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SYMMETRIC VARIETIES

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1. INTRODUCTION

The purpose of this article is to summarize some of the results the author has obtained concerning symmetric pairs.

Let G be an reductive algebraic group over a field k, and let σ be an involutive automorphism of G defined over k. Such a pair (G, σ) is said to be a symmetric pair. A classical example of a symmetric pair arises from Riemannian symmetric spaces: the group G is a real reductive group, the involution σ is a Cartan involution of G. Given a symmetric pair (G, σ) we have the associated homogeneous variety G/G^{σ} which we call the (associated) symmetric variety.

We shall first show that the theory of restricted root systems carry over to the case of symmetric pairs over arbitrary fields. This generalization has been carried out by Th.Vust, R.W.Richardson and Helminck-Wang for fields of characteristic not equal to two. We shall show that the theory carries over to fields of characteristic two by utilizing an argument based on buildings; the proof is new, and simpler.

It should be noted that centralizers of involutions in Chevalley groups over finite fields have been studied extensively by finite group theorists, including the $p = 2$ case.

As an application, we give a model of symmetric varieties over Spec($\mathbb{Z}$); this model allows us to produce a flat deformation of spherical unipotent varieties, where the fibre at $p = 2$ is a symmetric variety.

We then give decomposition theorems for symmetric varieties over p-adic fields. Let F be a p-adic field. These theorems describe orbits of parahoric subgroups of $G(F)$ acting on $G/G^{\sigma}(F)$. A novelty here is the definition of a "moment map" which greatly simplifies our statement of the theorem.

Details of the results summarized here will appear in a paper in preperation.

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2. STRUCTURE THEORY FOR INVOLUTIONS OVER ARBITRARY FIELDS

Let G be a reductive group defined over a field k, and let σ be an involutivie automorphism of G defined over k.

In the case of automorphisms of arbitrary order, it is customary to first appeal to Steinberg's theorem to conclude the existence of a σ-stable Borel subgroup, and then to use the fact that σ is a semisimple automorphism to show that there exists a σ-stable maximal torus.

This argument fails for involutions, since involutions are unipotent over fields of characteristic two. Indeed, we shall see that it is impossible, in general, to find a σ-stable maximal torus inside a stable Borel subgroup.

Let us consider an example. Let $G = GL(2)$ and let σ be an involution. Maximal tori of G are in one-to-one correspondence with unordered pairs of points on $\mathbb{P}^1(k)$. Thus a maximal torus is σ-stable if and only if the corresponding point pair is σ-stable. If σ acts as the identity map on $\mathbb{P}^1(k)$, then σ is also the identity map. Let $p \in \mathbb{P}^1(k)$ such that $\sigma(p) \neq p$. The point pair $(p, \sigma(p))$ produces a σ-stable maximal torus T. Since σ acts by interchanging two fixed points of T, σ acts by inversion on T; $\sigma(a) = a^{-1}$ for $a \in T$.

Let us see for this example the non-existence of σ-stable maximal tori inside a stable Borel subgroup. For $\sigma(X) = {}^tX^{-1}$, there exists a unique Borel subgroup B fixed by σ: thus σ has a unique fixed point on $\mathbb{P}^1(k)$. Any maximal torus T contained in B corresponds to a point pair (p, q) where p is the point corresponding to B. Since p is the unique fixed point of σ, we see that $\sigma(q) \neq q$; thus $\sigma(T) \neq T$.

We have thus seen that for a characteristic free theory of involutions, it is beneficial to look for tori on which σ acts by inversion. Thus the following definition.

Definition 2.1. A torus A is said to be σ-split if σ acts by inversion on A.

By a modification of the argument sketched above, one obtains the following theorem, which is due to Th. Vust for char$(k) \neq 2$.

Proposition 2.2. *Assume that the involution σ is not the identity map. Let k be an algebraically closed field. Then*

(1) *There exists a non-trivial σ-split torus.*
(2) *Let A be a maximal σ-split torus of G. Then A is the unique maximal σ-split torus of $Z_G(A)$.*
(3) *The commutator of $Z_G(A)$ is contained in G^σ: $[Z_G(A), Z_G(A)] \subset G^\sigma$.*
(4) $Z_G(A) = (Z_G(A) \cap H)^0 A$.
(5) *Let T be a torus of G such that $A \subset T$. Then T is σ-stable.*
(6) *Let P be a σ-split minimal k-parabolic subgroup. Then there is a unique maximal σ-split torus A in $L = P \cap P^\sigma$.*
(7) *One has $L - Z_G(A)$.*

We can then define a restricted root system for the symmetric pair (G, σ). Let A be a maximal σ-split maximal torus. Let $T \supset A$ be a maximal torus; it is automatically σ-stable, by virtue of 2.2.

The inclusion $A \to T$ induces the projection map $\pi : X^*(T) \to X^*(A)$. Let $R(G,T)$ be the root system of G with respect to T, and let $R = \pi(R(G,T)) - \{0\}$. The argument of Richardson [?] carries over to show that the pair $(R, X^*(A) \otimes \mathbb{R})$ is a root system, possibly non-reduced.

Proposition 2.3. (1) *The pair* $(R, X^*(A) \otimes \mathbb{R})$ *is a root system.*

(2) *Let* $W = N_G(A)/Z_G(A)$. *Then the image of* W *in* $\mathrm{End}(X^*(A)\otimes \mathbb{R})$ *coincides with the group generated by reflections with respect to elements of* R.

3. A model over $\mathbb{Z}$

The purpose of this section is to give a model for a symmetric variety defined over the ring of integers. In the group case, where $G \times G$ acts on G, a model has been constructed by Chevalley. The resulting model has a maximal torus split over the ring of integers. For the model we construct here, we assume that the symmetric pair is as "split" as possible. Namely, we assume that G is split over k, and that there exists a maximal σ-split torus split over k.

We first lift the action of σ to the Chevalley group $\mathcal{G}$ over $\mathbb{Z}$. Let us start with a connected reductive group G defined over a field k of char $\neq 2$, equipped with an involution σ defined over k. Let us assume that G is maximally split; there exists a maximal σ-split torus A split over k and a maximal torus T containing A split over k.

Let $R = R(G,T)$ denote the root system of G with respect to the maximal torus T. Fix a system of positive roots R^+ so that the following holds: for $\alpha \in R^+$ if $\sigma(\alpha) \neq \pm 1$, then $\sigma(\alpha) \in (-R^+)$. Let D be the set of simple roots corresponding to the choice of R^+. By the theory of Chevalley groups, there exists a reductive group scheme $\mathcal{G}$ defined over $\mathbb{Z}$ such that $\mathcal{G} \otimes k \cong G$. Let $\mathfrak{g}$ be the Lie algebra of $\mathcal{G}$. Let X_α, H_α be a Chevalley system of $\mathfrak{g}$, corresponding to the choice R^+ of positive roots. By virtue of ??, the root system $R(G,T)$ contains only real, complex or compact imaginary roots with respect to the σ-action. Define the action of σ on simple root vectors by

(1) If $\alpha \in D$ is a complex root, then $\sigma(X_\alpha) = -X_{\sigma(\alpha)}$ and $\sigma(H_\alpha) = H_{\sigma(\alpha)}$.
(2) If $\alpha \in D$ is a real root, then $\sigma(X_\alpha) = X_{-\alpha}$ and $\sigma(H_\alpha) = -H_\alpha$.
(3) If α is a compact imaginary root, then $\sigma(X_\alpha) = X_\alpha$ and $\sigma(H_\alpha) = H_\alpha$.

The following proposition holds.

Proposition 3.1. *Under the assumptions of this section:*

(1) *The morphism* σ *extends to an involutive automorphism of* $\mathfrak{g}$ *which extends to an involutive automorphism of* $\mathcal{G}$ *defined over* $\mathbb{Z}$.

(2) *Let k be the original field of definition. Then the pair $(\mathcal{G}\otimes k, \sigma\otimes k)$ is isomorphic to (G, σ).*

The pair $(\mathcal{G}\otimes\mathbb{Q}, \sigma\otimes\mathbb{Q})$ gives a symmetric pair defined over $\mathbb{Q}$. This symmetric pair is maximally split over $\mathbb{Q}$ in the sense that there exists a $\mathbb{Q}$-split maximal torus T of $\mathcal{G}$, and a maximal σ-split torus $A \subset T$ split over $\mathbb{Q}$. Let $G = \mathcal{G}\otimes\mathbb{Q}$. Then the homogeneous space G/G^σ is an affine variety defined over $\mathbb{Q}$. The Lang map $\Lambda(g) = \sigma(g)g^{-1}$ gives an isomorphism of G/G^σ with a subvariety S of G. The scheme theoretic closure of S in $\mathcal{G}$ gives an affine scheme $\mathcal{S}'$ over $\mathbb{Z}$ with twisted $\mathcal{G}$-action. However, the fiber of $\mathcal{S}'$ over 2 is not a homogeneous space of $\mathcal{G}\otimes\overline{\mathbb{F}_2}$ in general. This happens because in some cases, the fixed point set of σ at the prime 2 is not reductive. In order to remedy the situation, it is necessary to replace $\mathcal{S}'$ by its dilation with respect to the closed orbit in the special fibre over 2.

Theorem 3.1. *Let (G, σ) be a symmetric pair defined over a field k of characteristic not equal to 2. Let $(\mathcal{G}, \sigma)$ be its lift to $\mathbb{Z}$ defined above. Then there exists a closed smooth affine scheme $\mathcal{S}$ flat over* $\mathrm{Spec}(\mathbb{Z})$, *equipped with an action of $\mathcal{G}$ such that*

(1) *For a field k of characteristic not equal to 2, $\mathcal{S}\otimes k$ is isomorphic to $\mathcal{G}\otimes k/(\mathcal{G}\otimes k)^\sigma$.*

(2) *For a field k of characteristic 2, $\mathcal{S}\otimes k$ is a homogeneous space with reductive isotropy subgroup.*

It should be noted that $\mathcal{S}(k) \neq \mathcal{G}(k)/\mathcal{G}(k)^\sigma$ in general.

It is instructive to look at the case of non-singular conics to understand the construction. Let $G = GL(3)$, and let $\sigma(X) = {}^tX^{-1}$. Then the Lang map $L : G \to G$ defined by $L(X) = X\sigma(X)^{-1}$ allows us to identify G/G^σ with the space of non-degenerate 3×3 symmetric matrices over fields of characteristic not equal to two. For geometric reasons, one is interested in the space of non-singular conics. The correspondence between non-singular symmetric matrices and non-singular conics break down over fields of characteristic two. The correspondence between (a_{ij}) and $q = \sum b_{ij}x_ix_j$ has denominators since $a_{ij} = \frac{b_{ij}+b_{jk}}{2}$. The remedy here is to consider the rational map $a_{ij} \to a_{ij}$ if $i \neq j$ and $a_{ii} \to 2a_{ii}$ for the diagonal entries. The proper transform of non-singular matrices gives the desired model.

Let us give a brief description of the construction. In order to construct an affine scheme over the ring of integers, it is sufficient to give the coordinate ring as a $\mathbb{Z}$-algebra. Consider the G-action on X. There is the multiplication map $G\times X \to X$. Let us see how one can embed the coordinate ring of X into the dual of the universal envelopping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. Let $P \in U(\mathfrak{g})$ and let $f \in \mathbb{Q}[X]$. Then $Pf \in \mathbb{Q}[X]$, and $Pf(e) \in \mathbb{Q}$. Hence given f, we can associate a linear form $P \to Pf(e)$. The problem now is to find a subring A of $U(\mathfrak{g})^* = \mathrm{Hom}(U(frg), \mathbb{Q})$ which is finitely generated over $\mathbb{Z}$ satisfying the following conditions. Let μ denote the comultiplication of the Hopf algebra $U(\mathfrak{g})^*$. Then $\mu(A) \subset A \otimes B$, where B denotes the $\mathbb{Z}$-form of G. Chasing through duality, this becomes equivalent to the fact that A is stable under the action of $\mathfrak{g}_{\mathbb{Z}}$, which acts by differentiation.

$\mathbb{Q} \otimes A$ is the ring of elements of $B \otimes \mathbb{Q}$ killed by $(\mathfrak{g} \otimes \mathbb{Q})^\sigma$. The sublattice A is given by a divided-power construction for elements of $\mathfrak{g}$ corresponding to real roots.

4. The canonical compactification

We make a brief interlude to explain the notion of a canonical compactification of semisimple symmetric varieties of adjoint type.

Let k be an algebraically closed field, and let G be a semisimple group scheme of adjoint type defined over k equipped with an involutive automorphism σ, also defined over k.

Theorem 4.1. *There exists a smooth $\mathcal{G}$-scheme X over* $\mathrm{Spec}(\mathbb{Z})$ *such that:*

1. *X is a smooth scheme over* $\mathrm{Spec}(\mathbb{Z})$ *with a $\mathcal{G}$-action.*
2. *There is an open subscheme X_0 of X such that the complement of X_0 in X is a divisor with only normal crossings. The number of irreducible components of the divisor is equal to ℓ, the rank of the symmetric pair $(\mathcal{G}, \sigma)$.*
3. *Let F be an algebraically closed field. Then $X \otimes F$ is an equivariant smooth compactification of $\mathcal{G}/\mathcal{G}^\sigma \otimes F$. If* $\mathrm{char}(F) = 0$, *then it is isomorphic to the wonderful compactification of DeConcini-Procesi*[1].
4. *There is an action of Γ on $X \otimes k_s$ such that the fixed point scheme $(X \otimes k_s)^\Gamma$ is a smooth compactification of the homogeneous space G/G^σ.*

Remark 4.2. *The space of non-singular conics in the projective plane $\mathbb{P}^2$ is a homogeneous space under the action of* $\mathrm{PGL}(3)$. *The stabilizer of the form $xy + z^2$ is $O(3)$; it is the fixed point of an involution of* $\mathrm{PGL}(3)$. *The compactification X above has the following interpretation in classical terms. Let C denote a nonsingular conic in $\mathbb{P}^2$, and let $\check{C}$ denote the dual curve of C. The dual curve lies in $\check{\mathbb{P}}^2$, the dual projective space. This is the space of complete conics, first considered by Chasles.*

For an algebraically closed field F of characteristic not equal to 2, *$X \otimes F$ is the closure of the correspondence $(C, \check{(C)})$ in $\mathbb{P}^2 \times \check{\mathbb{P}}^2$.*

For an algebraically closed field F of characteristic 2, *the dual curve $\check{(C)}$ is a pencil of lines. The center of the pencil is called the strange point of C, denoted by* $\mathrm{st}(C)$. *The compactification $X \otimes F$ is the closure of the correspondence* $(C, \mathrm{st}(C))$. *This coincides with the compactification defined by Vainsencher*[2].

5. Decomposition theorems

The purpose of this section is to give decomposition theorems for symmetric varieties over p-adic fields. Let us give a heuristic argument to show what time of results are expected. Let X be a variety defined over $\mathbb{Q}_p$. Let $G(\mathbb{Z}_p)$ be a good maximal compact subgroup of G. The space $X(\mathbb{Q}_p)$ can be thought of as the space of loops in X. Equivalence under $G(\mathbb{Z}_p)$ simply deforms the loop within its homotopy class. Therefore, it is natural to look for geodesics within each homotopy class as candidates for representatives. In the case of compact groups, given a biinvariant Riemannian metric, the geodesics are one parameter subgroups.

There are two ways to formulate "homotopy" in the context of p-adic groups. One is to define an analogue of length of paths for p-adic points. This we call the moment map method. Another is to define numerical invariants for p-adic points which are invariant under homotopy; the invariants are defined as intersection numbers with boundary divisors.

The following construction is crucial for the second approach.

Let us see how one can understand elementary divisors through the theory of compactifications.

Let G be a group of adjoint type. Let $\overline{G}$ be the canonical compactification of G. Let $\mathrm{Spec}\mathbb{Q}_p \to G$ be a $\mathbb{Q}_p$-rational point of G. Then by the valuative criterion of completeness, we see that there is a $\mathbb{Z}_p$-valued rational point of $\overline{G}$ such that the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{Q}_p) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{Z}_p) & \longrightarrow & \overline{X} \end{array}$$

Let D be a Cartier divisor on $\overline{X}$. Let us recall the definition of the intersection number of $\mathrm{Spec}(\mathbb{Z}_p) \to \overline{X}$ with D.

Definition 5.1. Let (f, V) be a local coordinate of D such that the closed point P of $\mathrm{Spec}(\mathbb{Z}_p)$ is contained in V. Define $w(P, D)$ to be the order of vanishing of f at P.

Let $\overline{X}$ be the canonical compactification of $X = G/G^\sigma$.

We define the following map.

$\mu : X(\mathbb{Q}_p) \to X_*(A)$

Let α_i be a set of simple roots. Then each α_i corresponds to a divisor D_i of the canonical compactification $\overline{X}$. For a point p of $X(\mathbb{Q}_p)$, let $< \alpha_i, p >= w(p, D_i)$. Then this pairing allows us to associate a one parameter subgroup $\mu(p)$ to p.

As a formal consequences of this definition, we see that the map μ is $G(\mathbb{Z}_p)$-invariant.

We are now in position to give the decomposition theorem. Let $\lambda \in X_*(A)$.

The following definition is standard.

Proposition 5.2. *Let λ be a one parameter subgroup (1PS for short) of G. Let $P(\lambda) = \{s \mid \lim_{t\to 0} \lambda(t)s\lambda(t)^{-1}$ exists in $G\}$ and let $L(\lambda)$ be the centralizer of λ in G. Then $P(\lambda)$ is a parabolic subgroup of G, and $L(\lambda)$ is its Levi subgroup.*

Then we can state the following theorem.

Theorem 5.3. *Let F be a local field of residual characteristic not equal to 2. Let $\mu : X(F) \to X_*(A)$ be the map defined above and let O_F denote the ring of integers of F. Then μ is $G(O_F)$-invariant, and the fiber has the following structure. Let k_F denote the residue class field. Then $\mu^{-1}(\lambda) = \{L(\lambda)(k_F)\backslash(L(\lambda)/L(\lambda)^\sigma)(k_F)\}$.*

We remark that the case where the residual characteristic is equal to 2 can also be treated in a similar way; in this case, we need to allow for ramification, and the fiber will be of the form

$$L(\lambda)(O_F/\pi^e))\backslash(L(\lambda)/L(\lambda)^\sigma)(O_F/\pi^e)$$

for a suitable integer e (the ramification index).

References

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