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# A Numerical Methodology for a Singular Limit Problem

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## 1 Introduction

In ecological and chemical problems, we encounter nonlinear equations of the form

(1) $$w_t = \nabla(d(w)\nabla w) + h(w) \qquad x \in \Omega, \ \ t > 0,$$

(2) $$\frac{\partial w}{\partial \nu} = 0 \qquad x \in \partial\Omega, \ \ t > 0,$$

(3) $$w(0,x) = w_0(x) \in L^\infty(\Omega) \qquad x \in \Omega,$$

where $d(s)$ is a step function:

(4) $$d(s) = \begin{cases} d_1 & (s \geq 0), \\ d_2 & (s < 0). \end{cases}$$

The function $w = w(t,x)$ is a real-valued function, $d_1$ and $d_2$ are positive constants, $\Omega$ is a bounded region in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $\nu$ is the unit outer normal to $\partial\Omega$. The aim of this article is to present a time-discrete scheme for (1)–(3), which is based on the operator-splitting methodology.

The problem (1)–(3) is obtained as a singular limit of the following system:

(5) $$u_t = d_1\Delta u + f(u)u - kuv \qquad x \in \Omega,\ t > 0,$$

(6) $$v_t = d_2\Delta v + g(v)v - \alpha kuv \qquad x \in \Omega,\ t > 0,$$

(7) $$\frac{\partial u}{\partial \nu} = 0, \quad \frac{\partial v}{\partial \nu} = 0 \qquad x \in \partial\Omega,\ t > 0,$$

(8) $$u(0,x) = u_0(x), \quad v(0,x) = v_0(x) \qquad x \in \Omega.$$

Here $\alpha$ and $k$ are positive constants. The functions $f$ and $g$ are written as $f(s) = a_u(R_u - s)$ and $g(s) = a_v(R_v - s)$, where $a_u$, $a_v$, $R_u$ and $R_v$ are positive constants. Let $(u^{(k)}, v^{(k)})$ be a solution to (5)–(8). Dancer et al. have shown that as $k \to \infty$ the function $w^{(k)} = u^{(k)} - v^{(k)}/\alpha$ converges to a weak solution $w$ to (1)–(3) with a function $h$ such that

(9) $$h(s) = \begin{cases} f(s)s & (s \geq 0), \\ g(-\alpha s)s & (s < 0). \end{cases}$$

Furthermore they proved that $u^{(k)} \to [w]^+$ and $v^{(k)} \to \alpha[w]^-$ respectively, where $[a]^\pm$ is $\max\{\pm a, 0\}$; see Proposition 2.1 in [1] for the detail. Thus (1)–(3) describes the asymptotic behavior of (5)–(8).

The system (5)–(8) models behavior of competing biological species. On the other hand, when $f = g \equiv 0$, (5)–(8) is a model of chemical reactions. In this case the singular limit also leads to (1)–(3). Evans [2] has given convergence proof for restrictive initial data. Tonegawa [6] proved regularity properties of solutions to the limiting problem.

## 2 A Time-Discrete Scheme

We present a time-discrete scheme to integrate (1)–(3) numerically.

Put

(10) $$F(s) := f(s)s, \quad G(s) := g(s)s.$$

Then the scheme that we propose is written as follows.

**Threshold Competition Dynamics (TCD)**

Let $M$ be a positive integer. The approximate solution $(u_M(t,x), v_M(t,x))$ by TCD to the limiting problem of (5)–(8) as $k \to \infty$ is defined by

(11) $$u_M(0,x) = u_0(x), \quad v_M(0,x) = v_0(x) \quad \text{for } x \in \Omega,$$

(12) $$u_M(t,x) = \overline{u}^j_M(t,x), \quad v_M(t,x) = \overline{v}^j_M(t,x), \quad \text{for } t \in (t_j, t_{j+1}],\ x \in \Omega,$$

$$\tau := T/M, \quad t_j := j\tau \ \ (j = 0, 1, \ldots, M). \tag{13}$$

The functions $\overline{u}^j_M(t, x)$ and $\overline{v}^j_M(t, x)$ are constructed by the following steps:

Step 1. Put $u^0_M(x) = u_0(x)$, $v^0_M(x) = v_0(x)$ $(x \in \Omega)$.

Step 2. For given $u^j_M(x)$ and $v^j_M(x)$,

(i) Find $\overline{u}^j_M(t, x)$ and $\overline{v}^j_M(t, x)$ such that

$$\begin{cases} \dfrac{\partial \overline{u}^j_M}{\partial t} = d_1 \Delta \overline{u}^j_M + F(\overline{u}^j_M) & x \in \Omega, \quad t_j < t < t_{j+1}, \\ \dfrac{\partial \overline{u}^j_M}{\partial \nu} = 0 & x \in \partial\Omega, \quad t_j < t < t_{j+1}, \\ \overline{u}^j_M(t_j, x) = u^j_M(x) & x \in \Omega, \end{cases} \tag{14}$$

$$\begin{cases} \dfrac{\partial \overline{v}^j_M}{\partial t} = d_2 \Delta \overline{v}^j_M + G(\overline{v}^j_M) & x \in \Omega, \quad t_j < t < t_{j+1}, \\ \dfrac{\partial \overline{v}^j_M}{\partial \nu} = 0 & x \in \partial\Omega, \quad t_j < t < t_{j+1}, \\ \overline{v}^j_M(t_j, x) = v^j_M(x) & x \in \Omega. \end{cases} \tag{15}$$

(ii) Define $u^{j+1}_M(x)$ and $v^{j+1}_M(x)$ by

$$u^{j+1}_M(x) = \lim_{\theta \to \infty} \widehat{u}^j_M(\theta; x), \quad v^{j+1}_M(x) = \lim_{\theta \to \infty} \widehat{v}^j_M(\theta; x), \tag{16}$$

where $\widehat{u}^j_M$ and $\widehat{v}^j_M$ solve

$$\begin{cases} \dfrac{d\widehat{u}^j_M}{d\theta} = -\widehat{u}^j_M \widehat{v}^j_M & x \in \Omega, \quad 0 < \theta < k\tau, \\ \dfrac{d\widehat{v}^j_M}{d\theta} = -\alpha \widehat{u}^j_M \widehat{v}^j_M & x \in \Omega, \quad 0 < \theta < k\tau, \\ \widehat{u}^j_M(0; x) = \overline{u}^j_M(x, t_{j+1}), \quad \widehat{v}^j_M(0; x) = \overline{v}^j_M(x, t_{j+1}), & x \in \Omega. \end{cases} \tag{17}$$

We note that an operator-splitting method is used in Step 2, that is, (5) and (6) are splitted into

$$u_t = d_1 \Delta u + F(u), \quad v_t = d_2 \Delta v + G(v), \tag{18}$$

$$\frac{du}{dt} = -kuv, \quad \frac{dv}{dt} = -k\alpha uv. \tag{19}$$

The main idea of TCD is Step 2 (ii). Let $\theta = kt$; then (19) are rewritten to (17). Instead of passing to the limit $k \to \infty$ in (19), we use the asymptotic limit $\theta \to \infty$ in a solution to (17). The limit is easily obtained. In fact, by using the fact that $d(u - v/\alpha)/d\theta = 0$, it follows that

$$\lim_{\theta \to \infty} \widehat{u}_M^j(\theta; x) = \left[\overline{u}_M^j(t_{j+1}, x) - \overline{v}_M^j(t_{j+1}, x)/\alpha\right]^+, \tag{20}$$

$$\lim_{\theta \to \infty} \widehat{v}_M^j(\theta; x) = \alpha \left[\overline{u}_M^j(t_{j+1}, x) - \overline{v}_M^j(t_{j+1}, x)/\alpha\right]^-. \tag{21}$$

# 3 Results

To state our results we need to define a weak solution to (1)–(3).

**Definition .** *We call $w$ a weak solution if it satisfies:*

$$w \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega)), \tag{22}$$

$$\int_\Omega w(T)\phi(T) - \iint_{Q_T} \{w\phi_t - d(w)\nabla w \nabla \phi + h(w)\phi\} = \int_\Omega w_0 \phi(0), \tag{23}$$

*for all $\phi \in C^1(\overline{Q}_T)$, where $Q_T = \Omega \times (0,T)$.*

**Remark 1.** *There exists a unique solution to (22)–(23) if $w_0 \in L^\infty(\Omega)$ [1].*

We are ready to state our results.

**Theorem A.** *Suppose $w_0 \in L^\infty(\Omega)$. Set $u_0 = [w_0]^+$ and $v_0 = \alpha[w_0]^-$. Let $w$ be a weak solution for the initial data $w_0$ and $(u_M, v_M)$ an approximate solution by Threshold Competition Dynamics for the initial data $(u_0, v_0)$. Then $u_M$, $v_M$ and $w_M = u_M - v_M/\alpha$ converge to $[w]^+$, $\alpha[w]^-$ and $w$ in $L^2(0,T; L^2(\Omega))$ respectively as $M$ tends to $\infty$.*

Moreover, if $d_1 = d_2$ we have information about the convergence rate. Let $\Omega^{\text{int}}$ be a Lipschitz domain such that $\Omega^{\text{int}} \subset\subset \Omega$. Denote $\Omega \setminus \overline{\Omega^{\text{int}}}$ by $\Omega^{\text{ext}}$.

**Theorem B.** *Functions $w_0$, $u_0$, $v_0$, $w_M$ and $w$ are the same as those in Theorem A. Assume that*

$$u_0 \in H^2(\Omega^{\text{int}}), \quad v_0 \in H^2(\Omega^{\text{ext}}), \quad u_0 = v_0 = 0 \ \text{ on } \partial\Omega^{\text{int}}, \tag{24}$$

*and that there exists a sequence of functions $\{a_n\}_{n=1}^{\infty}$ such that*

$$\left.\begin{array}{c} a_n \in C^2(\overline{\Omega^{\text{ext}}}), \quad (n = 1, 2, \dots), \\ 0 \le a_n \le R_0 \quad (n = 1, 2, \dots) \qquad \text{for some constant } R_0, \\ \left.\dfrac{\partial a_n}{\partial \nu}\right|_{\partial\Omega} = 0 \quad (n = 1, 2, \dots), \\ a_n \to v_0 \quad \text{as} \quad n \to \infty \quad \text{in } H^2(\Omega^{\text{ext}}). \end{array}\right\} \tag{25}$$

*In addition if $d_1 = d_2$, then*

$$\|(u_M(T) - v_M(T)/\alpha) - w(T)\|_{L^2(\Omega)} \le C(1/M)^{1/2}, \tag{26}$$

$$\|(u_M(T) - v_M(T)/\alpha) - w(T)\|_{L^1(\Omega)} \le C'(1/M), \tag{27}$$

*where $C$ and $C'$ are positive constants independent of $M$.*

# 4 Outline of the Proof

## 4.1 Theorem A

We work within the framework of the evolution triple $H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (H^1(\Omega))^*$ (see chapter 23 in [7] or chapter 3 in [5]).

The next lemmas play an important role in the proof.

**Lemma 4.1.** *Functions $u_M$ and $v_M$ are uniformly bounded in $L^2(0,T;H^1(\Omega))$ with respect to $M$.*

**Lemma 4.2.**

$$\iint_{Q_T} u_M v_M \to 0 \qquad \text{as } M \to \infty. \tag{28}$$

In view of Lemma 4.1 we observe $w_M$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$ with respect to $M$ and so is $\partial_t w_M$ in $L^2(0,T;H^1(\Omega)^*)$. Hence thanks to the compactness property (Theorem 2.1, chapter 3 in [5]) we obtain a subsequence from $\{w_M\}$, which is denoted by $\{w_M\}$ again, converging in $L^2(0,T;L^2(\Omega))$. We write the limit as $w_\infty$:

$$w_M \to w_\infty \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \quad \text{as} \quad M \to \infty. \tag{29}$$

With the aid of Lemma 4.2 we obtain, passing to a subsequence if necessary,

$$\begin{aligned} u_M &\to [w_\infty]^+ \qquad \text{in } L^2(0,T;L^2(\Omega)), \\ v_M/\alpha &\to [w_\infty]^- \qquad \text{in } L^2(0,T;L^2(\Omega)). \end{aligned}$$

Then $w_\infty$ turns out to be the weak solution to (1)–(3).

## 4.2 Theorem B

Throughout this subsection we assume that the conditions for Theorem B are satisfied. Set

$$e_j^{(p)} := \|w_M(t_j,\cdot) - w(t_j,\cdot)\|_{L^p(\Omega)} \qquad (p=1,2). \tag{30}$$

Our strategy is to deduce a recursive inequality for $e_j^{(p)}$.

To prove the theorem we use the following lemmas.

**Lemma 4.3.** *For any positive $\varepsilon$, there exist $k = k(\varepsilon) > 0$ and non-negative functions $u_0^{(\varepsilon)}(x)$, $v_0^{(\varepsilon)}(x)$ such that solutions $u^{(k(\varepsilon))}$, $v^{(k(\varepsilon))}$ to (5)–(7) with the initial conditions $u^{(k(\varepsilon))}(0,\cdot) = u_0^{(\varepsilon)}$ and $v^{(k(\varepsilon))} = v_0^{(\varepsilon)}$ satisfy the following:*

$$\|w^{(k(\varepsilon))} - w\|_{C([0,T];L^p(\Omega))} \le \varepsilon, \tag{31}$$

$$\|u^{(k(\varepsilon))} - [w]^+\|_{C([0,T];L^p(\Omega))} \le \varepsilon, \tag{32}$$

$$\|v^{(k(\varepsilon))}/\alpha - [w]^-\|_{C([0,T];L^p(\Omega))} \le \varepsilon, \tag{33}$$

*where $w^{(k(\varepsilon))} = u^{(k(\varepsilon))} - v^{(k(\varepsilon))}/\alpha$ and $p = 1, 2$.*

**Lemma 4.4.** *Consider the following equations in each interval $[t_j, t_{j+1}]$:*

$$\begin{aligned} \frac{\partial \bar{\bar{u}}_M^{j,\varepsilon}}{\partial t} &= d_1 \Delta \bar{\bar{u}}_M^{j,\varepsilon} + F(\bar{\bar{u}}_M^{j,\varepsilon}), \qquad t_j < t \le t_{j+1}, \quad x \in \Omega, \\ \frac{\partial \bar{\bar{v}}_M^{j,\varepsilon}}{\partial t} &= d_2 \Delta \bar{\bar{v}}_M^{j,\varepsilon} + G(\bar{\bar{v}}_M^{j,\varepsilon}), \qquad t_j < t \le t_{j+1}, \quad x \in \Omega, \\ \frac{\partial \bar{\bar{u}}_M^{j,\varepsilon}}{\partial \nu} &= \frac{\partial \bar{\bar{v}}_M^{j,\varepsilon}}{\partial \nu} = 0, \qquad t_j < t \le t_{j+1}, \quad x \in \partial\Omega, \\ \bar{\bar{u}}_M^{j,\varepsilon}(t_j,x) &= u^{(k(\varepsilon))}(t_j,x), \\ \bar{\bar{v}}_M^{j,\varepsilon}(t_j,x) &= v^{(k(\varepsilon))}(t_j,x). \end{aligned}$$

$$\bar{\bar{w}}_M^{j,\varepsilon} = \bar{\bar{u}}_M^{j,\varepsilon} - \bar{\bar{v}}_M^{j,\varepsilon}/\alpha. \tag{34}$$

*If $d_1 = d_2$, the following inequality holds:*

$$\|w_M(t_{j+1},\cdot) - \bar{\bar{w}}_M^{j,\varepsilon}(t_{j+1},\cdot)\|_{L^p(\Omega)} \le (e_j^{(p)} + \varepsilon)(1 + E\tau), \quad (p = 1, 2), \tag{35}$$

*where $E$ is independent of $M$, $j$ and $\varepsilon$.*

**Lemma 4.5.** *Suppose $\bar{\bar{w}}_M^{j,\varepsilon}$ is given by (34). If $d_1 = d_2$, $\bar{\bar{w}}_M^{j,\varepsilon}$ satisfies*

$$\|\bar{\bar{w}}_M^{j,\varepsilon}(t_{j+1},\cdot) - w^{(k(\varepsilon))}(t_{j+1},\cdot)\|_{L^2(\Omega)} \le C_1\tau^{3/2}, \tag{36}$$

$$\|\bar{\bar{w}}_M^{j,\varepsilon}(t_{j+1},\cdot) - w^{(k(\varepsilon))}(t_{j+1},\cdot)\|_{L^1(\Omega)} \le C_2\tau^2, \tag{37}$$

*where $C_1$ and $C_2$ are independent of $M$, $j$ and $\varepsilon$.*

Now we are in a position to prove Theorem B.

*Proof of Theorem B.* From (31), (35) and (36) we observe

$$\begin{aligned} e_{j+1}^{(2)} &= \|w_M(t_{j+1},\cdot) - w(t_{j+1},\cdot)\|_{L^2(\Omega)} \\ &\le \|w_M(t_{j+1},\cdot) - \bar{\bar{w}}_M^{j,\varepsilon}(t_{j+1},\cdot)\|_{L^2(\Omega)} \\ &\qquad + \|\bar{\bar{w}}_M^{j,\varepsilon}(t_{j+1},\cdot) - w^{(k(\varepsilon))}\|_{L^2(\Omega)} \\ &\qquad\qquad + \|w^{(k(\varepsilon))}(t_{j+1},\cdot) - w(t_{j+1},\cdot)\|_{L^2(\Omega)} \\ &\le (e_j^{(2)} + \varepsilon)(1 + E\tau) + C_1\tau^{3/2} + \varepsilon. \end{aligned}$$

Since $\varepsilon$ is arbitrary, we have

$$e_{j+1}^{(2)} \le (1 + E\tau)e_j^{(2)} + C_1\tau^{3/2}.$$

Consequently we are led to $e_M^{(2)} \le C(1/M)^{1/2}$. In a similar way we arrive at $e_M^{(1)} \le C'(1/M)$, thereby completing the proof. □

# 5 Concluding Remarks

In [4] we have applied TCD to three-component competition-diffusion systems.

Some numerical experiments show that TCD converges in practical computations for the singular limit of the two-component system. For the completeness, further experiments are under way.

It is interesting to investigate whether the idea of TCD is applicable or not to other problems that are characterized as singular limits of reaction-diffusion systems. In fact, numerical experiments suggest that it is applicable to certain problems.

To integrate (1)–(3) numerically, we can also use the Finite Volume Method (FVM). Eymard et al. have given a convergence proof of FVM for the following:

$$u_t(t,x) - \Delta\Phi(u(t,x)) = v(t,x), \quad x \in \Omega, \ t > 0,$$
$$\frac{\partial \Phi(u)}{\partial \nu} = 0, \quad x \in \partial\Omega, \ t > 0,$$
$$u(0,x) = u_0(x), \quad x \in \Omega,$$

where $\Phi \in C(\mathbb{R})$ is a non decreasing locally Lipschitz continuous function and $u_0 \in L^\infty(\Omega)$, $v \in L^\infty(\Omega \times (0,T))$. See [3] for the detail.

To prove Lemma 4.5, we resort to the Duhamel formula. We write $u^{(k(\varepsilon))}$ and $v^{(k(\varepsilon))}$ with the formula. Then the terms related to $k(\varepsilon)$ disappear from the expression of $w^{(k(\varepsilon))}$ when $d_1 = d_2$.

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