数理解析研究所講究録 1319 巻 2003 年 40-50

An inductive method for proving the transcendence of certain power series

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1 Introduction

If α is an algebraic number, we denote by $\overline{|\alpha|}$ the maximum of the absolute values of the conjugates of α and by $\mathrm{den}(\alpha)$ the least positive integer such that $\mathrm{den}(\alpha)\alpha$ is an algebraic integer, and we set $\|\alpha\| = \max\{\overline{|\alpha|}, \mathrm{den}(\alpha)\}$. Then for nonzero algebraic α, we have the fundamental inequalities

$$|\alpha| \geq \|\alpha\|^{-2[\mathbf{Q}(\alpha):\mathbf{Q}]} \text{ and } \|\alpha^{-1}\| \leq \|\alpha\|^{2[\mathbf{Q}(\alpha):\mathbf{Q}]}$$

(cf. Lemma 2.10.2 in [12]).

Let K be an algebraic number field, O_K be the ring of integers in K. Let r and L be integers such that $r \geq 2$ and $L \geq 1$. We consider the function:

$$\Phi_0(x) = \sum_{k=0}^{\infty} \frac{E_k(x^{r^k})}{F_k(x^{r^k})},$$

where

$$E_k(x) = a_{k1}x + a_{k2}x^2 + \dots + a_{kL}x^L \in K[x],$$

$$F_k(x) = 1 + b_{k1}x + b_{k2}x^2 + \dots + b_{kL}x^L \in O_K[x],$$

$$\log\|a_{k\ell}\|,\ \log\|b_{k\ell}\| = o(r^k), \quad 1 \leq \ell \leq L.$$

The aim of this paper is to study the arithmetical nature of $\Phi_0(\alpha)$ when $\alpha \in K$, $0 < |\alpha| < 1$, and $F_k(\alpha^{r^k}) \neq 0$ for every $k \geq 0$.

It should be noticed that in some cases $\Phi_0(x)$ can be explicitely computed as a rational function. Specific examples are, with r=2

$$\sum_{k=0}^{\infty} \frac{x^{2^k}}{1-x^{2^{k+1}}} = \frac{x}{1-x},$$

$$\sum_{k=0}^{\infty} \frac{2^k x^{2^k}}{1+x^{2^k}} = \frac{x}{1-x},$$

$$\sum_{k=0}^{\infty} \frac{(-2)^k x^{2^k}}{x^{2^{k+1}} - x^{2^k} + 1} = \frac{x}{x^2+x+1}.$$

The first equality is due to Lucas [9]. The latter two equalities are proved in Duverney [4] but are evidently older. In the case where r=3, we have for example

$$\sum_{k=0}^{\infty} \frac{3^k x^{3^k}(1-x^{2\cdot 3^k})}{x^{4\cdot 3^k} + x^{2\cdot 3^k} + 1} = \frac{x}{1-x^2}.$$

This equality is proved in Duverney and Shiokawa [7]. Clearly for these examples, $\Phi_0(\alpha) \in K$ if $\alpha \in K$.

Our main result will be the

Transcendence Criterion. $\Phi_0(\alpha)$ is algebraic if and only if $\Phi_0(x) \in K(x)$.

The proof of Transcendence Criterion relies on Mahler's transcendence method, more precisely on the following result, which is a special case of a theorem of Loxton and van der Poorten [8] (cf. Theorem 2.9.1 in [12]).

Theorem 1. Let K be an algebraic number field, $r \geq 2$ be an integer, $\{\Phi_n(x)\}_{n\geq 0}$ be a sequence in the ring of formal power series $K[[x]]$ and $\alpha \in K$ with $0 < |\alpha| < 1$. If the following three properties are satisfied, then $\Phi_0(\alpha)$ is transcendental.

(I) $\Phi_n(\alpha^{r^n}) = a_n \Phi_0(\alpha) + b_n$,
where $a_n, b_n \in K$, and $\log \|a_n\|, \log \|b_n\| = O(r^n)$.

(II) If $\Phi_n(x) = \sum_{\ell=0}^{\infty} \sigma_\ell^{(n)} x^\ell$, then for any $\varepsilon > 0$ there is a positive integer n_0 such that

$$\log \|\sigma_\ell^{(n)}\| \leq \varepsilon r^n (1+\ell)$$

for any $n \geq n_0$ and $\ell \geq 0$.

(III) Let $\{s_\ell\}_{\ell\geq 0}$ be variables and

$$F(x;s) = F(x;\{s_\ell\}_{\ell\geq 0}) = \sum_{\ell=0}^{\infty} s_\ell x^\ell,$$

in such a way that

$$F(x;\sigma^{(n)}) = F(x;\{\sigma_\ell^{(n)}\}_{\ell\geq 0}) = \Phi_n(x).$$

Then for any polynomials $P_0(x,s),\ldots,P_d(x,s) \in K[x,\{s_\ell\}_{\ell\geq 0}]$ and

$$E(x,s) = \sum_{j=0}^{d} P_j(x,s)F(x;s)^j,$$

there is a positive integer I with the following property: if n is sufficiently large and $P_0(x,\sigma^{(n)}),\ldots,P_d(x,\sigma^{(n)})$ are not all zero, then ord $E(x,\sigma^{(n)}) \leq I$, where ord denotes the zero order at 0.

However, applying Theorem 1 to $\Phi_0(x)$ will not be an easy task, because of condition (III). Thus the second section will be devoted to the proof of Theorem 2, in which condition (III) will be replaced by a simpler one, namely, some kind of irrationality measure of the function $\Phi_0(x)$. The tool in this section is an inductive method developed in Duverney [5]. By introducing low-order Padé-approximants of the functions $\Phi_n(x)$ connected to $\Phi_0(x)$, we will arrive to Transcendence Criterion.

2 An inductive method

Theorem 2. Let K be an algebraic number field, r and L be integers such that $r \geq 2$ and $L \geq 1$, and

$$S = \Phi_0(x) = \sum_{k=0}^{\infty} \frac{E_k(x^{r^k})}{F_k(x^{r^k})},$$

$$E_k(x) = a_{k1}x + a_{k2}x^2 + ... + a_{kL}x^L \in K[x],$$
$$F_k(x) = 1 + b_{k1}x + b_{k2}x^2 + ... + b_{kL}x^L \in K[x].$$

Suppose that there is a positive constant c_1 such that for any polynomials $A_0, A_1 \in K[x]$, not both zero, satisfying $\deg A_0, \deg A_1 \leq M$,

$$\mathrm{ord}(A_0 + A_1 S) \leq c_1 M. \tag{1}$$

Then for any positive integer d there is a positive constant c_d such that for any polynomials $A_0, A_1, ..., A_d \in K[x]$, not all zero, satisfying $\deg A_i \leq M, 0 \leq i \leq d$,

$$\mathrm{ord}(A_0 + A_1 S + ... + A_d S^d) \leq c_d M. \tag{2}$$

Proof. Let

$$\Phi_n(x) = \sum_{k=0}^{\infty} \frac{E_{n+k}(x^{r^k})}{F_{n+k}(x^{r^k})},$$
$$R_n = \Phi_n(x^{r^n}),$$
$$T_n = \sum_{k=0}^{n-1} \frac{E_k(x^{r^k})}{F_k(x^{r^k})}.$$

Then $S = T_n + R_n$. We prove (2) by induction on d. If $d = 1$, (2) is the same as (1). Suppose that for a given $d \geq 2$, we have

$$\mathrm{ord}(B_0 + B_1 S + ... + B_{d-1} S^{d-1}) \leq c_{d-1} M, \tag{3}$$

for every $B_0, ..., B_{d-1} \in K[x]$, not all zero, $\deg B_i \leq M$, $0 \leq i \leq d-1$. We may assume $c_{d-1} \geq 1$ and $A_d \neq 0$. Let $e = dL$. For every $n > 0$, there exist $Q_n(x) \in K[x]$ with $Q_n(x) \neq 0$, and $P_{n1}(x), ..., P_{nd}(x) \in K[x]$ such that

$$\deg Q_n \leq de, \quad \deg P_{ni} \leq de, \quad 1 \leq i \leq d,$$
$$Q_n(x)\Phi_n(x)^i - P_{ni}(x) = x^{de+e+1} G_{ni}(x), \quad 1 \leq i \leq d, \tag{4}$$

where

$$G_{ni}(x) = \sum_{\ell=0}^{\infty} g_{ni\ell} x^{\ell} \in K[[x]].$$

For this we choose $Q_n(x)$ in such a way that the terms of degrees $de+1, \ldots, de+e$ vanish in the Taylor expansion of $Q_n(x)\Phi_n(x)^i$ for $i=1,2,\ldots,d$. We only have to solve a linear homogeneous system which has de equations and $de+1$ unknowns.

Lemma 1. ord $G_{n1}(x) \le \gamma$, where $\gamma = c_1(de+L)-(de+e+1)$.

Proof. In (4) replacing x by x^{r^n} , we have

$$Q_n(x^{r^n})(S-T_n)-P_{n1}(x^{r^n}) = x^{(de+e+1)r^n}G_{n1}(x^{r^n}).$$

Multiplying both sides by $D_n = \prod_{k=0}^{n-1} F_k(x^{r^k})$, we have

$$D_nQ_n(x^{r^n})S - Q_n(x^{r^n})D_nT_n - D_nP_{n1}(x^{r^n}) = x^{(de+e+1)r^n}D_nG_{n1}(x^{r^n}).$$

Since $\deg D_n, \deg D_nT_n \le Lr^n$,

$$\deg D_nQ_n(x^{r^n}),\ \deg(Q_n(x^{r^n})D_nT_n + D_nP_{n1}(x^{r^n})) \le (L+de)r^n.$$

By (1) we have

$$\text{ord } G_{n1}(x^{r^n}) \le (c_1(de+L)-(de+e+1))r^n,$$

which implies the lemma.

We define $P_{n0}(x) = Q_n(x)$, $G_{n0}(x) = 0$. In (4) replacing x by x^{r^n}, we obtain for every $i = 0,1,\ldots,d$,

$$Q_n(x^{r^n})(S-T_n)^i - P_{ni}(x^{r^n}) = x^{(de+e+1)r^n}G_{ni}(x^{r^n}). \qquad (5)$$

We develop $(S-T_n)^i$ and write the equality (5) in matricial form. Then we get

$$Q_n(x^{r^n})\mathcal{M}_n \begin{pmatrix} 1 \\ S \\ \vdots \\ S^d \end{pmatrix} - \begin{pmatrix} P_{n0}(x^{r^n}) \\ P_{n1}(x^{r^n}) \\ \vdots \\ P_{nd}(x^{r^n}) \end{pmatrix} = x^{(de+e+1)r^n} \begin{pmatrix} 0 \\ G_{n1}(x^{r^n}) \\ \vdots \\ G_{nd}(x^{r^n}) \end{pmatrix}, \qquad (6)$$

$$\mathcal{M}_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ -T_n & 1 & 0 & \cdots & \cdots & 0 \\ T_n^2 & -2T_n & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ (-1)^d T_n^d & (-1)^{d-1}\binom{d}{1} T_n^{d-1} & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$

In [5] it is shown that

$$\mathcal{M}_n^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ T_n & 1 & 0 & \cdots & \cdots & 0 \\ T_n^2 & 2T_n & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ T_n^d & \binom{d}{1} T_n^{d-1} & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$

Note that $D_n^d \mathcal{M}_n^{-1}$ has its elements in $K[x]$. Multiplying (6) on the left by $\mathcal{M}_n^{-1}$, we get

$$Q_n(x^{r^n}) \begin{pmatrix} 1 \\ S \\ \vdots \\ S^d \end{pmatrix} - \mathcal{M}_n^{-1} \begin{pmatrix} P_{n0}(x^{r^n}) \\ P_{n1}(x^{r^n}) \\ \vdots \\ P_{nd}(x^{r^n}) \end{pmatrix} = x^{(de+e+1)r^n} \mathcal{M}_n^{-1} \begin{pmatrix} 0 \\ G_{n1}(x^{r^n}) \\ \vdots \\ G_{nd}(x^{r^n}) \end{pmatrix}. \quad (7)$$

Multiplying (7) on the left by the row matrix $D_n^d(A_0, ..., A_d)$ we obtain

$$U_n \left(\sum_{h=0}^{d} A_h S^h \right) - V_n = x^{(de+e+1)r^n} H_n, \quad (8)$$

where

$$U_n = D_n^d Q_n(x^{r^n}) \in K[x],$$

$$V_n = (A_0, ..., A_d) D_n^d \mathcal{M}_n^{-1} \begin{pmatrix} P_{n0}(x^{r^n}) \\ P_{n1}(x^{r^n}) \\ \vdots \\ P_{nd}(x^{r^n}) \end{pmatrix} \in K[x],$$

$$H_n = (A_0, ..., A_d) D_n^d \mathcal{M}_n^{-1} \begin{pmatrix} 0 \\ G_{n1}(x^{r^n}) \\ \vdots \\ G_{nd}(x^{r^n}) \end{pmatrix} \in K[[x]].$$

Let n be the positive integer such that

$$r^{n-1} \le c_{d-1} M < r^n. \tag{9}$$

Then, as $e = dL$ and $c_{d-1} \ge 1$,

$$\deg V_n \le M + dLr^n + der^n < (de + e + 1)r^n. \tag{10}$$

Let m be the least integer such that $(0, g_{n1m}, ..., g_{ndm}) \neq \mathbf{0}$. By Lemma 1, $m \le \gamma$. Let

$$\begin{pmatrix} 0 \\ g_{n1m} \\ \vdots \\ \vdots \\ \vdots \\ g_{ndm} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g_{nim} \\ \vdots \\ g_{ndm} \end{pmatrix}, \quad g_{nim} \neq 0.$$

Then under mod $x^{(m+1)r^n}$, we have

$$\begin{aligned} H_n &\equiv D_n^d (A_0, ..., A_d) \mathcal{M}_n^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g_{nim} x^{mr^n} \\ \vdots \\ g_{ndm} x^{mr^n} \end{pmatrix} \\ &\equiv D_n^d (A_0, ..., A_d) \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \\ \binom{d-1}{i} T_n^{d-i-1} & \cdots & \cdots & 1 & 0 \\ \binom{d}{i} T_n^{d-i} & \cdots & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} g_{nim} \\ \vdots \\ \vdots \\ g_{ndm} \end{pmatrix} x^{mr^n} \end{aligned}$$

$$\equiv \quad D_n^d(A_0, ..., A_d) \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \\ \binom{d-1}{i} S^{d-i-1} & \cdots & \cdots & 1 & 0 \\ \binom{d}{i} S^{d-i} & \cdots & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} g_{nim} \\ \vdots \\ \vdots \\ g_{ndm} \end{pmatrix} x^{mr^n}$$

$$\equiv \quad D_n^d(B_0 + B_1 S + ... + B_{d-i} S^{d-i}) x^{mr^n},$$

where $B_0, ..., B_{d-i} \in K[x]$ and

$$B_{d-i} = A_d \binom{d}{i} g_{nim} \neq 0, \quad \deg B_h \leq M, \quad 0 \leq h \leq d-i.$$

Since ord $D_n = 0$, by (3), (9) we obtain

$$\text{ord}\left(D_n^d(B_0 + B_1 S + ... + B_{d-i} S^{d-i}) x^{mr^n}\right) \leq c_{d-1} M + mr^n < (1+m) r^n.$$

Hence $H_n \not\equiv 0 \bmod x^{(m+1)r^n}$. Suppose that $V_n \neq 0$. By (10) we get

$$\text{ord } V_n < (de + e + 1) r^n.$$

Therefore by (8), (9) we obtain

$$\text{ord}\left(\sum_{h=0}^{d} A_h S^h\right) < (de + e + 1) r^n \leq (de + e + 1) r c_{d-1} M.$$

If $V_n = 0$, by (8), (9) we obtain

$$\text{ord}\left(\sum_{h=0}^{d} A_h S^h\right) < (de + e + 1) r^n + (m+1) r^n \leq (de + e + 2 + \gamma) r c_{d-1} M.$$

Letting $c_d = (de + e + 2 + \gamma) r c_{d-1}$, we obtain (2).

Examples involving Fibonacci and Lucas numbers

Let $\alpha = \dfrac{1-\sqrt{5}}{2}$ and $\beta = \dfrac{1+\sqrt{5}}{2}$. Then nth Fibonacci number F_n and nth Lucas number L_n are written as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\alpha - \beta},$$

$$L_n = \alpha^n + \beta^n = \alpha^n + (-1)^n \alpha^{-n}.$$

Let $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ be sequences in K and O_K respectively. Then

$$\sum_{k=1}^{\infty} \frac{a_k}{F_{2^k} + b_k} = (\beta - \alpha) \sum_{k=1}^{\infty} \frac{a_k \alpha^{2^k}}{1 + (\beta - \alpha) b_k \alpha^{2^k} - (\alpha^{2^k})^2}, \qquad (11)$$

$$\sum_{k=1}^{\infty} \frac{a_k}{L_{2^k} + b_k} = \sum_{k=1}^{\infty} \frac{a_k \alpha^{2^k}}{1 + b_k \alpha^{2^k} + (\alpha^{2^k})^2}. \qquad (12)$$

Mignotte [11] proved that $\sum_{k=0}^{\infty} \frac{1}{k! F_{2^k}}$ is transcendental by using Schmidt's theorem on approximations of an algebraic number by algebric numbers. Later Mahler [10] proved it without using Schmidt's theorem and Loxton and van der Poorten [8] generalized Mahler's method. Becker and Töpfer [1] and Nishioka [13] studied the arithmetical nature of the series (11) and (12) when $b_k = 0$ for every k, $\{a_k\}$ is a periodic sequence and a linear recurence sequence of algebraic numbers respectively. Duverney, Kanoko and Tanaka [5] studied the case $b_k = b$ for every k and $\{a_k\}$ is a linear recurrence sequence of algebraic numbers.

We have the following.

Theorem 3. Assume there exist infinitely many k such that $a_k \neq 0$, and that $\log \|a_k\|, \log \|b_k\| = o(2^k)$. Let

$$\Phi_0(x) = \sum_{k=0}^{\infty} \frac{a_k x^{2^k}}{1 + (\beta - \alpha) b_k x^{2^k} - x^{2^{k+1}}}.$$

If $\Phi_0(x) \in K(x)$, then there exist $N \in \mathrm{N}$ and $a \in K$ such that $b_k = 0$ and $a_k = a$ for every $k \geq N$.

In particular, $\sum_{k=1}^{\infty} \frac{a_k}{F_{2^k} + b_k}$ is algebraic if and only if $a_k = a$ and $b_k = 0$ for every $k \geq N$.

Theorem 4. Assume that there exist infinitely many k such that $a_k \neq 0$, and $\log \|a_k\|, \log \|b_k\| = o(2^k)$. Let

$$\Phi_0(x) = \sum_{k=0}^{\infty} \frac{a_k x^{2^k}}{1 + b_k x^{2^k} + x^{2^{k+1}}}.$$

If $\Phi_0(x) \in K(x)$, then one of the following two conditions is satisfied.
(i) There exist $N \in \mathbf{N}$ and $a \in K$ such that $b_k = 2$ and $a_k = a4^k$ for every $k \geq N$.
(ii) There exist a constant a, $p, q \in \mathbf{N}$, $q \neq 0$, and $N \in \mathbf{N}$ such that $b_k = 2\cos\left(2^k \cdot \frac{p}{q}\pi\right)$, $a_k = a2^k \sin\left(2^k \cdot \frac{p}{q}\pi\right)$ for every $k \geq N$.

In particular, $\sum_{k=1}^{\infty} \frac{a_k}{L_{2^k} + b_k}$ is algebraic if and only if (i) or (ii) holds.

Corollary. Assume that there exist infinitely many k such that $a_k \neq 0$, and $\log\|a_k\|, \log\|b_k\| = o(2^k)$. If $\sum_{k=1}^{\infty} \frac{a_k}{L_{2^k} + b_k}$ is algebraic, then $\{b_k\}$ is eventually periodic, $|b_k| \leq 2$ and $a_{k+1} = 2a_k b_k$ for every large k.

Example . Under the assumptions of Theorem 11, $\sum_{k=1}^{\infty} \frac{a_k}{L_{2^k}}$ is transcendental.

Moreover if $|b_k| > 2$ for infinitely many k, then $\sum_{k=1}^{\infty} \frac{a_k}{L_{2^k} + b_k}$ is transcendental.

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