coefficients are defined in [4] and [5] by

$$\begin{cases} A_k(\rho) = \sum_{l=0}^{2k} (-1) b_l \rho^{-\frac{3}{2}l} U_{2k-l} \\ \rho^{\frac{1}{2}} B_k(\rho) = \sum_{l=0}^{2k+1} (-1) a_l \rho^{-\frac{3}{2}l} U_{2k-l+1}. \end{cases}$$

Here a_l, b_l are real coefficients and U_k is a polynomial in $\frac{1}{\sqrt{r^2-1}}$. The error term ε_{2m+1} is defined in [6] and [7].

The asymptotic expansion (3) is useful near the transition point $t = \tau$. Expansions for the cases $\tau/t \not\sim 1$ or $\tau - t = o(\tau^{\frac{1}{3}})$ are already obtained by using saddle point method. (See [8].)

In the following we assume $\sigma = \frac{1}{2}$. Applying asymptotic expansions of the Bessel function to the Eisenstein series, we have the following theorem.

Theorem 1 *For any positive constant $\varepsilon > 0$, we have*

$$\begin{aligned}
&\zeta(1+2it)E(z,\tfrac{1}{2}+it) \\
&= 4\pi^{it}\sqrt{y}\sum_{n=1}^{N_1} n^{-it}\sigma_{2it}(n)\left\{t^2-(2\pi ny)^2\right\}^{-\frac{1}{4}}\cos(2\pi nx)\left\{\sqrt{2}\sin(f(t,n))+O\left(\frac{1}{2\pi ny}\right)\right\} \\
&\quad +4\pi^{1+it}\sqrt{y}\sum_{n=N_1+1}^{N_2} n^{-it}\sigma_{2it}(n)\left\{t^2-(2\pi ny)^2\right\}^{-\frac{1}{4}}|\xi|^{\frac{1}{4}}\cos(2\pi nx) \\
&\quad \times\left\{\mathrm{Ai}(\xi)\left(1+t^{-2}A_1(\rho)-\xi^{\frac{1}{2}}t^{-\frac{4}{3}}B_0(\rho)\right)+\exp(-\tfrac{2}{3}\xi^{\frac{3}{2}})(1+|\xi|)^{-\frac{1}{4}}O\left(\frac{1}{t}\right)\right\} \\
&\quad \times\left\{1+O\left(\frac{1}{t}\right)\right\}+O(t^{\varepsilon}).
\end{aligned} \tag{4}$$

Here

$$N_1 = \left[\frac{1}{1+\delta}\cdot\frac{t}{2\pi y}\right], \qquad N_2 = \left[\frac{t}{2y}\right]$$

for some positive constant $\delta > 0$ and

$$f(t,n) = t\,\mathrm{arccosh}\left(\frac{t}{2\pi ny}\right) - \left\{t^2-(2\pi ny)^2\right\}^{\frac{1}{2}} + \frac{1}{4}\pi.$$

In (4), we are able to describe error terms more precisely. In that sense we call (4) the asymptotic expansion of the Eisenstein series.

3 Remark

Let $D < 0$ be the discriminant of an imaginary quadratic field K and $\zeta_K(s)$ be the Dedekind zeta-function of K. Let $h(D)$ be the ideal class number of K and $f_1, f_2, \cdots, f_{h(D)}$ be the equivalence classes of binary qudratic forms of discriminant D. Then we have

$$\sum_{i=1}^{h(D)} \zeta(2s)E(z(f_i),s) = 2^{-s-1}w|D|^{s/2}\zeta_K(s).$$

Here $z(f) = (-b+\sqrt{D})/2a \in H$ is the associated root of the quadratic form $f(X,Y) = aX^2+bXY+cY^2$, and w is the number of roots of unity in K. (Cf. [9], [10, Sect.8, Sect.11].) It is also well known that $\zeta_K(s) = \zeta(s)L(s)$ for the L-function with Kronecker's symbol. This shows that Theorem 1 is one approach to the study of the product of classical zeta-functions. Especially we will find an analogy between Theorem 1 and formulas of Voronoi-Atkinson type for $\zeta^2(s)$ proved by Jutila [3].

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