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The nonstationary Stokes and Navier-Stokes flows through an aperture

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1 Introduction

We study the global existence and asymptotic behavior of a strong solution to the Navier-Stokes initial value problem in an aperture domain $\Omega \subset \boldsymbol{R}^n$ with smooth boundary $\partial\Omega$:

$$\begin{cases} \partial_t u + u\cdot\nabla u &= \Delta u - \nabla p \qquad (x\in\Omega,\ t>0), \\ \nabla\cdot u &= 0 \qquad (x\in\Omega,\ t\geq 0), \\ u|_{\partial\Omega} &= 0 \qquad (t>0), \\ u|_{t=0} &= a \qquad (x\in\Omega), \end{cases} \tag{1.1}$$

where $u(x,t)$ and $p(x,t)$ denote the unknown velocity and pressure of a fluid, respectively, while $a(x)$ is a prescribed initial velocity. The aperture domain Ω is a compact perturbation of two separated half spaces $H_+\cup H_-$, where $H_\pm = \{x\in \boldsymbol{R}^n; \pm x_n > 1\}$; to be precise, we call a connected open set $\Omega\subset \boldsymbol{R}^n$ an aperture domain if there is a ball $B\subset\boldsymbol{R}^n$ such that $\Omega\setminus B = (H_+\cup H_-)\setminus B$. Thus the upper and lower half spaces $H_\pm$ are connected by an aperture (hole) $M\subset\Omega\cap B$, which is a smooth $(n-1)$-dimensional manifold so that Ω consists of upper and lower disjoint subdomains $\Omega_\pm$ and M: $\Omega=\Omega_+\cup M\cup\Omega_-$.

The aperture domain is a particularly interesting class of domains with noncompact boundaries because of the following remarkable feature, which was in 1976 pointed out by Heywood [11]: the solution is not uniquely determined by usual boundary conditions even for the stationary Stokes system in this domain and therefore, in order to single out a unique solution, we have to prescribe either the flux through the aperture M

$$\phi(u) = \int_M N\cdot u\,d\sigma,$$

or the pressure drop at infinity (in a sense) between the upper and lower subdomains $\Omega_\pm$

$$[p] = \lim_{|x|\to\infty, x\in\Omega_+} p(x) - \lim_{|x|\to\infty, x\in\Omega_-} p(x),$$

as an additional boundary condition. Here, N denotes the unit normal vector on M directed to Ω_- and the flux $\phi(u)$ is independent of the choice of M since $\nabla\cdot u = 0$ in Ω.

The results of Farwig and Sohr [6] are the first step to discuss the nonstationary problem (1.1) in the L^q space. They, as well as Miyakawa [22], showed the Helmholtz decomposition of the L^q space of vector fields $L^q(\Omega) = L^q_\sigma(\Omega)\oplus L^q_\pi(\Omega)$ for $n\geq 2$ and $1<q<\infty$, where $L^q_\sigma(\Omega)$ is the completion in $L^q(\Omega)$ of the class of all smooth, solenoidal and compactly supported vector fields, and $L^q_\pi(\Omega) = \{\nabla p\in L^q(\Omega); p\in L^q_{loc}(\overline{\Omega})\}$. The space $L^q_\sigma(\Omega)$ is characterized as

$$L^q_\sigma(\Omega) = \{u\in L^q(\Omega); \nabla\cdot u = 0, \nu\cdot u|_{\partial\Omega} = 0, \phi(u) = 0\}, \tag{1.2}$$

where ν is the unit outer normal vector on $\partial\Omega$. Here, the condition $\phi(u) = 0$ follows from the other ones and may be omitted if $q \le n/(n-1)$, but otherwise, the element of $L^q_\sigma(\Omega)$ must possess this additional property. Using the projection P_q from $L^q(\Omega)$ onto $L^q_\sigma(\Omega)$ associated with the Helmholtz decomposition, we can define the Stokes operator $A = A_q = -P_q\Delta$, which generates a bounded analytic semigroup e^{-tA} in each $L^q_\sigma(\Omega), 1 < q < \infty$, for $n \ge 2$ ([6, Theorem 2.5]).

We are interested in strong solutions to (1.1). However, there are no results on the global existence of such solutions in the L^q framework unless $q = 2$, while a few local existence theorems are known. In the 3-dimensional case, Heywood [11], [12] first constructed a local solution to (1.1) with a prescribed either $\phi(u(t))$ or $[p(t)]$ when $a \in H^2(\Omega)$ fulfills some compatibility conditions. Franzke [7] has recently developed the L^q theory of local solutions via the approach of [10] with use of fractional powers of the Stokes operator. When a suitable $\phi(u(t))$ is prescribed, his assumption on initial data is for instance that $a \in L^q(\Omega), q > n$, together with some compatibility conditions.

It is possible to discuss the L^2 theory of global strong solutions for an arbitrary unbounded domain (with smooth boundary) in a unified way since the Stokes operator is a nonnegative selfadjoint one in L^2_σ; see Heywood [13] ($n = 3$), Kozono and Ogawa [18] ($n = 3$) and Kozono and Sohr [19] ($n = 4, 5$). Especially, from the viewpoint of the class of initial data, optimal results were given by [18] and [19]. In fact, they constructed a global solution with various decay properties for small $a \in D(A_2^{n/4-1/2})$. For the aperture domain Ω their solutions $u(t)$ should satisfy the hidden flux condition $\phi(u(t)) = 0$ on account of $u(t) \in L^2_\sigma(\Omega)$ together with (1.2).

Our purpose is to provide the global existence theorem for a unique strong solution $u(t)$ of (1.1), which satisfies the flux condition $\phi(u(t)) = 0$ and some sharp decay properties as $t \to \infty$, when the initial velocity a is small enough in $L^n_\sigma(\Omega), n \ge 3$. Up to now we have the same global existence result for the whole space (Kato [16]), the half space (Ukai [25]), bounded domains (Giga and Miyakawa [10]) and exterior domains (Iwashita [15]). For the proof, as is well known, it is crucial to establish the L^q-L^r estimates of the Stokes semigroup

$$\|e^{-tA}f\|_{L^r(\Omega)} \le Ct^{-\alpha}\|f\|_{L^q(\Omega)}, \tag{1.3}$$

$$\|\nabla e^{-tA}f\|_{L^r(\Omega)} \le Ct^{-\alpha-1/2}\|f\|_{L^q(\Omega)}, \tag{1.4}$$

for all $t > 0$ and $f \in L^q_\sigma(\Omega)$, where $\alpha = (n/q - n/r)/2 \ge 0$. Recently for $n \ge 3$ Abels [1] has proved some partial results: (1.3) for $1 < q \le r < \infty$ and (1.4) for $1 < q \le r < n$. However, because of the lack of (1.4) for the most important case $q = r = n$, his results are not satisfactory for the construction of the global strong solution possessing various time-asymptotic behaviors as long as one follows the straightforward method of Kato [16]. In this article we consider the case $n \ge 3$ and prove (1.3) for $1 \le q \le r \le \infty$ ($q \ne \infty, r \ne 1$) and (1.4) for $1 \le q \le r \le n$ ($r \ne 1$) or $1 \le q < n < r < \infty$; here, when $q = 1$, f should be taken from $L^1(\Omega) \cap L^s_\sigma(\Omega)$ for some $s \in (1, \infty)$. The result on (1.4) is better than that for exterior Stokes flows [15]; in fact, Maremonti and Solonnikov [20] clarified that one cannot remove the restriction $r \le n$ for the exterior problem.

In the proof of the L^q-L^r estimates, it seems to be heuristically reasonable to combine some local decay properties near the aperture with the L^q-L^r estimates of the Stokes semigroup for the half space by means of a localization procedure. Indeed, Abels [1] used this idea that had been well developed by Iwashita [15] and, later, Kobayashi and Shibata [17] in the case of exterior domains. We should however note that the boundary $\partial\Omega$ is noncompact; thus, a difficulty is to deduce the sharp local energy decay estimate

$$\|e^{-tA}f\|_{W^{1,q}(\Omega_R)} \le Ct^{-n/2q}\|f\|_{L^q(\Omega)}, \quad t \ge 1, \tag{1.5}$$

for $f \in L^q_\sigma(\Omega), 1 < q < \infty$, where $\Omega_R = \{x \in \Omega; |x| < R\}$, but this is the essential part of our proof. Estimate (1.5) improves the local energy decay given by Abels [1], in which a little slower rate $t^{-n/2q+\varepsilon}$ was shown. In [1], similarly to Iwashita [15], a resolvent expansion around the origin $\lambda = 0$ was derived in some weighted function spaces. To this end, Abels made use of the Ukai formula of the Stokes semigroup for the half space ([25]) and, in order to estimate the Riesz operator appearing in this formula, he had to introduce Muckenhoupt weights, which caused some restrictions. On the other hand, Kobayashi and Shibata [17] refined the proof of Iwashita in some sense and obtained the L^q-L^r estimates of the Oseen semigroup for the 3-dimensional exterior domain. In this article we employ in principle the strategy developed by [17] and extend the method to general $n \geq 3$ to prove (1.3) and (1.4) for the Stokes flow (we omit the proof of the global existence and decay properties of the Navier-Stokes flow [14]).

After stating our main theorems in the next section, section 3 is devoted to the investigation of the Stokes resolvent for the half space $H = H_+$ or H_-. We derive some regularity estimates near the origin $\lambda = 0$ of $(\lambda + A_H)^{-1}P_H f$ when $f \in L^q(H)$ has a bounded support, where $A_H = -P_H\Delta$ is the Stokes operator for the half space H. Although the obtained estimates do not seem to be optimal compared with those shown by [17] for the whole space, the results are sufficient for our aim and the proof is rather elementary; in fact, we represent the resolvent $(\lambda + A_H)^{-1}$ in terms of the semigroup e^{-tA_H} and, with the aid of local energy decay properties of this semigroup, we have only to perform several integrations by parts and to estimate the resulting formulae.

In section 4, based on the results for the half space, we proceed to the analysis of the Stokes resolvent for the aperture domain Ω. To do so, in an analogous way to [15], [17] and [1], we first construct the resolvent $(\lambda + A)^{-1}Pf$ near the origin $\lambda = 0$ for $f \in L^q(\Omega)$ with bounded support by use of the operator $(\lambda + A_H)^{-1}P_H$, the Stokes flow in a bounded domain and a cut-off function together with the result of Bogovskiĭ [2] on the boundary value problem for the equation of continuity. And then, for the same f as above, we deduce essentially the same regularity estimates near the origin $\lambda = 0$ of $(\lambda + A)^{-1}Pf$ as shown in section 3.

In the final section we prove (1.5) and thereby (1.4) for $q = r \in (1, n]$ as well as (1.3) for $r = \infty$, from which the other cases follow. Some of the estimates obtained in section 4 enable us to justify a representation formula of the semigroup $e^{-tA}Pf$ in $W^{1,q}(\Omega_R)$ in terms of the Fourier inverse transform of $\partial_s^m(is + A)^{-1}Pf$ when $f \in L^q(\Omega)$ has a bounded support, where $n = 2m + 1$ or $n = 2m + 2$. We then appeal to the lemma due to Shibata [23], which tells us a relation between the regularity of a function at the origin and the decay property of its Fourier inverse image, so that we obtain another local energy decay estimate

$$\|e^{-tA}Pf\|_{W^{1,q}(\Omega_R)} \leq Ct^{-n/2+\varepsilon}\|f\|_{L^q(\Omega)}, \quad t \geq 1, \tag{1.6}$$

for $f \in L^q(\Omega), 1 < q < \infty$, with bounded support, where $\varepsilon > 0$ is arbitrary. Estimate (1.6) was shown in [1] only for solenoidal data $f \in L^q_\sigma(\Omega)$ with bounded support, from which (1.5) with the rate replaced by $t^{-n/2q+\varepsilon}$ follows through an interpolation argument. But it is crucial for the proof of (1.5) to use (1.6) even for data which are not solenoidal. In order to deduce (1.5) from (1.6), we develop the method in [15] and [17] based on a localization argument. In fact, we regard the Stokes flow for the aperture domain Ω as the sum of the Stokes flows for the half spaces $H_\pm$ and a certain perturbed flow. Since the Stokes flow for the half space enjoys the L^q-L^∞ decay estimate with the rate $t^{-n/2q}$ ([3]), our main task is to show (1.5) for the perturbation part. In contrast to the case of exterior domains, the support of the derivative of the cut-off function touches the boundary $\partial\Omega$ and thus we have to carry out a localization procedure carefully. Furthermore, the remainder term arising from such a procedure involves the pressure of the nonstationary Stokes system in the half space and, therefore, does not belong to any solenoidal function space. Hence, in order to treat this term, (1.6) is necessary for non-solenoidal data, while that is not the case for the exterior problem.

2 Results

We denote upper and lower half spaces by $H_\pm = \{x \in \boldsymbol{R}^n; \pm x_n > 1\}$, and sometimes write $H = H_+$ or H_- to state some assertions for the half space. Set $B_R = \{x \in \boldsymbol{R}^n; |x| < R\}$ for $R > 0$. Let $\Omega \subset \boldsymbol{R}^n$ be a given aperture domain with smooth boundary $\partial\Omega$, namely, there is $R_0 > 1$ so that $\Omega \setminus B_{R_0} = (H_+ \cup H_-) \setminus B_{R_0}$; in what follows we fix such R_0. Since Ω should be connected, there are some apertures and one can take two disjoint subdomains $\Omega_\pm$ and a smooth $(n-1)$-dimensional manifold M such that $\Omega = \Omega_+ \cup M \cup \Omega_-, \Omega_\pm \setminus B_{R_0} = H_\pm \setminus B_{R_0}$ and $M \cup \partial M = \partial\Omega_+ \cap \partial\Omega_- \subset \overline{B_{R_0}}$. We set $\Omega_R = \Omega \cap B_R$ and $H_R = H \cap B_R$, which is one of $H_{\pm,R} = H_\pm \cap B_R$, for $R > 1$.

For a domain $G \subset \boldsymbol{R}^n$, integer $j \geq 0$ and $1 \leq q \leq \infty$, we denote by $W^{j,q}(G)$ the standard L^q-Sobolev space with norm $\|\cdot\|_{j,q,G}$ so that $L^q(G) = W^{0,q}(G)$ with norm $\|\cdot\|_{q,G}$. The space $W_0^{j,q}(G)$ is the completion of $C_0^\infty(G)$, the class of C^∞ functions having compact support in G, in the norm $\|\cdot\|_{j,q,G}$, and $W^{-j,q}(G)$ stands for the dual space of $W_0^{j,q/(q-1)}(G)$ with norm $\|\cdot\|_{-j,q,G}$. For simplicity, we use the abbreviations $\|\cdot\|_q$ for $\|\cdot\|_{q,\Omega}$ and $\|\cdot\|_{j,q}$ for $\|\cdot\|_{j,q,\Omega}$ when $G = \Omega$. We often use the same symbols for denoting the vector and scalar function spaces if there is no confusion. It is convenient to introduce a Banach space

$$L^q_{[R]}(G) = \{u \in L^q(G); \operatorname{supp} u \subset \overline{G_R}\}, \quad G = \Omega \text{ or } H,$$

for $R > 1$, where supp u denotes the support of the function u. For a Banach space X we denote by $B(X)$ the Banach space which consists of all bounded linear operators from X into itself.

Given $R \geq R_0$, we take (and fix) two cut-off functions $\psi_{\pm,R}$ satisfying

$$\psi_{\pm,R} \in C^\infty(\boldsymbol{R}^n; [0,1]), \quad \psi_{\pm,R}(x) = \begin{cases} 1 & \text{in } H_\pm \setminus B_{R+1}, \\ 0 & \text{in } H_\mp \cup B_R. \end{cases} \tag{2.1}$$

In some localization procedures with use of the cut-off functions above, the bounded domain of the form $D_{\pm,R} = \{x \in H_\pm; R < |x| < R+1\}$ appears, and for this we need the following result of Bogovskiĭ [2] which provides a certain solution having an optimal regularity of the boundary value problem for $\nabla \cdot u = f$ with $u = 0$ on the boundary (see also Borchers and Sohr [4] and Galdi [9]): there is a linear operator $S_{\pm,R}$ from $C_0^\infty(D_{\pm,R})$ to $C_0^\infty(D_{\pm,R})^n$ such that for $1 < q < \infty$ and integer $j \geq 0$

$$\|\nabla^{j+1} S_{\pm,R} f\|_{q,D_{\pm,R}} \leq C \|\nabla^j f\|_{q,D_{\pm,R}}, \tag{2.2}$$

with $C = C(R,q,j) > 0$ independent of $f \in C_0^\infty(D_{\pm,R})$ (where ∇^j denotes all the j-th derivatives); and $\nabla \cdot S_{\pm,R} f = f$ for all $f \in C_0^\infty(D_{\pm,R})$ with $\int_{D_{\pm,R}} f(x)dx = 0$. By (2.2) the operator $S_{\pm,R}$ extends uniquely to a bounded operator from $W_0^{j,q}(D_{\pm,R})$ to $W_0^{j+1,q}(D_{\pm,R})^n$.

For $G = \Omega, H$ and a smooth bounded domain $(n \geq 2)$, let $C_{0,\sigma}^\infty(G)$ be the set of all solenoidal (divergence free) vector fields whose components belong to $C_0^\infty(G)$, and $L^q_\sigma(G)$ the completion of $C_{0,\sigma}^\infty(G)$ in the norm $\|\cdot\|_{q,G}$. If, in particular, $G = \Omega$, then the space $L^q_\sigma(\Omega)$ is characterized as (1.2). The space $L^q(G)$ of vector fields admits the Helmholtz decomposition

$$L^q(G) = L^q_\sigma(G) \oplus L^q_\pi(G), \quad 1 < q < \infty,$$

with $L^q_\pi(G) = \{\nabla p \in L^q(G); p \in L^q_{loc}(\overline{G})\}$; see [8], [24] for bounded domains, [3], [21] for $G = H$ and [6], [22] for $G = \Omega$. Let $P_{q,G}$ be the projection operator from $L^q(G)$ onto $L^q_\sigma(G)$ associated with the decomposition above. Then the Stokes operator $A_{q,G}$ is defined by the solenoidal part of the Laplace operator, that is,

$$D(A_{q,G}) = W^{2,q}(G) \cap W_0^{1,q}(G) \cap L^q_\sigma(G), \quad A_{q,G} = -P_{q,G}\Delta,$$

for $1 < q < \infty$. The dual operator $A^*_{q,G}$ of $A_{q,G}$ coincides with $A_{q/(q-1),G}$ on $L^q_\sigma(G)^* = L^{q/(q-1)}_\sigma(G)$. We use, for simplicity, the abbreviations P_q for $P_{q,\Omega}$ and A_q for $A_{q,\Omega}$, and the subscript q is also often omitted if there is no confusion. The Stokes operator enjoys the parabolic resolvent estimate

$$\|(\lambda + A_G)^{-1}\|_{B(L^q_\sigma(G))} \leq C_\varepsilon/|\lambda|, \tag{2.3}$$

for $|\arg\lambda| \leq \pi - \varepsilon$ $(\lambda \neq 0)$, where $\varepsilon > 0$ is arbitrarily small; [21], [3] for $G = H$ and [6] for $G = \Omega$. Estimate (2.3) implies that the operator $-A_G$ generates a bounded analytic semigroup $\{e^{-tA_G}; t \geq 0\}$ of class (C_0) in each $L^q_\sigma(G), 1 < q < \infty$. We write $E(t) = e^{-tA_H}$, which is one of $E_\pm(t) = e^{-tA_{H_\pm}}$.

The first theorem provides the L^q-L^r estimates of the Stokes semigroup e^{-tA} for the aperture domain Ω.

Theorem 2.1 *Let $n \geq 3$.*

1. *Let $1 \leq q \leq r \leq \infty$ $(q \neq \infty, r \neq 1)$. There is a constant $C = C(\Omega, n, q, r) > 0$ such that (1.3) holds for all $t > 0$ and $f \in L^q_\sigma(\Omega)$ unless $q = 1$; when $q = 1$, the assertion remains true if f is taken from $L^1(\Omega) \cap L^s_\sigma(\Omega)$ for some $s \in (1, \infty)$.*
2. *Let $1 \leq q \leq r \leq n$ $(r \neq 1)$ or $1 \leq q < n < r < \infty$. There is a constant $C = C(\Omega, n, q, r) > 0$ such that (1.4) holds for all $t > 0$ and $f \in L^q_\sigma(\Omega)$ unless $q = 1$; when $q = 1$, the assertion remains true if f is taken from $L^1(\Omega) \cap L^s_\sigma(\Omega)$ for some $s \in (1, \infty)$.*

By use of the Stokes operator A, one can formulate the problem (1.1) subject to the vanishing flux condition

$$\phi(u(t)) = \int_M N \cdot u(t) d\sigma = 0, \quad t \geq 0, \tag{2.4}$$

as the Cauchy problem

$$\partial_t u + Au + P(u \cdot \nabla u) = 0, \quad t > 0; u(0) = a, \tag{2.5}$$

in $L^q_\sigma(\Omega)$. Given $a \in L^n_\sigma(\Omega)$ and $0 < T \leq \infty$, a measurable function u defined on $\Omega \times (0,T)$ is called a strong solution of (1.1) with (2.4) on $(0,T)$ if u is of class $u \in C([0,T); L^n_\sigma(\Omega)) \cap C(0,T; D(A_n)) \cap C^1(0,T; L^n_\sigma(\Omega))$ together with $\lim_{t\to 0} \|u(t) - a\|_n = 0$ and satisfies (2.5) for $0 < t < T$ in $L^n_\sigma(\Omega)$.

The next theorem tells us the global existence of a strong solution with several decay properties provided that $\|a\|_n$ is small enough.

Theorem 2.2 *Let $n \geq 3$. There is a constant $\delta = \delta(\Omega, n) > 0$ with the following property: if $a \in L^n_\sigma(\Omega)$ satisfies $\|a\|_n \leq \delta$, then the problem (1.1) with (2.4) admits a unique strong solution $u(t)$ on $(0, \infty)$, which enjoys*

$$\|u(t)\|_r = o\left(t^{-1/2+n/2r}\right) \quad \textit{for } n \leq r \leq \infty,$$

$$\|\nabla u(t)\|_n = o\left(t^{-1/2}\right), \qquad \|\partial_t u(t)\|_n + \|Au(t)\|_n = o\left(t^{-1}\right),$$

as $t \to \infty$.

The final theorem shows further decay properties of the global solution when we additionally impose L^1-summability on the initial data.

Theorem 2.3 *Let $n \geq 3$. There is a constant $\eta = \eta(\Omega, n) \in (0, \delta]$ with the following property: if $a \in L^1(\Omega) \cap L^n_\sigma(\Omega)$ satisfies $\|a\|_n \leq \eta$, then the solution $u(t)$ obtained in Theorem 2.2 and the associated pressure $p(t)$ enjoy*

$$\|u(t)\|_r = O\left(t^{-(n-n/r)/2}\right) \quad \text{for } 1 < r \leq \infty,$$

$$\|\nabla u(t)\|_r = O\left(t^{-(n-n/r)/2-1/2}\right) \quad \text{for } 1 < r < \infty,$$

$$\|\partial_t u(t)\|_r + \|Au(t)\|_r = O\left(t^{-(n-n/r)/2-1}\right) \quad \text{for } 1 < r < \infty,$$

$$\|\nabla^2 u(t)\|_r + \|\nabla p(t)\|_r = O\left(t^{-(n-n/r)/2-1}\right) \quad \text{for } 1 < r < n,$$

as $t \to \infty$. Moreover, for each $t > 0$ there exist two constants $p_\pm(t) \in \boldsymbol{R}$ such that $p(t) - p_\pm(t) \in L^r(\Omega_\pm)$ with

$$\|p(t) - p_\pm(t)\|_{r,\Omega_\pm} = O\left(t^{-(n-n/r)/2-1/2}\right) \quad \text{for } n/(n-1) < r < \infty,$$

$$|p_+(t) - p_-(t)| = O\left(t^{-n/2-1/2+\varepsilon}\right),$$

as $t \to \infty$, where $\varepsilon > 0$ is arbitrarily small.

3 The Stokes resolvent for the half space

The resolvent $v = (\lambda + A_H)^{-1} P_H f$ together with the associated pressure π solves the system $\lambda v - \Delta v + \nabla \pi = f$, $\nabla \cdot v = 0$ in the half space $H = H_+$ or H_- subject to $v|_{\partial H} = 0$ for the external force $f \in L^q(H), 1 < q < \infty$, and $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$. In this section we are concerned with the analysis of v near $\lambda = 0$. One needs the following local energy decay estimate of the semigroup $E(t) = e^{-tA_H}$, which is a simple consequence of (1.3) for $\Omega = H$ together with

$$\|\nabla^j u\|_{r,H} \leq C\|A_H^{j/2} u\|_{r,H}, \quad u \in D(A_{r,H}^{j/2}), \tag{3.1}$$

for $1 < r < \infty$ and $j = 1, 2$ (Borchers and Miyakawa [3]).

Lemma 3.1 *Let $n \geq 2, 1 < q < \infty, d > 1$ and $R > 1$. For any small $\varepsilon > 0$ and integer $k \geq 0$ there is a constant $C = C(n, q, d, R, \varepsilon, k) > 0$ such that*

$$\|\nabla^j \partial_t^k E(t) P_H f\|_{q,H_R} \leq C t^{-j/2-k}(1+t)^{-n/2+\varepsilon}\|f\|_{q,H}, \tag{3.2}$$

for $t > 0, f \in L^q_{[d]}(H)$ and $j = 0, 1, 2$.

Lemma 3.1 is sufficient for our analysis of the resolvent in this section, but the local energy decay estimate of the following form will be used in section 5.

Lemma 3.2 *Let $n \geq 2, 1 < q < \infty$ and $R > 1$. Then there is a constant $C = C(n, q, R) > 0$ such that*

$$\|E(t) f\|_{2,q,H_R} + \|\partial_t E(t) f\|_{q,H_R} \leq C(1+t)^{-n/2q}\|f\|_{D(A_{q,H})}, \tag{3.3}$$

for $t \geq 0$ and $f \in D(A_{q,H})$.

We next employ Lemma 3.1 to show some regularity estimates near $\lambda = 0$ of the Stokes resolvent in the localized space $W^{2,q}(H_R)$.

Lemma 3.3 *Let $n \geq 3, 1 < q < \infty, d > 1$ and $R > 1$. Given $f \in L^q_{[d]}(H)$, set $v(\lambda) = (\lambda + A_H)^{-1} P_H f$. For any small $\varepsilon > 0$ there is a constant $C = C(n, q, d, R, \varepsilon) > 0$ such that*

$$|\lambda|^\beta \|\partial_\lambda^m v(\lambda)\|_{2,q,H_R} + \sum_{k=0}^{m-1} \|\partial_\lambda^k v(\lambda)\|_{2,q,H_R} \leq C\|f\|_{q,H}, \tag{3.4}$$

for Re $\lambda \geq 0$ $(\lambda \neq 0)$ and $f \in L^q_{[d]}(H)$, where

$$m = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd,} \\ n/2 - 1 & \text{if } n \text{ is even,} \end{cases}$$

$$\beta = \beta(\varepsilon) = 1 + m - \frac{n}{2} + \varepsilon = \begin{cases} 1/2 + \varepsilon & \text{if } n \text{ is odd,} \\ \varepsilon & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, we have

$$\sup\left\{\frac{\|v(\lambda) - w\|_{2,q,H_R}}{\|f\|_{q,H}};\ f \neq 0, f \in L^q_{[d]}(H)\right\} \to 0, \tag{3.5}$$

as $\lambda \to 0$ with Re $\lambda \geq 0$, where $w = \int_0^\infty E(t) P_H f dt$.

Proof. We recall the formula

$$v(\lambda) = (\lambda + A_H)^{-1} P_H f = \int_0^\infty e^{-\lambda t} E(t) P_H f dt, \tag{3.6}$$

which is valid in $L^q_\sigma(H)$ for Re $\lambda > 0$ and $f \in L^q(H)$. In the other region $\{\lambda \in C \setminus (-\infty, 0]; \text{Re } \lambda \leq 0\}$ we usually utilize the analytic extension of the semigroup $\{E(t); \text{Re } t > 0\}$ to obtain the similar formula. For the case Re $\lambda = 0$ $(\lambda \neq 0)$ which is important for us, however, thanks to the local energy decay property (3.2), the formula (3.6) remains valid in the localized space $L^q(H_R)$ for $f \in L^q_{[d]}(H)$ (the function w in (3.5) is well-defined in $L^q(H_R)$ by the same reasoning). We thus obtain from (3.2)

$$\|\nabla^j \partial_\lambda^k v(\lambda)\|_{q,H_R} \leq \int_0^\infty t^k \|\nabla^j E(t) P_H f\|_{q,H_R} dt \leq C\|f\|_{q,H},$$

provided that

$$\begin{aligned} &j = 0,1 \quad \text{if } k = 0; \quad j = 0,1,2 \quad \text{if } n \geq 5,\ 1 \leq k \leq m-1; \\ &j = 2 \quad \text{if } k = m, n = 2m+1; \quad j = 1,2 \quad \text{if } k = m, n = 2m+2. \end{aligned}$$

For $\{k, j\} = \{0, 2\}$ we have only to use (3.1) together with (2.3) to see that

$$\|\nabla^2 v(\lambda)\|_{q,H_R} \leq C\|A_H(\lambda + A_H)^{-1} P_H f\|_{q,H} \leq C\|f\|_{q,H}.$$

The remaining case $k = m$ is the most important part of (3.4). Since

$$\|\partial_\lambda^m v(\lambda)\|_{2,q,H_R} \leq C m! \{|\lambda|^{-m} + |\lambda|^{-(m+1)}\} \|f\|_{q,H},$$

we have the assertion for $|\lambda| \geq 1$. For $0 < |\lambda| < 1$ and odd n (resp. even n), we have already shown the estimate as above when $j = 2$ (resp. $j = 1, 2$). Thus, let $j = 0$ or 1 for $n = 2m+1$ and $j = 0$ for $n = 2m+2$. We divide the integral of (3.6) into two parts

$$\partial_\lambda^m v(\lambda) = \left\{\int_0^{1/|\lambda|} + \int_{1/|\lambda|}^\infty\right\} e^{-\lambda t} (-t)^m E(t) P_H f dt = w_1(\lambda) + w_2(\lambda).$$

Then (3.2) implies
$$\|\nabla^j w_1(\lambda)\|_{q,H_R} \le C|\lambda|^{-\beta+j/2}\|f\|_{q,H},$$
for $f \in L^q_{[d]}(H)$. On the other hand, by integration by parts we get
$$w_2(\lambda) = \frac{e^{-\lambda/|\lambda|}}{\lambda}\left(\frac{-1}{|\lambda|}\right)^m E\left(\frac{1}{|\lambda|}\right) P_H f + \int_{1/|\lambda|}^{\infty} \frac{e^{-\lambda t}}{\lambda}\partial_t\left[(-t)^m E(t)P_H f\right]dt,$$
in $L^q(H_R)$ since (3.2) implies $\lim_{t\to\infty} t^m\|E(t)P_H f\|_{q,H_R} = 0$. With the aid of (3.2) again we see that
$$\|\nabla^j w_2(\lambda)\|_{q,H_R} \le C|\lambda|^{-\beta+j/2}\|f\|_{q,H},$$
for $f \in L^q_{[d]}(H)$. Collecting the estimates above leads us to (3.4). We next show (3.5). Since $|e^{-\lambda t} - 1| \le 2^{1-\theta}|\lambda|^\theta t^\theta$ for $\mathrm{Re}\,\lambda \ge 0$ and $\theta \in (0,1]$, we have
$$\|\nabla^j(v(\lambda) - w)\|_{q,H_R} \le 2^{1-\theta}|\lambda|^\theta \int_0^\infty t^\theta \|\nabla^j E(t)P_H f\|_{q,H_R}dt,$$
for $j = 0,1,2$. From (3.2) together with a suitable choice of θ (for instance, $\theta < 1/2$ for $n = 3$), we conclude (3.5). □

Finally, we derive further information on the regularity of the resolvent along the imaginary axis.

Lemma 3.4 *Let $n \ge 3, 1 < q < \infty, d > 1$ and $R > 1$. Set*
$$\Phi_H^{(k)}(s) = \partial_s^k(is + A_H)^{-1}P_H \quad (s \in \boldsymbol{R}\setminus\{0\},\ k = m \text{ or } m-1),$$
where $i = \sqrt{-1}$. Then, for any small $\varepsilon > 0$, there is a constant $C = C(n,q,d,R,\varepsilon) > 0$ such that
$$\|\Phi_H^{(m)}(s+h)f - \Phi_H^{(m)}(s)f\|_{2,q,H_R} \le C|h||s|^{-\beta-1}\|f\|_{q,H}, \tag{3.7}$$
$$\|\Phi_H^{(m-1)}(s+h)f - \Phi_H^{(m-1)}(s)f\|_{2,q,H_R} \le C|h||s|^{-\beta}\|f\|_{q,H}, \tag{3.8}$$
for $h \in \boldsymbol{R}, |s| > 2|h|$ and $f \in L^q_{[d]}(H)$, where m and $\beta = \beta(\varepsilon)$ are the same as in Lemma 3.3.

Proof. Estimate (3.8) is a direct consequence of (3.4). In fact, we see that
$$\|\Phi_H^{(m-1)}(s+h)f - \Phi_H^{(m-1)}(s)f\|_{2,q,H_R} \le \left|\int_s^{s+h}\|\Phi_H^{(m)}(\tau)f\|_{2,q,H_R}d\tau\right|,$$
which together with the relation $|s+h| \ge |s| - |h| \ge |s|/2$ implies (3.8). We next show (3.7). By (3.6) with $\mathrm{Re}\,\lambda = 0$ in $L^q(H_R)$ we have
$$\begin{aligned}&\Phi_H^{(m)}(s+h)f - \Phi_H^{(m)}(s)f\\ &= (-i)^m\left\{\int_0^{1/|s|} + \int_{1/|s|}^\infty\right\}e^{-ist}(e^{-iht}-1)t^m E(t)P_H f dt = (-i)^m(w_1 + w_2).\end{aligned}$$
For the convenience we introduce the function
$$F_k(t) = \partial_t^k[t^m E(t)P_H f], \quad k \ge 0.$$

We then deduce from (3.2)

$$\|F_k(t)\|_{2,q,H_R} \le Ct^{-k+m-1}(1+t)^{-n/2+1+\varepsilon}\|f\|_{q,H}, \tag{3.9}$$

for $t>0$ and $f \in L^q_{[d]}(H)$. Taking $|e^{-iht}-1| \le |h|t$ into account, we see from (3.9) that

$$\|w_1\|_{2,q,H_R} \le |h|\int_0^{1/|s|} t\|F_0(t)\|_{2,q,H_R}dt \le C|h||s|^{-\beta-1}\|f\|_{q,H},$$

for $f \in L^q_{[d]}(H)$. By integration by parts we split $w_2 = w_{21}+w_{22}+w_{23}$, where

$$w_{21} = \frac{ih}{s(s+h)}e^{-i(s+h)/|s|}F_0\left(\frac{1}{|s|}\right) - \frac{i}{s}e^{-is/|s|}(e^{-ih/|s|}-1)F_0\left(\frac{1}{|s|}\right),$$

$$w_{22} = \frac{ih}{s(s+h)}\int_{1/|s|}^\infty e^{-i(s+h)t}F_1(t)dt, \qquad w_{23} = \frac{-i}{s}\int_{1/|s|}^\infty e^{-ist}(e^{-iht}-1)F_1(t)dt.$$

Since $1/|s(s+h)| \le 2/|s|^2$ for $|s|>2|h|$, it follows from (3.9) that

$$\|w_{21}\|_{2,q,H_R} \le 3|h||s|^{-2}\|F_0(1/|s|)\|_{2,q,H_R} \le C|h||s|^{-\beta-1}\|f\|_{q,H},$$

and that

$$\|w_{22}\|_{2,q,H_R} \le 2|h||s|^{-2}\int_{1/|s|}^\infty \|F_1(t)\|_{2,q,H_R}dt \le C|h||s|^{-\beta-1}\|f\|_{q,H},$$

for $f \in L^q_{[d]}(H)$. We perform integration by parts once more to obtain $w_{23} = w_{231}+w_{232}+w_{233}$ with

$$w_{231} = \frac{h}{s^2(s+h)}e^{-i(s+h)/|s|}F_1\left(\frac{1}{|s|}\right) - \frac{1}{s^2}e^{-is/|s|}(e^{-ih/|s|}-1)F_1\left(\frac{1}{|s|}\right),$$

$$w_{232} = \frac{h}{s^2(s+h)}\int_{1/|s|}^\infty e^{-i(s+h)t}F_2(t)dt, \qquad w_{233} = \frac{-1}{s^2}\int_{1/|s|}^\infty e^{-ist}(e^{-iht}-1)F_2(t)dt.$$

By the same way as in $w_{21}+w_{22}$ we find

$$\begin{aligned}\|w_{231}+w_{232}\|_{2,q,H_R} &\le 3|h||s|^{-3}\left\{\|F_1(1/|s|)\|_{2,q,H_R} + \int_{1/|s|}^\infty \|F_2(t)\|_{2,q,H_R}dt\right\}\\ &\le C|h||s|^{-\beta-1}\|f\|_{q,H},\end{aligned}$$

for $f \in L^q_{[d]}(H)$. Finally, we use (3.9) again to get

$$\|w_{233}\|_{2,q,H_R} \le |h||s|^{-2}\int_{1/|s|}^\infty t\|F_2(t)\|_{2,q,H_R}dt \le C|h||s|^{-\beta-1}\|f\|_{q,H},$$

for $f \in L^q_{[d]}(H)$. We gather all the estimates above to conclude (3.7). □

4 The Stokes resolvent

In this section, based on the results for the half space obtained in the previous section, we address ourselves to analogous regularity estimates near $\lambda=0$ of the Stokes resolvent $u=(\lambda+A)^{-1}Pf$, which together with the associated pressure p satisfies the system $\lambda u - \Delta u + \nabla p = f$, $\nabla\cdot u = 0$ in an aperture domain Ω subject to $u|_{\partial\Omega}=0$ and $\phi(u)=0$, where $f\in L^q(\Omega), 1<q<\infty$ and $\lambda \in \boldsymbol{C}\setminus(-\infty,0]$. To this end, as in [15], [17] and [1], we start with the construction of the resolvent near $\lambda=0$ for $f\in L^q(\Omega)$ with bounded support. We fix a smooth bounded subdomain

D so that $\Omega_{R_0+3} \subset D \subset \Omega$. Given $f \in L^q(\Omega)$, we set $v_0 = A_{q,D}^{-1} P_{q,D} f$ and take a pressure π_0 associated to v_0; they solve the Stokes system $-\Delta v_0 + \nabla \pi_0 = f$, $\nabla \cdot v_0 = 0$ in D subject to $v_0|_{\partial D} = 0$, where f is understood as the restriction of f on D. We further set

$$v_\pm(x,\lambda) = (\lambda + A_{q,H_\pm})^{-1} P_{q,H_\pm}[\psi_{\pm,R_0} f],$$

where $\psi_{\pm,R_0}$ are the cut-off functions given by (2.1). One needs also the case $\lambda = 0$

$$v_\pm(x,0) = \int_0^\infty E_\pm(t) P_{q,H_\pm}[\psi_{\pm,R_0} f]dt,$$

which is the solution written by the Green tensor for the Stokes problem in $H_\pm$. We take the pressures $\pi_\pm$ in $H_\pm$ associated to $v_\pm$ so that

$$\int_{D_{\pm,R_0+1}} \{\pi_\pm(x,\lambda) - \pi_0(x)\}dx = 0, \tag{4.1}$$

for each λ. In this section, for simplicity, we use the abbreviations $\psi_\pm$ for the cut-off functions $\psi_{\pm,R_0+1}$ given by (2.1) and $S_\pm$ for the Bogovskiĭ operators $S_{\pm,R_0+1}$ introduced in section 2. With use of $\{v_\pm, \pi_\pm\}$, $\{v_0, \pi_0\}$ and $\psi_\pm$ together with $S_\pm$, we set

$$\left\{\begin{array}{ll} v & = T(\lambda)f \\ & = \psi_+ v_+ + \psi_- v_- + (1 - \psi_+ - \psi_-)v_0 \\ & \quad -S_+[(v_+ - v_0)\cdot\nabla\psi_+] - S_-[(v_- - v_0)\cdot\nabla\psi_-], \\ \pi & = \psi_+\pi_+ + \psi_-\pi_- + (1-\psi_+-\psi_-)\pi_0. \end{array}\right. \tag{4.2}$$

We here note that $\int_{D_{\pm,R_0+1}} (v_\pm - v_0)\cdot\nabla\psi_\pm dx = 0$ since $\nabla\cdot v_\pm = \nabla\cdot v_0 = 0$. An elementary calculation shows that the pair $\{v,\pi\}$ satisfies

$$\lambda v - \Delta v + \nabla\pi = f + Q(\lambda)f, \quad \nabla\cdot v = 0, \tag{4.3}$$

in Ω subject to $v|_{\partial\Omega} = 0$ and $\phi(v) = \int_M N\cdot v_0 d\sigma = \int_{\Omega_+\cap D} \nabla\cdot v_0 dx = 0$, where

$$Q(\lambda)f = Q_1(\lambda)f + Q_2(\lambda)f \tag{4.4}$$

with

$$\begin{aligned} Q_1(\lambda)f &= \lambda(1-\psi_+-\psi_-)v_0 - 2\nabla\psi_+\cdot\nabla(v_+-v_0) - 2\nabla\psi_-\cdot\nabla(v_--v_0) \\ &\quad -(\Delta\psi_+)(v_+-v_0) - (\Delta\psi_-)(v_--v_0) \\ &\quad +(\nabla\psi_+)(\pi_+-\pi_0) + (\nabla\psi_-)(\pi_--\pi_0) \\ &\quad -\lambda S_+[(v_+-v_0)\cdot\nabla\psi_+] - \lambda S_-[(v_--v_0)\cdot\nabla\psi_-], \end{aligned}$$

and

$$Q_2(\lambda)f = \Delta S_+[(v_+-v_0)\cdot\nabla\psi_+] + \Delta S_-[(v_--v_0)\cdot\nabla\psi_-].$$

By (2.2) we have $S_\pm[(v_\pm - v_0)\cdot\nabla\psi_\pm] \in W_0^{2,q}(D_{\pm,R_0+1})$. But one can obtain the regularity of this term only up to $W_0^{2,q}$ (while the $W_0^{3,q}$-regularity of the corresponding term is available for the exterior problem). This is the reason why the remaining term $Q(\lambda)$ has been divided into two parts. We first derive the regularity estimates near $\lambda = 0$ of $T(\lambda)$ and $Q(\lambda)$.

Lemma 4.1 *Let $n \geq 3, 1 < q < \infty, d \geq R_0$ and $R \geq R_0$. For any small $\varepsilon > 0$ there are constants $C_1 = C_1(\Omega,n,q,d,R,\varepsilon) > 0$ and $C_2 = C_2(\Omega,n,q,d,\varepsilon) > 0$ such that*

$$|\lambda|^\beta \|\partial_\lambda^m T(\lambda)f\|_{2,q,\Omega_R} + \sum_{k=0}^{m-1} \|\partial_\lambda^k T(\lambda)f\|_{2,q,\Omega_R} \leq C_1\|f\|_q, \tag{4.5}$$

for $Re\ \lambda \geq 0$ ($\lambda \neq 0$) and $f \in L^q_{[d]}(\Omega)$; and

$$|\lambda|^\beta \|\partial_\lambda^m Q(\lambda)f\|_q + \sum_{k=0}^{m-1} \|\partial_\lambda^k Q(\lambda)f\|_q \le C_2\|f\|_q, \tag{4.6}$$

for Re $\lambda \ge 0$ *with* $0 < |\lambda| \le 2$ *and* $f \in L^q_{[d]}(\Omega)$*, where* m *and* $\beta = \beta(\varepsilon)$ *are the same as in Lemma 3.3.*

Proof. In view of (4.2), we deduce (4.5) immediately from (3.4) together with (2.2). One can show (4.6) likewise, but it remains to estimate the pressures $\pi_\pm$ contained in (4.4). By (4.1) we have

$$\int_{D_{\pm,R_0+1}} \partial_\lambda^k \pi_\pm(x,\lambda)dx = 0, \quad 1 \le k \le m. \tag{4.7}$$

On the other hand, from the Stokes resolvent system we obtain $\lambda\partial_\lambda^k v_\pm + k\partial_\lambda^{k-1}v_\pm - \Delta\partial_\lambda^k v_\pm + \nabla\partial_\lambda^k\pi_\pm = 0$ $(1 \le k \le m)$ in $H_\pm$. This combined with (4.7) gives

$$\begin{aligned}
&\|(\nabla\psi_\pm)\partial_\lambda^k\pi_\pm(\lambda)\|_q \\
\le\ & C\|\nabla\partial_\lambda^k\pi_\pm(\lambda)\|_{-1,q,D_{\pm,R_0+1}} \\
\le\ & C\|\nabla\partial_\lambda^k v_\pm(\lambda)\|_{q,H_{\pm,R_0+2}} + C|\lambda|\|\partial_\lambda^k v_\pm(\lambda)\|_{q,H_{\pm,R_0+2}} + Ck\|\partial_\lambda^{k-1}v_\pm(\lambda)\|_{q,H_{\pm,R_0+2}},
\end{aligned}$$

for $1 \le k \le m$. Similarly, for $k = 0$, we use (4.1) to get

$$\begin{aligned}
\|(\nabla\psi_\pm)(\pi_\pm(\lambda) - \pi_0)\|_q &\le C\|\nabla(\pi_\pm(\lambda) - \pi_0)\|_{-1,q,D_{\pm,R_0+1}} \\
&\le C\|\nabla v_\pm(\lambda)\|_{q,H_{\pm,R_0+2}} + C|\lambda|\|v_\pm(\lambda)\|_{q,H_{\pm,R_0+2}} + C\|f\|_q.
\end{aligned}$$

It thus follows from (3.4) that

$$|\lambda|^\beta\|(\nabla\psi_\pm)\partial_\lambda^m\pi_\pm(\lambda)\|_q + \sum_{k=0}^{m-1}\|(\nabla\psi_\pm)\partial_\lambda^k(\pi_\pm(\lambda) - \pi_0)\|_q \le C\|f\|_q,$$

for Re $\lambda \ge 0$ with $0 < |\lambda| \le 2$ and $f \in L^q_{[d]}(\Omega)$. This completes the proof. □

Let us consider the case $\lambda = 0$ and simply write $v_\pm = v_\pm(x,0)$. Since $\|(v_\pm - v_0)\cdot\nabla\psi_\pm\|_{2,q} \le C\|f\|_q$, the operator $[f \mapsto (v_\pm - v_0)\cdot\nabla\psi_\pm] : L^q(\Omega) \to W_0^{1,q}(D_{\pm,R_0+1})$ is compact, which combined with (2.2) implies that so is the operator $Q_2(0) : L^q(\Omega) \to L^q_{[d]}(\Omega)$, where $d \ge R_0 + 2$. The other part $Q_1(0)f$ fulfills $\|Q_1(0)f\|_{1,q} \le C\|f\|_q$, from which the compactness of $Q_1(0) : L^q(\Omega) \to L^q_{[d]}(\Omega)$ follows; as a consequence, $Q(0) = Q_1(0) + Q_2(0)$ is a compact operator from $L^q_{[d]}(\Omega)$, $d \ge R_0+2$, into itself. We will show that $1+Q(0)$ is injective in $L^q_{[d]}(\Omega)$. Let $f \in L^q_{[d]}(\Omega)$ satisfy $(1 + Q(0))f = 0$. In view of (4.3), the pair $\{v,\pi\}$ given by (4.2) for such f should obey $-\Delta v + \nabla\pi = 0$, $\nabla\cdot v = 0$ in Ω subject to $v|_{\partial\Omega} = 0$ and $\phi(v) = 0$. Since $f \in L^r_{[d]}(\Omega)$ for $1 < r < \min\{n,q\}$, we have $\nabla^2 v, \nabla\pi \in L^r(\Omega)$, $\nabla v \in L^{nr/(n-r)}(\Omega)$, $v,\pi \in L^r_{loc}(\overline{\Omega})$. It thus follows from Theorem 1.4 (i) of Farwig [5] that $v = \nabla\pi = 0$; here, it should be remarked that the uniqueness holds without any radiation condition (unlike the exterior problem discussed in [15] and [17]). We go back to (4.2) to see that $v_\pm = \nabla\pi_\pm = f = 0$ in $H_\pm \setminus B_{R_0+2}$ and that $v_0 = \nabla\pi_0 = f = 0$ in Ω_{R_0+1}. Set $U_\pm = (D \cup B_{R_0}) \cap H_\pm$. Both $\{v_\pm,\pi_\pm\}$ and $\{v_0,\pi_0\}$ then belong to $W^{2,q}(U_\pm) \times W^{1,q}(U_\pm)$ and are the solutions of the Stokes system in $U_\pm$ with zero boundary condition for the external force f. They thus coincide with each other and, in view of (4.2) again, we have $v_0 = \nabla\pi_0 = f = 0$ in D; after all, $f = 0$ in Ω. Owing to the Fredholm theorem, $1 + Q(0)$ has a bounded inverse $(1 + Q(0))^{-1}$ on $L^q_{[d]}(\Omega)$.

Set $\Sigma_\eta = \{\lambda \in \boldsymbol{C}; \text{Re}\,\lambda \ge 0, 0 < |\lambda| \le \eta\}$ for $\eta > 0$. Since

$$\begin{aligned}
\|Q(\lambda)f - Q(0)f\|_q \le\ & C\|v_+(\lambda) - v_+(0)\|_{1,q,H_{+,R_0+2}} + C\|v_-(\lambda) - v_-(0)\|_{1,q,H_{-,R_0+2}} \\
&+ C|\lambda|\{\|v_+(\lambda)\|_{q,H_{+,R_0+2}} + \|v_-(\lambda)\|_{q,H_{-,R_0+2}} + \|v_0\|_{q,D}\},
\end{aligned}$$

we obtain from (3.5)

$$\|Q(\lambda) - Q(0)\|_{B(L^q_{[d]}(\Omega))} \to 0,$$

as $\lambda \to 0$ with Re $\lambda \geq 0$, which implies the existence of a constant $\eta > 0$ such that $1 + Q(\lambda)$ has also a bounded inverse (in terms of the Neumann series) on $L^q_{[d]}(\Omega)$ with uniform bounds

$$\|(1 + Q(\lambda))^{-1}\|_{B(L^q_{[d]}(\Omega))} \leq C, \tag{4.8}$$

for $\lambda \in \Sigma_\eta \cup \{0\}$. Since the resolvent is uniquely determined, one can represent it for $\lambda \in \Sigma_\eta$ and $f \in L^q_{[d]}(\Omega), d \geq R_0 + 2$, as

$$(\lambda + A)^{-1}Pf = T(\lambda)(1 + Q(\lambda))^{-1}f. \tag{4.9}$$

We are in a position to show an analogous result for the resolvent to (3.4).

Lemma 4.2 *Let $n \geq 3, 1 < q < \infty, d \geq R_0$ and $R \geq R_0$. Given $f \in L^q_{[d]}(\Omega)$, set $u(\lambda) = (\lambda + A)^{-1}Pf$. For any small $\varepsilon > 0$ there is a constant $C = C(\Omega, n, q, d, R, \varepsilon) > 0$ such that*

$$|\lambda|^\beta \|\partial^m_\lambda u(\lambda)\|_{2,q,\Omega_R} + \sum_{k=0}^{m-1} \|\partial^k_\lambda u(\lambda)\|_{2,q,\Omega_R} \leq C\|f\|_q, \tag{4.10}$$

for Re $\lambda \geq 0$ ($\lambda \neq 0$) and $f \in L^q_{[d]}(\Omega)$, where m and $\beta = \beta(\varepsilon)$ are the same as in Lemma 3.3.

Proof. The problem is only near $\lambda = 0$ because we have (2.3) for $G = \Omega$. We may also assume $d \geq R_0 + 2$ since $L^q_{[R_0]}(\Omega) \subset L^q_{[d]}(\Omega)$ for such d. It thus suffices to show (4.10) for $\lambda \in \Sigma_\eta$ by use of (4.9). For such λ and $0 \leq k \leq m$ we see that $\partial^k_\lambda(1 + Q(\lambda))^{-1} \in B(L^q_{[d]}(\Omega))$; furthermore,

$$|\lambda|^\beta \|\partial^m_\lambda(1 + Q(\lambda))^{-1}f\|_q + \sum_{k=0}^{m-1} \|\partial^k_\lambda(1 + Q(\lambda))^{-1}f\|_q \leq C\|f\|_q, \tag{4.11}$$

for $f \in L^q_{[d]}(\Omega)$. In fact, we have the representation

$$\partial^k_\lambda(1 + Q(\lambda))^{-1}f = -(1 + Q(\lambda))^{-1}[\partial^k_\lambda Q(\lambda)](1 + Q(\lambda))^{-1}f + L_k(\lambda)(1 + Q(\lambda))^{-1}f, \tag{4.12}$$

for $k \geq 1$ and $f \in L^q_{[d]}(\Omega)$, where $L_1(\lambda) = 0$ and $L_k(\lambda)$ with $k \geq 2$ consists of finite sums of finite products of $(1 + Q(\lambda))^{-1}, \partial_\lambda Q(\lambda), \cdots, \partial^{k-1}_\lambda Q(\lambda)$. Consequently, (4.6) together with (4.8) implies (4.11). In view of

$$\partial^k_\lambda u(\lambda) = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \partial^{k-j}_\lambda T(\lambda)\ \partial^j_\lambda(1 + Q(\lambda))^{-1}f,$$

we conclude (4.10) from (4.5) and (4.11). □

In the last part of this section we will complete the regularity estimate of the resolvent. To this end, we employ Lemma 3.4 to show the following lemma.

Lemma 4.3 *Let $n \geq 3, 1 < q < \infty, d \geq R_0$ and $R \geq R_0$. Set*

$$T^{(k)}(s) = \partial^k_s T(is), \quad Q^{(k)}(s) = \partial^k_s Q(is) \quad (s \in \boldsymbol{R} \setminus \{0\},\ 0 \leq k \leq m).$$

For any small $\varepsilon > 0$ there is a constant $C = C(\Omega, n, q, d, R, \varepsilon) > 0$ such that

$$\|T^{(k)}(s + h)f - T^{(k)}(s)f\|_{2,q,\Omega_R} + \|Q^{(k)}(s + h)f - Q^{(k)}(s)f\|_q$$

$$\leq \begin{cases} C|h||s|^{-\beta-1}\|f\|_q & \text{if } k=m, \\ C|h||s|^{-\beta}\|f\|_q & \text{if } k=m-1, \\ C|h|\|f\|_q & \text{if } n\geq 5,\ 0\leq k\leq m-2, \end{cases} \tag{4.13}$$

for $2|h| < |s| \leq 1$ *and* $f \in L^q_{[d]}(\Omega)$, *where* m *and* $\beta = \beta(\varepsilon)$ *are the same as in Lemma 3.3. Concerning the first term of the left-hand side, (4.13) holds true for* $h \in \boldsymbol{R}$ *and* $|s| > 2|h|$.

Proof. Set $v_\pm^{(k)}(s) = \partial_s^k v_\pm(is)$, $\pi_\pm^{(k)}(s) = \partial_s^k \pi_\pm(is)$ ($s \in \boldsymbol{R}\setminus\{0\}$, $k = m$ or $m-1$). It then follows from (4.2) together with (2.2) that

$$\begin{aligned} &\|T^{(m)}(s+h)f - T^{(m)}(s)f\|_{2,q,\Omega_R} \\ \leq\ & C\|v_+^{(m)}(s+h) - v_+^{(m)}(s)\|_{2,q,H_+,R} + C\|v_-^{(m)}(s+h) - v_-^{(m)}(s)\|_{2,q,H_-,R}. \end{aligned}$$

In order to estimate $Q^{(m)}$, let us investigate the pressures $\pi_\pm^{(m)}$. Similarly to the proof of Lemma 4.1 with the aid of (4.7), one can show

$$\begin{aligned} \|(\nabla\psi_\pm)\{\pi_\pm^{(m)}(s+h) - \pi_\pm^{(m)}(s)\}\|_q \leq\ & C\|\nabla\pi_\pm^{(m)}(s+h) - \nabla\pi_\pm^{(m)}(s)\|_{-1,q,D_\pm,R_0+1} \\ \leq\ & C\|\nabla v_\pm^{(m)}(s+h) - \nabla v_\pm^{(m)}(s)\|_{q,H_\pm,R_0+2} \\ & + C\|(s+h)v_\pm^{(m)}(s+h) - s v_\pm^{(m)}(s)\|_{q,H_\pm,R_0+2} \\ & + Cm\|v_\pm^{(m-1)}(s+h) - v_\pm^{(m-1)}(s)\|_{q,H_\pm,R_0+2}. \end{aligned}$$

This combined with estimates on the other terms by use of (2.2) yields

$$\begin{aligned} &\|Q^{(m)}(s+h)f - Q^{(m)}(s)f\|_q \\ \leq\ & C\|v_+^{(m)}(s+h) - v_+^{(m)}(s)\|_{1,q,H_+,R_0+2} + C\|v_-^{(m)}(s+h) - v_-^{(m)}(s)\|_{1,q,H_-,R_0+2} \\ & + C|s|\|v_+^{(m)}(s+h) - v_+^{(m)}(s)\|_{q,H_+,R_0+2} + C|s|\|v_-^{(m)}(s+h) - v_-^{(m)}(s)\|_{q,H_-,R_0+2} \\ & + C|h|\|v_+^{(m)}(s+h)\|_{q,H_+,R_0+2} + C|h|\|v_-^{(m)}(s+h)\|_{q,H_-,R_0+2} \\ & + Cm\|v_+^{(m-1)}(s+h) - v_+^{(m-1)}(s)\|_{q,H_+,R_0+2} + Cm\|v_-^{(m-1)}(s+h) - v_-^{(m-1)}(s)\|_{q,H_-,R_0+2}. \end{aligned}$$

Hence (3.7), (3.8) and (3.4) imply (4.13) for the case $k = m$. For $0 \leq k \leq m-1$ we have

$$\|T^{(k)}(s+h)f - T^{(k)}(s)f\|_{2,q,\Omega_R} \leq \left|\int_s^{s+h} \|T^{(k+1)}(\tau)f\|_{2,q,\Omega_R} d\tau\right|,$$

$$\|Q^{(k)}(s+h)f - Q^{(k)}(s)f\|_q \leq \left|\int_s^{s+h} \|Q^{(k+1)}(\tau)f\|_q d\tau\right|,$$

which together with (4.5) and (4.6) respectively lead us to (4.13). The proof is thus complete. □

The regularity of the resolvent along the imaginary axis given by the following lemma plays a crucial role in the next section.

Lemma 4.4 *Let* $n \geq 3, 1 < q < \infty, d \geq R_0$ *and* $R \geq R_0$. *Set*

$$\Phi^{(m)}(s) = \partial_s^m (is + A)^{-1} P \quad (s \in \boldsymbol{R}\setminus\{0\}).$$

For any small $\varepsilon > 0$ *there is a constant* $C = C(\Omega, n, q, d, R, \varepsilon) > 0$ *such that*

$$\int_{-\infty}^{\infty} \|\Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f\|_{2,q,\Omega_R} ds \leq C|h|^{1-\beta}\|f\|_q, \tag{4.14}$$

for $|h| < h_0 = \min\{\eta/4, 1/2\}$ *and* $f \in L^q_{[d]}(\Omega)$. *Here,* m *and* $\beta = \beta(\varepsilon)$ *are the same as in Lemma 3.3, and* $\eta > 0$ *is the constant such that (4.9) is valid for* $\lambda \in \Sigma_\eta$.

Proof. We may assume $d \geq R_0 + 2$ (as in the proof of Lemma 4.2). Given h satisfying $|h| < h_0$, we divide the integral into three parts

$$\int_{-\infty}^{\infty} \|\Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f\|_{2,q,\Omega_R} ds = \int_{|s|\leq 2|h|} + \int_{2|h|<|s|\leq 2h_0} + \int_{|s|>2h_0} = I_1 + I_2 + I_3.$$

With the aid of (4.10), we find

$$I_1 \leq 2\int_{|s|\leq 3|h|} \|\Phi^{(m)}(s)f\|_{2,q,\Omega_R} ds \leq C|h|^{1-\beta}\|f\|_q,$$

for $f \in L^q_{[d]}(\Omega)$. In order to estimate I_2, we use the representation

$$\Phi^{(m)}(s)f = \sum_{j=0}^{m} \binom{m}{j} T^{(m-j)}(s)\ V^{(j)}(s)f,$$

where $V^{(j)}(s) = \partial_s^j(1+Q(is))^{-1} \in B(L^q_{[d]}(\Omega))$ $(0 < |s| \leq \eta,\ 0 \leq j \leq m)$. Then,

$$\begin{aligned} \Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f = & \sum_{j=0}^{m} \binom{m}{j} [T^{(m-j)}(s+h) - T^{(m-j)}(s)]\ V^{(j)}(s+h)f \\ & + \sum_{j=0}^{m} \binom{m}{j} T^{(m-j)}(s)\ [V^{(j)}(s+h) - V^{(j)}(s)]f. \end{aligned}$$

We first show

$$\|V^{(j)}(s+h)f - V^{(j)}(s)f\|_q \leq \begin{cases} C|h||s|^{-\beta-1}\|f\|_q & \text{if } j = m, \\ C|h||s|^{-\beta}\|f\|_q & \text{if } j = m-1, \\ C|h|\|f\|_q & \text{if } n \geq 5,\ 0 \leq j \leq m-2, \end{cases} \tag{4.15}$$

for $2|h| < |s| \leq 2h_0$ and $f \in L^q_{[d]}(\Omega)$. Similarly to the proof of (4.13) for $0 \leq k \leq m-1$, (4.11) implies (4.15) for $0 \leq j \leq m-1$. As in (4.12), we have $V^{(m)}(s) = -V^{(0)}(s)Q^{(m)}(s)V^{(0)}(s) + W_m(s)V^{(0)}(s)$, where $W_1(s) = 0$ and, for $m \geq 2$, $W_m(s) = i^m L_m(is)$ consists of finite sums of finite products of $V^0(s), Q^{(1)}(s), \cdots, Q^{(m-1)}(s)$. Therefore, we collect (4.6), (4.8), (4.13) and (4.15) for $j = 0$ to arrive at (4.15) for $j = m$. It thus follows from (4.5), (4.11), (4.13) and (4.15) that

$$\|\Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f\|_{2,q,\Omega_R} \leq C|h||s|^{-\beta-1}\|f\|_q,$$

for $2|h| < |s| \leq 2h_0$ and $f \in L^q_{[d]}(\Omega)$. As a consequence, we are led to

$$I_2 \leq C|h|\|f\|_q \int_{|s|>2|h|} |s|^{-\beta-1} ds \leq C|h|^{1-\beta}\|f\|_q,$$

for $f \in L^q_{[d]}(\Omega)$. Finally, to estimate I_3, one does not need any localization. In fact, since

$$\Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f = (-i)^{m+1}(m+1)! \int_s^{s+h} (i\tau + A)^{-(m+2)} Pf d\tau,$$

(2.3) gives

$$\|\Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f\|_{2,q,\Omega_R} \leq C|h||s|^{-(m+1)}\|f\|_q,$$

for $|s| > 2h_0$ $(> 2|h|)$ and $f \in L^q(\Omega)$. Therefore, we obtain

$$I_3 \leq C|h|\|f\|_q \int_{|s|>2h_0} |s|^{-(m+1)} ds \leq C|h|\|f\|_q,$$

for $f \in L^q(\Omega)$. Collecting the estimates above on I_1, I_2 and I_3, we conclude (4.14). $\square$

5 L^q-L^r estimates of the Stokes semigroup

In this section we will prove Theorem 2.1. As explained in section 1, the first step is to derive (1.6) for non-solenoidal data with bounded support.

Lemma 5.1 *Let $n \geq 3, 1 < q < \infty, d \geq R_0$ and $R \geq R_0$. For any small $\varepsilon > 0$ there is a constant $C = C(\Omega, n, q, d, R, \varepsilon) > 0$ such that*

$$\|e^{-tA}Pf\|_{1,q,\Omega_R} \leq Ct^{-1/2}(1+t)^{-n/2+1/2+\varepsilon}\|f\|_q, \tag{5.1}$$

for $t > 0$ and $f \in L^q_{[d]}(\Omega)$.

For the proof, the following lemma due to Shibata is crucial since we know the regularity of the Stokes resolvent given by Lemmas 4.2 and 4.4.

Lemma 5.2 *Let X be a Banach space with norm $\|\cdot\|$ and $g \in L^1(\boldsymbol{R}; X)$. If there are constants $\theta \in (0,1)$ and $M > 0$ such that*

$$\int_{-\infty}^{\infty} \|g(s)\|ds + \sup_{h\neq 0} \frac{1}{|h|^\theta} \int_{-\infty}^{\infty} \|g(s+h) - g(s)\|ds \leq M,$$

then the Fourier inverse image $G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist}g(s)ds$ of g enjoys

$$\|G(t)\| \leq CM(1+|t|)^{-\theta},$$

with some $C > 0$ independent of $t \in \boldsymbol{R}$.

Proof. Although this lemma was already proved by Shibata [23], we give our different proof which seems to be simpler. Since $\|G(t)\| \leq M/2\pi$, it suffices to consider the case $|t| > 1$. It is easily seen that if $ht \neq 2j\pi$ $(j = 0, \pm 1, \pm 2, \cdots)$, then $G(t) = \frac{e^{iht}}{2\pi(1-e^{iht})}\int_{-\infty}^{\infty} e^{ist}(g(s+h) - g(s))ds$, from which the assumption leads us to $\|G(t)\| \leq M|h|^\theta/2\pi|1-e^{iht}|$. Taking $h = 1/t$ immediately implies the desired estimate. □

Proof of Lemma 5.1. Since

$$\|e^{-tA}Pf\|_{1,q} \leq C\|e^{-tA}Pf\|_{D(A_q)}^{1/2}\|e^{-tA}Pf\|_q^{1/2} \leq Ct^{-1/2}\|f\|_q, \tag{5.2}$$

for $0 < t < 1$ and $f \in L^q(\Omega)$, we will concentrate ourselves on the proof of (5.1) for $t \geq 1$, namely (1.6). Given $R \geq R_0$, we set $\psi = 1 - \psi_{+,R} - \psi_{-,R}$, where the cut-off functions $\psi_{\pm,R}$ are given by (2.1). One can justify the following representation formula of the semigroup for $f \in L^q_{[d]}(\Omega)$:

$$\psi e^{-tA}Pf = \frac{i^m}{2\pi t^m}\int_{-\infty}^{\infty} e^{ist}\psi\Phi^{(m)}(s)f ds, \tag{5.3}$$

where $\Phi^{(m)}(s) = \partial_s^m(is + A)^{-1}P$ and m is the same as in Lemma 3.3. In fact, starting from the standard Dunford integral representation, we perform m-times integrations by parts and then move the path of integration to the imaginary axis but avoid the origin $\lambda = 0$, so that

$$\psi e^{-tA}Pf = \frac{i^m}{2\pi t^m}\left\{\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right\} e^{ist}\psi\Phi^{(m)}(s)f ds + \frac{(-1)^m}{2\pi i t^m}\int_{\Gamma_\delta} e^{\lambda t}\psi\partial_\lambda^m(\lambda + A)^{-1}Pf d\lambda,$$

for any $\delta > 0$, where $\Gamma_\delta = \{\delta e^{i\theta}; -\pi/2 \leq \theta \leq \pi/2\}$. Owing to (4.10), the last integral vanishes in $L^q(\Omega)$ as $\delta \to 0$ for $f \in L^q_{[d]}(\Omega)$; thus, we arrive at (5.3). Now, it follows from (4.10) and $\|\Phi^{(m)}(s)f\|_{1,q} \leq C\|\Phi^{(m)}(s)f\|_{D(A_q)}^{1/2}\|\Phi^{(m)}(s)f\|_q^{1/2}$ together with (2.3) that

$$\int_{-\infty}^{\infty} \|\psi\Phi^{(m)}(s)f\|_{1,q}ds \leq C\int_{|s|\leq 1} \frac{\|f\|_q}{|s|^\beta}ds + C\int_{|s|>1} \frac{\|f\|_q}{|s|^{m+1/2}}ds \leq C\|f\|_q.$$

Further, (4.14) and the estimate above respectively imply that

$$\sup_{0<|h|<h_0} \frac{1}{|h|^{1-\beta}} \int_{-\infty}^{\infty} \|\psi\Phi^{(m)}(s+h)f - \psi\Phi^{(m)}(s)f\|_{1,q} ds \le C\|f\|_q,$$

and that

$$\sup_{|h|\ge h_0} \frac{1}{|h|^{1-\beta}} \int_{-\infty}^{\infty} \|\psi\Phi^{(m)}(s+h)f - \psi\Phi^{(m)}(s)f\|_{1,q} ds \le \frac{2}{h_0^{1-\beta}} \int_{-\infty}^{\infty} \|\psi\Phi^{(m)}(s)f\|_{1,q} ds \le C\|f\|_q.$$

Hence, we can apply Lemma 5.2 with $X = W^{1,q}(\Omega)$ and $g(s) = \psi\Phi^{(m)}(s)f$ to the formula (5.3); as a consequence, we obtain

$$\|e^{-tA}Pf\|_{1,q,\Omega_R} \le \|\psi e^{-tA}Pf\|_{1,q} \le Ct^{-m}(1+t)^{-1+\beta}\|f\|_q,$$

for $t > 0$, which implies (5.1) for $t \ge 1$ and $f \in L^q_{[d]}(\Omega)$. This completes the proof. □

The next step is to deduce the sharp local energy decay estimate (1.5) from Lemma 5.1.

Lemma 5.3 *Let $n \ge 3, 1 < q < \infty$ and $R \ge R_0$. Then there is a constant $C = C(\Omega, n, q, R) > 0$ such that*

$$\|e^{-tA}f\|_{1,q,\Omega_R} \le Ct^{-n/2q}\|f\|_q, \tag{5.4}$$

for $t \ge 2$ and $f \in L^q_\sigma(\Omega)$; and

$$\|e^{-tA}f\|_{1,q,\Omega_R} + \|\partial_t e^{-tA}f\|_{q,\Omega_R} \le C(1+t)^{-n/2q}\|f\|_{D(A_q)}, \tag{5.5}$$

for $t \ge 0$ and $f \in D(A_q)$.

Proof. Given $f \in L^q_\sigma(\Omega)$, we set $g = e^{-A}f \in D(A_q)$ and intend to derive the decay estimate of $u(t) = e^{-tA}g = e^{-(t+1)A}f$ in $W^{1,q}(\Omega_R)$ for $t \ge 1$. We denote by p the pressure associated to u. We make use of the cut-off functions given by (2.1) and the Bogovskiĭ operator introduced in section 2. Set $g_\pm = \psi_{\pm,R_0+1}\, g - S_{\pm,R_0+1}[g \cdot \nabla\psi_{\pm,R_0+1}]$ and $v_\pm(t) = E_\pm(t)g_\pm$. Note that $\int_{D_{\pm,R_0+1}} g \cdot \nabla\psi_{\pm,R_0+1}dx = 0$ and that $g_\pm \in D(A_{q,H_\pm})$ with

$$\|g_\pm\|_{D(A_{q,H_\pm})} \le C\|g_\pm\|_{2,q,H_\pm} \le C\|g\|_{2,q} \le C\|g\|_{D(A_q)} \le C\|f\|_q, \tag{5.6}$$

by (2.2). We take the pressures $\pi_\pm$ in $H_\pm$ associated to $v_\pm$ in such a way that

$$\int_{D_{\pm,R_0}} \pi_\pm(x,t)dx = 0, \tag{5.7}$$

for each t. In the course of the proof of this lemma, for simplicity, we abbreviate $\psi_{\pm,R_0}$ to $\psi_\pm$ and $S_{\pm,R_0}$ to $S_\pm$. We now define $\{u_\pm, p_\pm\}$ by $u_\pm(t) = \psi_\pm v_\pm(t) - S_\pm[v_\pm(t)\cdot\nabla\psi_\pm]$, $p_\pm(t) = \psi_\pm\pi_\pm(t)$. Then it follows from Lemma 3.2 together with (2.2) and (5.6) that

$$\|u_\pm(t)\|_{1,q,\Omega_R} \le C\|v_\pm(t)\|_{1,q,H_{\pm,L}} \le C(1+t)^{-n/2q}\|f\|_q, \tag{5.8}$$

for $t \ge 0$, where $L = \max\{R, R_0+1\}$. Thus, in order to estimate $u(t)$, let us consider $v(t) = u(t)-u_+(t)-u_-(t)$ and $\pi(t) = p(t)-p_+(t)-p_-(t)$, which should obey $\partial_t v - \Delta v + \nabla\pi = K$, $\nabla\cdot v = 0$ in Ω subject to $v|_{\partial\Omega} = 0$, $\phi(v) = \phi(u) = 0$ and $v|_{t=0} = v_0 = g - g_+ - g_- \in L^q_{[R_0+2]}(\Omega) \cap D(A_q)$, where

$$\begin{aligned} K = \;& 2\nabla\psi_+ \cdot \nabla v_+ + 2\nabla\psi_- \cdot \nabla v_- + (\Delta\psi_+)v_+ + (\Delta\psi_-)v_- \\ & -\Delta S_+[v_+ \cdot \nabla\psi_+] - \Delta S_-[v_- \cdot \nabla\psi_-] \\ & +S_+[\partial_t v_+ \cdot \nabla\psi_+] + S_-[\partial_t v_- \cdot \nabla\psi_-] - (\nabla\psi_+)\pi_+ - (\nabla\psi_-)\pi_-; \end{aligned}$$

we here note that $\nabla \cdot K \neq 0$ as well as $K|_{\partial\Omega} \neq 0$ and we can obtain the regularity of K only up to L^q (in contrast to the exterior problem discussed in [15] and [17]). By (5.7) and in view of the Stokes system in $H_\pm$ we have

$$\|(\nabla\psi_\pm)\pi_\pm(t)\|_q \leq C\|\nabla\pi_\pm(t)\|_{-1,q,D_{\pm,R_0}} \leq C\|\nabla v_\pm(t)\|_{q,H_{\pm,R_0+1}} + C\|\partial_t v_\pm(t)\|_{q,H_{\pm,R_0+1}},$$

which together with (2.2) implies $K(t) \in L^q_{[R_0+1]}(\Omega)$ and

$$\begin{aligned}\|K(t)\|_q \leq\ & C\|v_+(t)\|_{1,q,H_{+,R_0+1}} + C\|v_-(t)\|_{1,q,H_{-,R_0+1}} \\ & + C\|\partial_t v_+(t)\|_{q,H_{+,R_0+1}} + C\|\partial_t v_-(t)\|_{q,H_{-,R_0+1}}.\end{aligned}$$

Therefore, Lemma 3.2 and (5.6) yield

$$\|K(t)\|_q \leq C(1+t)^{-n/2q}\|f\|_q, \tag{5.9}$$

for $t \geq 0$. In order to estimate

$$v(t) = e^{-tA}v_0 + \int_0^t e^{-(t-\tau)A}PK(\tau)d\tau,$$

we employ Lemma 5.1. By (5.1) with a suitable $\varepsilon > 0$ and (5.6) we find

$$\|e^{-tA}v_0\|_{1,q,\Omega_R} \leq Ct^{-n/2+\varepsilon}\|v_0\|_q \leq Ct^{-n/2q}\|f\|_q,$$

for $t \geq 1$. We next combine (5.1) with (5.9) to get

$$\begin{aligned}\int_0^t \|e^{-(t-\tau)A}PK(\tau)\|_{1,q,\Omega_R}d\tau\ & \leq C\|f\|_q \int_0^t (t-\tau)^{-1/2}(1+t-\tau)^{-n/2+1/2+\varepsilon}(1+\tau)^{-n/2q}d\tau \\ & = C\|f\|_q(I_1 + I_2),\end{aligned}$$

where $I_1 = \int_0^{t/2}$ and $I_2 = \int_{t/2}^t$. An elementary calculation gives

$$I_1 \leq \left\{\begin{array}{ll} Ct^{-1/2}(1+t/2)^{-n/2-n/2q+3/2+\varepsilon} & \text{if } q > n/2 \\ Ct^{-1/2}(1+t/2)^{-n/2+1/2+\varepsilon}\log(1+t/2) & \text{if } q = n/2 \\ Ct^{-1/2}(1+t/2)^{-n/2+1/2+\varepsilon} & \text{if } q < n/2 \end{array}\right\} \leq Ct^{-n/2q},$$

for $t \geq 1$ and

$$I_2 \leq (1+t/2)^{-n/2q}\int_0^\infty \tau^{-1/2}(1+\tau)^{-n/2+1/2+\varepsilon}d\tau \leq C(1+t/2)^{-n/2q},$$

for $t > 0$. We collect the estimates above to obtain

$$\|v(t)\|_{1,q,\Omega_R} \leq Ct^{-n/2q}\|f\|_q, \tag{5.10}$$

for $t \geq 1$. From (5.8) and (5.10) we deduce

$$\|u(t)\|_{1,q,\Omega_R} = \|v(t) + u_+(t) + u_-(t)\|_{1,q,\Omega_R} \leq Ct^{-n/2q}\|f\|_q,$$

for $t \geq 1$ and $f \in L^q_\sigma(\Omega)$, which proves (5.4). Let $f \in D(A_q)$. Then we easily observe $\|e^{-tA}f\|_{1,q,\Omega_R} + \|\partial_t e^{-tA}f\|_{q,\Omega_R} \leq C\|e^{-tA}f\|_{D(A_q)} \leq C\|f\|_{D(A_q)}$ for $t \geq 0$ and also we can estimate $\partial_t e^{-tA}f$ for large t; in fact, by virtue of (5.4) just proved we get $\|\partial_t e^{-tA}f\|_{q,\Omega_R} = \|e^{-tA}Af\|_{q,\Omega_R} \leq Ct^{-n/2q}\|Af\|_q$ for $t \geq 2$. This implies (5.5). □

We are interested in the L^q estimate of ∇e^{-tA} for large t, in particular, the L^n estimate is quite important for us.

Lemma 5.4 *Let $n \geq 3$ and $1 < q < \infty$. Then there is a constant $C = C(\Omega, n, q) > 0$ such that*

$$\|\nabla e^{-tA} f\|_q \leq C t^{-\min\{1/2, n/2q\}} \|f\|_q, \tag{5.11}$$

for $t \geq 2$ and $f \in L^q_\sigma(\Omega)$.

Proof. We fix $R \geq R_0 + 1$. Since we have already known the decay rate $t^{-n/2q}$ of $\|\nabla e^{-tA} f\|_{q,\Omega_R}$ by Lemma 5.3, it suffices to derive the estimate outside Ω_R, that is,

$$\|\nabla e^{-tA} f\|_{q,\Omega_\pm \setminus \Omega_R} \leq C t^{-\min\{1/2, n/2q\}} \|f\|_q, \tag{5.12}$$

for $t \geq 2$ and $f \in L^q_\sigma(\Omega)$. In an analogous way to [15], [17] and [1], we make use of the decay properties of the semigroup $E_\pm(t)$ for the half space. Given $f \in L^q_\sigma(\Omega)$, we set $g = e^{-A} f \in D(A_q)$ and then $u(t) = e^{-tA} g = e^{-(t+1)A} f$. We choose two pressures $p_\pm$ in Ω associated to u in such a way that

$$\int_{D_{\pm,R-1}} p_\pm(x,t)\,dx = 0, \tag{5.13}$$

for each t (p_+ and p_- will be used independently). With use of the cut-off functions given by (2.1) and the Bogovskiĭ operator introduced in section 2, we define $\{v_\pm, \pi_\pm\}$ by $v_\pm(t) = \psi_\pm u(t) - S_\pm[u(t) \cdot \nabla \psi_\pm]$, $\pi_\pm(t) = \psi_\pm p_\pm(t)$. Here and in what follows, we use the abbreviations $\psi_\pm$ for $\psi_{\pm,R-1}$ and $S_\pm$ for $S_{\pm,R-1}$. Since $v_\pm = u$ for $x \in \Omega_\pm \setminus \Omega_R = H_\pm \setminus B_R$, we will show

$$\|\nabla v_\pm(t)\|_{q,H_\pm} \leq C t^{-\min\{1/2, n/2q\}} \|g\|_{D(A_q)}, \tag{5.14}$$

for $t \geq 1$, which combined with $\|g\|_{D(A_q)} \leq C\|f\|_q$ implies (5.12) for $t \geq 2$. It is easily observed that $\{v_\pm, \pi_\pm\}$ satisfies $\partial_t v_\pm - \Delta v_\pm + \nabla \pi_\pm = Z_\pm$, $\nabla \cdot v_\pm = 0$ in $H_\pm$ subject to $v_\pm|_{\partial H_\pm} = 0$ and $v_\pm|_{t=0} = a_\pm = \psi_\pm g - S_\pm[g \cdot \nabla \psi_\pm]$, where

$$Z_\pm = -2\nabla \psi_\pm \cdot \nabla u - (\Delta \psi_\pm) u + \Delta S_\pm[u \cdot \nabla \psi_\pm] - S_\pm[\partial_t u \cdot \nabla \psi_\pm] + (\nabla \psi_\pm) p_\pm.$$

Our task is now to estimate the gradient of

$$v_\pm(t) = E_\pm(t) a_\pm + \int_0^t E_\pm(t-\tau) P_{H_\pm} Z_\pm(\tau)\,d\tau. \tag{5.15}$$

By virtue of (5.13) we have

$$\|(\nabla \psi_\pm) p_\pm(t)\|_{q,H_\pm} \leq C\|\nabla p_\pm(t)\|_{-1,q,D_{\pm,R-1}} \leq C\|\nabla u(t)\|_{q,\Omega_R} + C\|\partial_t u(t)\|_{q,\Omega_R},$$

from which together with (2.2) it follows that

$$\|Z_\pm(t)\|_{q,H_\pm} \leq C\|u(t)\|_{1,q,\Omega_R} + C\|\partial_t u(t)\|_{q,\Omega_R}.$$

Hence, (5.5) implies

$$\|P_{H_\pm} Z_\pm(t)\|_{r,H_\pm} \leq C\|Z_\pm(t)\|_{q,H_\pm} \leq C(1+t)^{-n/2q} \|g\|_{D(A_q)}, \tag{5.16}$$

for $t \geq 0$ and $r \in (1, q]$ since $Z_\pm(t) \in L^q_{[R]}(H_\pm) \subset L^r_{[R]}(H_\pm)$ for such r. In view of (5.15), we deduce from (1.4) for $\Omega = H_\pm$ together with (5.16)

$$\begin{aligned}
&\|\nabla v_\pm(t)\|_{q,H_\pm} \\
\leq\ & C t^{-1/2} \|a_\pm\|_{q,H_\pm} + C\|g\|_{D(A_q)} \int_0^t (t-\tau)^{-1/2} (1+t-\tau)^{-(n/r-n/q)/2} (1+\tau)^{-n/2q}\,d\tau \\
\leq\ & C t^{-1/2} \|g\|_q + C\|g\|_{D(A_q)} (I_1 + I_2),
\end{aligned}$$

for $r \in (1, q]$, where $I_1 = \int_0^{t/2}$ and $I_2 = \int_{t/2}^t$. We take r so that $1 < r < \min\{n/2, q\}$. Then we see that

$$I_1 \le \left\{ \begin{array}{ll} Ct^{-1/2}(1+t/2)^{-n/2r+1} & \text{if } q > n/2 \\ Ct^{-1/2}(1+t/2)^{-n/2r+1}\log(1+t/2) & \text{if } q = n/2 \\ Ct^{-1/2}(1+t/2)^{-(n/r-n/q)/2} & \text{if } q < n/2 \end{array} \right\} \le Ct^{-1/2},$$

for $t > 0$ and that

$$I_2 \le \left\{ \begin{array}{ll} C(1+t/2)^{-n/2q} & \text{if } q > n, \\ C(1+t/2)^{-1/2} & \text{if } q \le n, \end{array} \right.$$

for $t > 0$. Collecting the estimates above concludes (5.14). This completes the proof. □

The following lemma is concerned with the L^∞ estimate of the semigroup (the restriction $q > n$ will be removed later).

Lemma 5.5 *Let $3 \le n < q < \infty$. There is a constant $C = C(\Omega, n, q) > 0$ such that*

$$\|e^{-tA}f\|_\infty \le Ct^{-n/2q}\|f\|_q, \tag{5.17}$$

for $t > 0$ and $f \in L^q_\sigma(\Omega)$.

Proof. For fixed $R \ge R_0 + 1$, estimate (5.4) together with the Sobolev embedding property implies $\|e^{-tA}f\|_{\infty,\Omega_R} \le Ct^{-n/2q}\|f\|_q$ for $t \ge 2$ and $f \in L^q_\sigma(\Omega)$ on account of $n < q < \infty$. Along the lines of the proof of Lemma 5.4, one can show

$$\|e^{-tA}f\|_{\infty,\Omega_\pm\setminus\Omega_R} \le Ct^{-n/2q}\|f\|_q, \tag{5.18}$$

for $t \ge 2$. In fact, given $f \in L^q_\sigma(\Omega)$, we take the same $g, \{u, p_\pm\}$ and $\{v_\pm, \pi_\pm\}$, and apply the L^q-L^∞ estimate (1.3) for $\Omega = H_\pm$ to (5.15). Then, taking (5.16) into account, we get

$$\|v_\pm(t)\|_{\infty,H_\pm} \le Ct^{-n/2q}\|a_\pm\|_{q,H_\pm} + C\|g\|_{D(A_q)}\int_0^t (t-\tau)^{-n/2q}(1+t-\tau)^{-(n/r-n/q)/2}(1+\tau)^{-n/2q}d\tau,$$

for $r \in (1, q]$; we now choose $r \in (1, n/2)$ to find $\|v_\pm(t)\|_{\infty,H_\pm} \le Ct^{-n/2q}\|g\|_{D(A_q)}$ for $t \ge 1$, which proves (5.18) for $t \ge 2$. We thus obtain (5.17) for $t \ge 2$. For $0 < t < 2$, we recall (5.2) to see $\|e^{-tA}f\|_\infty \le C\|e^{-tA}f\|_{1,q}^{n/q}\|e^{-tA}f\|_q^{1-n/q} \le Ct^{-n/2q}\|f\|_q$. The proof is complete. □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. The proof is divided into three steps.

Step 1. First of all, we observe (1.4) for $q = r \in (1, n]$. Indeed, it follows from (5.2) for $0 < t < 2$ and (5.11) for $t \ge 2$ that

$$\|\nabla e^{-tA}f\|_q \le Ct^{-1/2}\|f\|_q, \tag{5.19}$$

for $t > 0$ and $f \in L^q_\sigma(\Omega)$ provided $1 < q \le n$. In this step we accomplish the proof of (1.3) for $1 < q \le r \le \infty$ ($q \ne \infty$) and (1.4) for $1 < q \le r \le n$. We begin with the removal of the restriction $q > n$ in Lemma 5.5. In view of (5.19) and the Sobolev embedding property we have

$$\|e^{-tA}f\|_r \le Ct^{-1/2}\|f\|_q, \tag{5.20}$$

for $t > 0$ and $f \in L^q_\sigma(\Omega)$ when $1 < q < n$ and $1/r = 1/q - 1/n$. Let $n/(k+1) < q < n/k$ with $k = 1, 2, \cdots, n-1$. We put $\{q_j\}_{j=0}^k$ in such a way that $1/q_{j+1} = 1/q_j - 1/n$ ($j = 0, 1, \cdots, k-1$) with $q_0 = q$. Since $n < q_k < \infty$, we make use of (5.17) with $q = q_k$ and (5.20) to obtain $\|e^{-tA}f\|_\infty \le Ct^{-n/2q_k}\|e^{-(t/2)A}f\|_{q_k} \le Ct^{-n/2q_k - k/2}\|f\|_q$ for $t > 0$, which proves (5.17) except for $q = n, n/2, \cdots, n/(n-1)$. But the exceptional cases can be also deduced via interpolation.

Thus the L^q-L^∞ estimate (5.17) has been established for all $q \in (1, \infty)$. This together with the L^q boundedness immediately gives (1.3) for $1 < q \le r \le \infty$, from which combined with (5.19) we further obtain (1.4) for $1 < q \le r \le n$.
Step 2. In this step we prove (1.4) for $1 < q < n < r < \infty$. Given $r \in (n, \infty)$, we take $s \in (n/2, n)$ so that $1/s = 1/r + 1/n$. When $1 < q \le s$, an embedding relation given by Lemma 3.1 of [6] together with $\|\nabla^2 u\|_s \le C\|Au\|_s$ [6, Theorem 2.5] implies

$$\|\nabla e^{-tA} f\|_r \le C\|\nabla^2 e^{-tA} f\|_s \le C\|Ae^{-tA} f\|_s \le Ct^{-1}\|e^{-(t/2)A} f\|_s,$$

for $t > 0$, from which together with (1.3) we obtain (1.4). If $s < q < n$, which implies $r < q_*$ with $1/q_* = 1/q - 1/n$, then by the same reasoning as above

$$\|\nabla e^{-tA} f\|_r \le \|\nabla e^{-tA} f\|_{q_*}^{1-\theta} \|\nabla e^{-tA} f\|_q^\theta \le C\|Ae^{-tA} f\|_q^{1-\theta} \|\nabla e^{-tA} f\|_q^\theta,$$

for $t > 0$, where $1/r = (1-\theta)/q_* + \theta/q = 1/q - (1-\theta)/n$. Therefore, (5.19) yields (1.4).
Step 3. Let $f \in L^1(\Omega) \cap L^s_\sigma(\Omega)$ for some $s \in (1, \infty)$. This step is devoted to the case $q = 1$, namely L^1-L^r estimate. Let $1 < r < \infty$. We apply a simple duality argument; in fact, the L^q-L^∞ estimate implies

$$|(e^{-tA} f, g)| = |(f, e^{-tA} g)| \le \|f\|_1 \|e^{-tA} g\|_\infty \le Ct^{-(n-n/r)/2} \|f\|_1 \|g\|_{r/(r-1)},$$

for $g \in L^{r/(r-1)}_\sigma(\Omega)$, which gives (1.3) for $q = 1 < r < \infty$. Combining this with (5.17) and (1.4), respectively, we obtain (1.3) for $q = 1 < r = \infty$ and (1.4) for $q = 1 < r < \infty$. We have completed the proof. □

References

[1] H. Abels, L_q-L_r estimates for the non-stationary Stokes equations in an aperture domain, *Z. Anal. Anwendungen* **21** (2002), 159–178.

[2] M. E. Bogovskiĭ, Solution of the first boundary value problem for the equation of continuity of an incompressible medium, *Soviet Math. Dokl.* **20** (1979), 1094–1098.

[3] W. Borchers and T. Miyakawa, L^2 decay for the Navier-Stokes flow in halfspaces, *Math. Ann.* **282** (1988), 139–155.

[4] W. Borchers and H. Sohr, On the equations rot $v = g$ and div $u = f$ with zero boundary conditions, *Hokkaido Math. J.* **19** (1990), 67–87.

[5] R. Farwig, Note on the flux condition and pressure drop in the resolvent problem of the Stokes system, *Manuscripta Math.* **89** (1996), 139–158.

[6] R. Farwig and H. Sohr, Helmholtz decomposition and Stokes resolvent system for aperture domains in L^q-spaces, *Analysis* **16** (1996), 1–26.

[7] M. Franzke, Strong L^q-theory of the Navier-Stokes equations in aperture domains, Preprint Nr. 2139 TU Darmstadt (2001).

[8] D. Fujiwara and H. Morimoto, An L_r-theorem of the Helmholtz decomposition of vector fields, *J. Fac. Sci. Univ. Tokyo Sect. IA* **24** (1977), 685–700.

[9] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. I: Linearized Steady Problems, Vol. II: Nonlinear Steady Problems*, Springer, New York, 1994.

[10] Y. Giga and T. Miyakawa, Solutions in L_r of the Navier-Stokes initial value problem, *Arch. Rational Mech. Anal.* **89** (1985), 267–281.

[11] J. G. Heywood, On uniqueness questions in the theory of viscous flow, *Acta Math.* **136** (1976), 61–102.

[12] J. G. Heywood, Auxiliary flux and pressure conditions for Navier-Stokes problems, *Approximation Methods for Navier-Stokes Problems*, 223–234, *Lecture Notes in Math.* **771**, Springer, Berlin, 1980.

[13] J. G. Heywood, The Navier-Stokes equations: on the existence, regularity and decay of solutions, *Indiana Univ. Math. J.* **29** (1980), 639–681.

[14] T. Hishida, The nonstationary Stokes and Navier-Stokes flows through an aperture, Preprint Nr. 2184 TU Darmstadt (2001).

[15] H. Iwashita, L_q-L_r estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q spaces, *Math. Ann.* **285** (1989), 265–288.

[16] T. Kato, Strong L^p solutions of the Navier-Stokes equation in $\boldsymbol{R}^m$, with applications to weak solutions, *Math. Z.* **187** (1984), 471–480.

[17] T. Kobayashi and Y. Shibata, On the Oseen equation in the three dimensional exterior domains, *Math. Ann.* **310** (1998), 1–45.

[18] H. Kozono and T. Ogawa, Global strong solution and its decay properties for the Navier-Stokes equations in three dimensional domains with non-compact boundaries, *Math. Z.* **216** (1994), 1–30.

[19] H. Kozono and H. Sohr, Global strong solution of the Navier-Stokes equations in 4 and 5 dimensional unbounded domains, *Ann. Inst. H. Poincaré Anal. Nonlinéaire* **16** (1999), 535–561.

[20] P. Maremonti and V. A. Solonnikov, On nonstationary Stokes problem in exterior domains, *Ann. Sc. Norm. Sup. Pisa* **24** (1997), 395–449.

[21] M. McCracken, The resolvent problem for the Stokes equation on halfspaces in L_p, *SIAM J. Math. Anal.* **12** (1981), 201–228.

[22] T. Miyakawa, The Helmholtz decomposition of vector fields in some unbounded domains, *Math. J. Toyama Univ.* **17** (1994), 115–149.

[23] Y. Shibata, On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain, *Tsukuba J. Math.* **7** (1983), 1–68.

[24] C. G. Simader and H. Sohr, A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains, *Mathematical Problems Relating to the Navier-Stokes Equation*, 1–35, *Ser. Adv. Math. Appl. Sci.* **11**, World Sci., NJ, 1992.

[25] S. Ukai, A solution formula for the Stokes equation in $\boldsymbol{R}^n_+$, *Commun. Pure Appl. Math.* **40** (1987), 611–621.