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Bifurcation of and ghost effect on the temperature field in the Bénard problem of a gas in the continuum limit

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Abstract

A gas in a time-independent state under a uniform weak gravity in a general domain is considered. The asymptotic behavior of the gas in the limit that the Knudsen number of the system tends to zero (or in the continuum limit) is investigated on the basis of the Boltzmann system for the case where the flow velocity vanishes in this limit, and the fluid-dynamic-type equations and their associated boundary conditions describing the behavior of the gas in the continuum limit are derived. The equations, different from the Navier–Stokes ones, contain thermal stress and infinitesimal velocity amplified by the inverse of the Knudsen number. The system is applied to analysis of the behavior of a gas between two parallel plane walls heated from below (Bénard problem), and a bifurcated strongly distorted temperature field is found in infinitesimal velocity and gravity. This is an example showing that the Navier–Stokes system fails to describe the correct behavior of a gas in the continuum limit.

1 Introduction

The study of the relation of the two systems describing the behavior of a gas, the system of classical fluid dynamics and the Boltzmann system, has a long history (see, e.g., chapter 1 in Ref.1 and references therein). In these works, systems of fluid-dynamic-type equations and their associated boundary conditions describing the asymptotic behavior of a gas for small Knudsen numbers are derived from the system of the Boltzmann equation and its boundary condition. One of the striking results of the systematic theoretical analyses is that in some important class of problems, infinitesimal quantities in the continuum limit produce a finite effect on the behavior of a gas in the continuum limit (ghost effect).[2] Consider, for example, a gas in a time-independent state in a closed boundary at rest with nonuniform temperature. The temperature field of the gas in the continuum limit is not correctly described by the heat-conduction equation, contrary to the prevalent understanding. It is determined by a set of equations coupled with infinitesimal flow velocity amplified by the inverse of the Knudsen number. Thus, in problems where there is a finite temperature variation, careful consideration is required to investigate the behavior of a gas even in the continuum limit.

The Bénard problem of a gas between two parallel plane walls with different temperatures in a gravity field is one of the most famous problems in classical fluid dynamics and is studied by various authors (see Ref. 3). However, when we consider the Bénard problem with the ratio of the temperatures of the two walls being not close to unity, the asymptotic analysis of the Boltzmann system mentioned above indicates that some modification is required for the basic fluid-dynamic equations and that an infinitesimal velocity field in the Knudsen number in its vanishing limit, which cannot be perceptible in the continuum world, influences the temperature field in the limit. For complete understanding of the problem, the corresponding asymptotic theory of the Boltzmann system where the effect of the gravity is taken into account is required. Thus, we first carry out the asymptotic analysis of the Boltzmann system for small Knudsen numbers under a weak gravity for the situation where the velocity vanishes in the continuum limit and derive the fluid-dynamic-type equations and their associated boundary conditions that describe the behavior of the gas in the continuum limit. A weak gravity field is considered here to show that infinitesimal quantities in the Knudsen number in its vanishing limit in the Boltzmann equation influence the behavior of the gas in the limit. Then, this system of equations and boundary conditions is applied to the Bénard problem, and the infinitesimal velocity and gravity fields are shown to influence the temperature field and to be the source of bifurcation of the temperature field. The bifurcation and resulting behavior of the temperature field show the incompleteness of the classical fluid dynamics in describing the behavior of a gas in the continuum limit.

2 Asymptotic Theory in a Weak Gravity Field

2.1 Formulation of Problem

Consider a gas in a time-independent state under a uniform weak gravity in a general domain. We will investigate the asymptotic behavior of the gas in the limit that the Knudsen number of the system tends to zero (or in the continuum limit) under the assumption that (i) the behavior of the gas is described by the Boltzmann equation; (ii) the gas molecules make the diffuse reflection on a boundary of the gas; (iii) the gravity is uniform and weak of the order of the square of the Knudsen number (the second-order infinitesimal); and (iv) the flow velocity vanishes in the limit that the Knudsen number vanishes (the first-order infinitesimal).

Let L, T_0, ρ_0, and g_i be, respectively, the reference length, the reference temperature, the reference density, and the gravity of the gas system. The nondimensional space coordinates x_i, the nondimensional molecular velocity ζ_i, the nondimensional velocity distribution function $\hat{f}$, and the nondimensional gravity $\hat{g}_i$ are defined from the corresponding dimensional variables X_i, ξ_i, f, and g_i as follows:

$$x_i = \frac{X_i}{L},\ \zeta_i = \frac{\xi_i}{(2RT_0)^{1/2}},\ \hat{f} = \frac{f}{\rho_0/(2RT_0)^{3/2}},\ \hat{g}_i = \frac{g_i}{(2RT_0)/L}, \tag{1}$$

where R is the specific gas constant [the Boltzmann constant $(1.3806503 \times 10^{-23}\,\mathrm{J\,K^{-1}})$ divided by the mass of a molecule]. Let the mean free path of the gas in the equilibrium state at rest and at temperature T_0 and density ρ_0 be ℓ_0. For a gas molecule with a finite influence range, $\ell_0 = 1/\sqrt{2}\pi d_m^2(\rho_0/m)$, where m is the mass of a molecule and d_m is the radius of the influence range of the intermolecular force (this corresponds to the diameter of a hard-sphere molecule). The Knudsen number Kn of the system is defined by

$$\mathrm{Kn} = \frac{\ell_0}{L}, \tag{2}$$

which characterizes the degree of rarefaction of the gas. Let

$$k = \frac{\sqrt{\pi}}{2}\mathrm{Kn} \quad \text{and} \quad \hat{g}_{i2} = \frac{\hat{g}_i}{k^2}. \tag{3}$$

The case where $\hat{g}_{i2}$ is of the order of unity (or $\hat{g}_i$ is of the order of k^2) is of our interest in the present paper [see the assumption (iii)].

The Boltzmann equation for a time-independent state is expressed with the above nondimensional variables in the following nondimensional form:

$$\zeta_i \frac{\partial \hat{f}}{\partial x_i} + k^2 \hat{g}_{i2} \frac{\partial \hat{f}}{\partial \zeta_i} = \frac{1}{k}\hat{J}(\hat{f}, \hat{f}), \tag{4a}$$

$$\hat{J}(\hat{f}, \hat{f}) = \int_{\text{all }\alpha_i,\,\text{all }\zeta_{i*}} (\hat{f}'\hat{f}'_* - \hat{f}\hat{f}_*)\hat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\mathrm{d}\boldsymbol{\zeta}_*, \tag{4b}$$

where

$$\left.\begin{aligned} &\hat{B} = \hat{B}(|\alpha_j(\zeta_{j*} - \zeta_j)|/|\zeta_{i*} - \zeta_i|, |\zeta_{i*} - \zeta_i|), \\ &\hat{f} = \hat{f}(x_i, \zeta_i), \quad \hat{f}_* = \hat{f}(x_i, \zeta_{i*}), \quad \hat{f}' = \hat{f}(x_i, \zeta_i'), \quad \hat{f}'_* = \hat{f}(x_i, \zeta_{i*}'), \\ &\zeta_i' = \zeta_i + \alpha_i\alpha_j(\zeta_{j*} - \zeta_j), \quad \zeta_{i*}' = \zeta_{i*} - \alpha_i\alpha_j(\zeta_{j*} - \zeta_j), \end{aligned}\right\} \tag{5}$$

and α_i (or $\boldsymbol{\alpha}$) is a unit vector, expressing the variation of the direction of the molecular velocity owing to a intermolecular collision, $\mathrm{d}\Omega(\boldsymbol{\alpha})$ is the solid-angle element in the direction of $\boldsymbol{\alpha}$, and $\hat{B}(|\alpha_j(\zeta_{j*} - \zeta_j)|/|\zeta_{i*} - \zeta_i|, |\zeta_{i*} - \zeta_i|)$ is a nonnegative function of $|\alpha_j(\zeta_{j*} - \zeta_j)|/|\zeta_{i*} - \zeta_i|$ and $|\zeta_{i*} - \zeta_i|$, whose functional form is determined by the intermolecular force [e.g., for a gas consisting of hard-sphere molecules, $\hat{B} = |\alpha_j(\zeta_{j*} - \zeta_j)|/4\sqrt{2\pi}$]. The integrations with respect to ζ_{i*} and α_i are carried out over the whole space of ζ_{i*} and over the whole direction of α_i (the whole spherical surface) respectively.

Let the temperature and velocity of the boundary be, respectively, T_w and v_{wi}. The corresponding nondimensional variables $\hat{T}_w$ and $\hat{v}_{wi}$ be defined, respectively, by T_w/T_0 and $v_{wi}/(2RT_0)^{1/2}$. The diffuse reflection boundary condition is given with these variables by

$$\hat{f}(x_i,\zeta_i) = \frac{\hat{\sigma}_w}{(\pi\hat{T}_w)^{3/2}} \exp\left(-\frac{(\zeta_i - \hat{v}_{wi})^2}{\hat{T}_w}\right) \quad (\zeta_j n_j > 0), \tag{6a}$$

$$\hat{\sigma}_w = -2\left(\frac{\pi}{\hat{T}_w}\right)^{1/2} \int_{\zeta_j n_j < 0} \zeta_j n_j \hat{f}(x_i,\zeta_i)\mathbf{d}\boldsymbol{\zeta}, \tag{6b}$$

where n_i is the unit normal vector to the boundary, pointed to the gas region and the condition required for a time-independent problem, $\hat{v}_{wi}n_i = 0$, is used here. The boundary parameters $\hat{T}_w$ and $\hat{v}_{wi}$ may depend on k and can be expanded in power series of k. Corresponding to the assumption (iv), the series of $\hat{v}_{wi}$ starts from the term of k, that is,

$$\hat{T}_w = \hat{T}_{w0} + \hat{T}_{w1}k + \cdots,$$
$$\hat{v}_{wi} = \hat{v}_{wi1}k + \cdots.$$

In the following sections, the asymptotic behavior of the solution $\hat{f}(x_i,\zeta_i)$ of the boundary-value problem (4a) with (6a) for small k (or $k \ll 1$) is studied under the assumption that

$$\int \zeta_i \hat{f}\mathbf{d}\boldsymbol{\zeta} = O(k). \tag{7}$$

This is the extension of Ref. 2 to the case with gravity. It will be made clear that a slight gravity influences the behavior of a gas drastically.

The macroscopic variables, the density ρ, the velocity v_i, the temperature T, the pressure p, the stress tensor p_{ij}, and the heat-flow vector q_i are defined by the velocity distribution function f. The corresponding nondimensional variables $\hat{\rho}$, $\hat{v}_i$, $\hat{T}$, $\hat{p}$, $\hat{p}_{ij}$, and $\hat{q}_i$ are defined, respectively, by ρ/ρ_0, $v_i/(2RT_0)^{1/2}$, T/T_0, p/p_0, p_{ij}/p_0, and $q_i/p_0(2RT_0)^{1/2}$, where $p_0 = R\rho_0 T_0$. They are related to $\hat{f}$ as follows:

$$\hat{\rho} = \int \hat{f}\mathbf{d}\boldsymbol{\zeta}, \tag{8a}$$

$$\hat{\rho}\hat{v}_i = \int \zeta_i \hat{f}\mathbf{d}\boldsymbol{\zeta}, \tag{8b}$$

$$\frac{3}{2}\hat{\rho}\hat{T} = \int (\zeta_i - \hat{v}_i)^2 \hat{f}\mathbf{d}\boldsymbol{\zeta}, \tag{8c}$$

$$\hat{p} = \hat{\rho}\hat{T}, \tag{8d}$$

$$\hat{p}_{ij} = 2\int (\zeta_i - \hat{v}_i)(\zeta_j - \hat{v}_j)\hat{f}\mathbf{d}\boldsymbol{\zeta}, \tag{8e}$$

$$\hat{q}_i = \int (\zeta_i - \hat{v}_i)(\zeta_j - \hat{v}_j)^2 \hat{f}\mathbf{d}\boldsymbol{\zeta}. \tag{8f}$$

2.2 SB Solution

Putting aside the boundary condition, we look for a moderately varying solution of Eq. (4a), whose length scale of variation is of the order of the reference length L of the system $[\partial\hat{f}/\partial x_i = O(\hat{f})]$, in a power series of k:

$$\hat{f}_{SB} = \hat{f}_{SB0} + \hat{f}_{SB1}k + \hat{f}_{SB2}k^2 + \cdots, \tag{9}$$

where the subscript *SB* is attached to discriminate the moderately varying solution satisfying the condition (7). This type of solution (or expansion) will be called SB solution (or expansion). The condition (7) is reduced to the following condition on the component function $\hat{f}_{SB0}$ of the expansion (9):

$$\int \zeta_i \hat{f}_{SB0}\mathbf{d}\boldsymbol{\zeta} = 0. \tag{10}$$

The relation between the macroscopic variables and the velocity distribution function is given by Eqs. (8a)–(8f) with the subscript SB attached. Corresponding to the expansion (9), the macroscopic variable $\hat{h}_{SB}$, where $\hat{h}$ represents $\hat{\rho}$, $\hat{v}_i$, $\hat{T}$, etc., is also expanded in k:

$$\hat{h}_{SB} = \hat{h}_{SB0} + \hat{h}_{SB1}k + \hat{h}_{SB2}k^2 + \cdots .$$

The component function h_{SBm} is related to the component function of the velocity distribution function as follows:

$$\hat{\rho}_{SB0} = \int \hat{f}_{SB0}\mathbf{d}\boldsymbol{\zeta}, \tag{11a}$$

$$\hat{\rho}_{SB0}\hat{v}_{iSB0} = \int \zeta_i \hat{f}_{SB0}\mathbf{d}\boldsymbol{\zeta} = 0, \tag{11b}$$

$$\frac{3}{2}\hat{\rho}_{SB0}\hat{T}_{SB0} = \int \zeta_i{}^2 \hat{f}_{SB0}\mathbf{d}\boldsymbol{\zeta}, \tag{11c}$$

$$\hat{p}_{SB0} = \hat{\rho}_{SB0}\hat{T}_{SB0}, \tag{11d}$$

$$\hat{p}_{ijSB0} = 2\int \zeta_i\zeta_j \hat{f}_{SB0}\mathbf{d}\boldsymbol{\zeta}, \tag{11e}$$

$$\hat{q}_{iSB0} = \int \zeta_i\zeta_j{}^2 \hat{f}_{SB0}\mathbf{d}\boldsymbol{\zeta}, \tag{11f}$$

$$\hat{\rho}_{SB1} = \int \hat{f}_{SB1}\mathbf{d}\boldsymbol{\zeta}, \tag{12a}$$

$$\hat{\rho}_{SB0}\hat{v}_{iSB1} = \int \zeta_i \hat{f}_{SB1}\mathbf{d}\boldsymbol{\zeta}, \tag{12b}$$

$$\frac{3}{2}\hat{\rho}_{SB0}\hat{T}_{SB1} = \int \zeta_i^2 \hat{f}_{SB1}\mathbf{d}\boldsymbol{\zeta} - \frac{3}{2}\hat{\rho}_{SB1}\hat{T}_{SB0}, \tag{12c}$$

$$\hat{p}_{SB1} = \hat{\rho}_{SB0}\hat{T}_{SB1} + \hat{\rho}_{SB1}\hat{T}_{SB0}, \tag{12d}$$

$$\hat{p}_{ijSB1} = 2\int \zeta_i\zeta_j \hat{f}_{SB1}\mathbf{d}\boldsymbol{\zeta}, \tag{12e}$$

$$\hat{q}_{iSB1} = \int \zeta_i\zeta_j^2 \hat{f}_{SB1}\mathbf{d}\boldsymbol{\zeta} - \frac{3}{2}\hat{\rho}_{SB0}\hat{T}_{SB0}\hat{v}_{iSB1} - \hat{p}_{ijSB0}\hat{v}_{jSB1}, \tag{12f}$$

$$\cdots\cdots\cdots\cdots ,$$

where the condition (10) is used.

Now return to obtaining the SB solution. Substituting Eq. (9) into the Boltzmann equation (4a) and arranging the same order terms of k, we obtain a series of integral equations for the component function $\hat{f}_{SBm}$:

$$\hat{J}(\hat{f}_{SB0}, \hat{f}_{SB0}) = 0, \tag{13}$$

$$2\hat{J}(\hat{f}_{SB0}, \hat{f}_{SBm}) = \zeta_i \frac{\partial \hat{f}_{SBm-1}}{\partial x_i} - \sum_{r=1}^{m-1} \hat{J}(\hat{f}_{SBr}, \hat{f}_{SBm-r}) + \mathcal{H}_3 \hat{g}_{i2} \frac{\partial \hat{f}_{SBm-3}}{\partial \zeta_i} \quad (m \geq 1), \tag{14}$$

where the $\sum$ term is absent when $m = 1$, and $\mathcal{H}_3 = 1$ for $m \geq 3$ and $\mathcal{H}_3 = 0$ for $m \leq 2$.

The solution $\hat{f}_{SB0}$ of the integral equation (13) satisfying the condition (10) is given by

$$\hat{f}_{SB0} = \frac{\hat{\rho}_{SB0}}{(\pi \hat{T}_{SB0})^{3/2}} \exp\left(-\frac{\zeta_i^2}{\hat{T}_{SB0}}\right), \tag{15}$$

where the relations (11a) and (11c) are used. The solution (15) is incomplete to determine $\hat{f}_{SB0}$, because the spatial variations of the parameter functions $\hat{\rho}_{SB0}$ and $\hat{T}_{SB0}$ are not specified. With this $\hat{f}_{SB0}$, the equation (14) is the inhomogeneous linear integral equation for $\hat{f}_{SBm}$ $(m \geq 1)$. The homogeneous equation corresponding to Eq. (14), i.e.,

$$\hat{J}(\hat{f}_{SB0}, \hat{f}_{SB0}\psi) = 0, \tag{16}$$

has five independent solutions:

$$\psi = 1, \quad \zeta_i, \quad \zeta_i^2, \tag{17}$$

which is seen from the relations $\psi' + \psi'_* = \psi + \psi_*$ and $\hat{f}'_{SB0}\hat{f}'_{SB0*} = \hat{f}_{SB0}\hat{f}_{SB0*}$. From the general relation $\int \psi \hat{J}(\hat{f}_{SBm}, \hat{f}_{SBn})\mathbf{d}\boldsymbol{\zeta} = 0$ of the collision integral $\hat{J}$, the inhomogeneous term of the integral equation (14) must satisfy the following relation (solvability condition) for Eq. (14) to have a solution:

$$\int (1, \ \zeta_i, \ \zeta_j^2)\zeta_k \frac{\partial \hat{f}_{SBm-1}}{\partial x_k}\mathbf{d}\boldsymbol{\zeta} - \mathcal{H}_3(0, \ \hat{g}_{i2}\hat{p}_{SBm-3}, \ 2\hat{g}_{j2}(\hat{p}_{SB}\hat{v}_{jSB})_{m-3}) = 0, \tag{18}$$

where the notation $(\cdots)_m$ indicates the m-th order component function of the SB expansion, for example $(\hat{\rho}_{SB}\hat{v}_{iSB}^2)_3 = 2\hat{\rho}_{SB0}\hat{v}_{iSB1}\hat{v}_{iSB2} + \hat{\rho}_{SB1}\hat{v}_{iSB1}^2$.

The solvability condition (18) being satisfied, the solution of the integral equation (14) is expressed in the form:

$$\hat{f}_{SBm} = \hat{f}_{SB0}(c_{0m} + c_{im}\zeta_i + c_{4m}\zeta_i^2) + \hat{f}_{SBPm}, \tag{19}$$

where $\hat{f}_{SBPm}$ is the particular solution satisfying the orthogonal relation

$$\int \psi \hat{f}_{SBPm}\mathbf{d}\boldsymbol{\zeta} = 0, \tag{20}$$

and

$$\left.\begin{aligned} c_{0m} &= \frac{5\hat{\rho}_{SBm}}{2\hat{\rho}_{SB0}} - \frac{3\hat{p}_{SBm}}{2\hat{p}_{SB0}} - \frac{(\hat{\rho}_{SB}\hat{v}_{iSB}^2)_m}{\hat{p}_{SB0}}, \quad c_{im} = \frac{2(\hat{\rho}_{SB}\hat{v}_{iSB})_m}{\hat{p}_{SB0}}, \\ c_{4m} &= \frac{1}{\hat{T}_{SB0}}\left[\frac{\hat{p}_{SBm}}{\hat{p}_{SB0}} - \frac{\hat{\rho}_{SBm}}{\hat{\rho}_{SB0}} + \frac{2(\hat{\rho}_{SB}\hat{v}_{iSB}^2)_m}{3\hat{p}_{SB0}}\right]. \end{aligned}\right\} \tag{21}$$

More explicitly, the inhomogeneous term of Eq. (14) for $m = 1$ is

$$\zeta_i \frac{\partial \hat{f}_{SB0}}{\partial x_i} = \zeta_i \left[\frac{1}{\hat{p}_{SB0}}\frac{\partial \hat{p}_{SB0}}{\partial x_i} + \frac{1}{\hat{T}_{SB0}}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\left(\frac{\zeta_j^2}{\hat{T}_{SB0}} - \frac{3}{2}\right)\right]\hat{f}_{SB0}. \tag{22}$$

The two relations for $\psi = 1$ and $\psi = \zeta_i^2$ in the solvability condition (18) for $m = 1$ are reduced to identities, and the relation for $\psi = \zeta_i$ is

$$\frac{\partial \hat{p}_{SB0}}{\partial x_i} = 0. \tag{23}$$

Then, the inhomogeneous term (22) is reduced to

$$\zeta_i \frac{\partial \hat{f}_{SB0}}{\partial x_i} = \frac{\zeta_i}{\hat{T}_{SB0}}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\left(\frac{\zeta_j^2}{\hat{T}_{SB0}} - \frac{5}{2}\right)\hat{f}_{SB0} = \frac{\hat{p}_{SB0}}{\hat{T}_{SB0}^2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\tilde{\zeta}_i\left(\tilde{\zeta}^2 - \frac{5}{2}\right)E(\tilde{\zeta}), \tag{24}$$

where

$$\tilde{\zeta}_i = \frac{\zeta_i}{\hat{T}_{SB0}^{1/2}}, \quad \tilde{\zeta} = (\tilde{\zeta}_j^2)^{1/2}, \qquad E(\tilde{\zeta}) = \frac{1}{\pi^{3/2}}\exp(-\tilde{\zeta}^2).$$

Now putting $\hat{f}_{SBm}$ in the form

$$\hat{f}_{SBm} = \hat{f}_{SB0}\phi_m(x_i, \tilde{\zeta}_i) = \frac{\hat{p}_{SB0}}{\hat{T}_{SB0}^{3/2}}E(\tilde{\zeta})\phi_m(x_i, \tilde{\zeta}_i), \tag{25}$$

we express the collision integral $\hat{J}(\hat{f}_{SB0}, \hat{f}_{SBm})$ in Eq. (14) in terms of the linearized collision integral of the function of $\phi_m(x_i, \tilde{\zeta}_i)$, that is,

$$\hat{J}(\hat{f}_{SB0}, \hat{f}_{SBm}) = \frac{\hat{p}_{SB0}^2}{2\hat{T}_{SB0}}E(\tilde{\zeta})\mathcal{L}_{\hat{T}_{SB0}}(\phi_m(x_i, \tilde{\zeta}_i)), \tag{26}$$

where $\mathcal{L}_{\hat{T}_{SB0}}(\phi_m(x_i, \tilde{\zeta}_i))$ is the linearized collision integral defined by

$$\mathcal{L}_{\hat{T}_{SB0}}(\phi(\tilde{\zeta}_i)) = \int E(\tilde{\zeta}_*)(\phi' + \phi'_* - \phi - \phi_*)\hat{B}_{\hat{T}_{SB0}}(|\alpha_i(\tilde{\zeta}_{i*} - \tilde{\zeta}_i)|/|\tilde{\zeta}_{j*} - \tilde{\zeta}_j|, \ |\tilde{\zeta}_{i*} - \tilde{\zeta}_i|)\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\tilde{\boldsymbol{\zeta}}_*, \tag{27a}$$

$$\hat{B}_{\hat{T}_{SB0}}(|\alpha_i(\tilde{\zeta}_{i*} - \tilde{\zeta}_i)|/|\tilde{\zeta}_{j*} - \tilde{\zeta}_j|, \ |\tilde{\zeta}_{i*} - \tilde{\zeta}_i|) = \hat{T}_{SB0}^{-1/2}\hat{B}(|\alpha_j(\zeta_{j*} - \zeta_j)|/|\zeta_{i*} - \zeta_i|, |\zeta_{i*} - \zeta_i|\hat{T}_{SB0}^{1/2}), \tag{27b}$$

$$\phi = \phi(\tilde{\zeta}_i),\quad \phi_* = \phi(\tilde{\zeta}_{i*}),\quad \phi' = \phi(\tilde{\zeta}_i'),\quad \phi_*' = \phi(\tilde{\zeta}_{i*}'), \tag{27c}$$

$$\tilde{\zeta}_i' = \tilde{\zeta}_i + \alpha_j(\tilde{\zeta}_{j*} - \tilde{\zeta}_j)\alpha_i,\quad \tilde{\zeta}_{i*}' = \tilde{\zeta}_{i*} - \alpha_j(\tilde{\zeta}_{j*} - \tilde{\zeta}_j)\alpha_i. \tag{27d}$$

Then, from Eqs. (14), (24), and (26) the equation for $\phi_1(x_i, \tilde{\zeta}_i)$ [or $\phi_1(\tilde{\zeta}_i)$ for short] is given in the following form:

$$\mathcal{L}_{\hat{T}_{SB0}}(\phi_1(x_i, \tilde{\zeta}_i)) = \frac{1}{\hat{p}_{SB0}}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\tilde{\zeta}_i\left(\tilde{\zeta}_j^2 - \frac{5}{2}\right). \tag{28}$$

The solution $\phi_1(x_i, \tilde{\zeta}_i)$ of this equation is expressed in the form

$$\phi_1(x_i, \tilde{\zeta}_i) = \frac{\hat{p}_{SB1}}{\hat{p}_{SB0}} + \frac{2\hat{T}_{SB0}^{1/2}\hat{p}_{SB0}\hat{v}_{iSB1}}{\hat{p}_{SB0}}\tilde{\zeta}_i + \frac{\hat{T}_{SB1}}{\hat{T}_{SB0}}\left(\tilde{\zeta}^2 - \frac{5}{2}\right) - \frac{1}{\hat{p}_{SB0}}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\tilde{\zeta}_i\mathcal{A}(\tilde{\zeta}, \hat{T}_{SB0}), \tag{29}$$

where $\mathcal{A}(\tilde{\zeta}, \hat{T}_{SB0})$ is the solution of the following integral equation:

$$\mathcal{L}_a[\zeta_i\mathcal{A}(\zeta, a)] = -\zeta_i\left(\zeta^2 - \frac{5}{2}\right), \tag{30}$$

$$\text{with the subsidiary condition: } \int_0^\infty \zeta^4\mathcal{A}(\zeta, a)E(\zeta)\mathrm{d}\zeta = 0.$$

The function $\mathcal{A}(\zeta, a)$ for a hard-sphere gas, which is independent of a, is tabuleted in Ref. 1. For the BKW (or BGK) model,

$$\mathcal{A}(\zeta, a) = \left(\zeta^2 - \frac{5}{2}\right)a^{1/2}.$$

From this $\hat{f}_{SB1}$, the first term of the inhomogeneous term of Eq. (14) for $m = 2$ is

$$\zeta_i\frac{\partial \hat{f}_{SB1}}{\partial x_i} = \hat{f}_{SB0}(I + II) = \frac{\hat{p}_{SB0}}{\hat{T}_{SB0}^{3/2}}E(\tilde{\zeta})(I + II), \tag{31}$$

where

$$\begin{aligned}
I = {} & \left(\frac{\hat{T}_{SB0}^{1/2}}{\hat{p}_{SB0}}\frac{\partial \hat{p}_{SB1}}{\partial x_i}\right)\tilde{\zeta}_i + \left(\frac{2}{\hat{p}_{SB0}}\frac{\partial \hat{p}_{SB0}\hat{v}_{jSB1}}{\partial x_i}\right)\tilde{\zeta}_i\tilde{\zeta}_j \\
& + \hat{T}_{SB0}^{1/2}\left[\frac{\hat{p}_{SB1}}{\hat{T}_{SB0}\hat{p}_{SB0}}\frac{\partial \hat{T}_{SB0}}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{\hat{T}_{SB1}}{\hat{T}_{SB0}}\right)\right]\tilde{\zeta}_i\left(\tilde{\zeta}^2 - \frac{5}{2}\right) \\
& + \left(\frac{2\hat{p}_{SB0}\hat{v}_{jSB1}}{\hat{p}_{SB0}}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right)\tilde{\zeta}_i\tilde{\zeta}_j\left(\tilde{\zeta}^2 - \frac{5}{2}\right) + \left(\frac{\hat{T}_{SB1}}{\hat{T}_{SB0}^{3/2}}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right)\tilde{\zeta}_i\left(\tilde{\zeta}^4 - 6\tilde{\zeta}^2 + \frac{25}{4}\right), \\
II = {} & -\left(\frac{1}{\hat{p}_{SB0}\hat{T}_{SB0}^{1/2}}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\frac{\partial \hat{T}_{SB0}}{\partial x_j}\right)\tilde{\zeta}_i\tilde{\zeta}_j\left[\left(\tilde{\zeta}^2 - 3\right)\mathcal{A}(\tilde{\zeta}, \hat{T}_{SB0}) - \frac{1}{2}\tilde{\zeta}\frac{\partial \mathcal{A}(\tilde{\zeta}, \hat{T}_{SB0})}{\partial \tilde{\zeta}} + \hat{T}_{SB0}\frac{\partial \mathcal{A}(\tilde{\zeta}, \hat{T}_{SB0})}{\partial \hat{T}_{SB0}}\right] \\
& - \left(\frac{\hat{T}_{SB0}^{1/2}}{\hat{p}_{SB0}}\frac{\partial^2 \hat{T}_{SB0}}{\partial x_i \partial x_j}\right)\tilde{\zeta}_i\tilde{\zeta}_j\mathcal{A}(\tilde{\zeta}, \hat{T}_{SB0}).
\end{aligned}$$

With this inhomogeneous term, the solvability condition (18) for $m = 2$ gives the following three equations:

$$\frac{\partial \hat{p}_{SB0}\hat{v}_{iSB1}}{\partial x_i} = 0, \tag{32}$$

$$\frac{\partial \hat{p}_{SB1}}{\partial x_i} = 0, \tag{33}$$

$$\hat{p}_{SB0}\hat{v}_{iSB1}\frac{\partial \hat{T}_{SB0}}{\partial x_i} = \frac{1}{2}\frac{\partial}{\partial x_i}\left(\hat{\gamma}_2(\hat{T}_{SB0})\hat{T}_{SB0}^{1/2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right), \tag{34}$$

where $\hat{\gamma}_2(\hat{T}_{SB0})$ is expressed in the following integral of $\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})$:

$$\hat{\gamma}_2(a) = 2I_6(\mathcal{A}(\zeta,a)), \tag{35a}$$

$$I_n(Z) = \frac{8}{15\sqrt{\pi}}\int_0^\infty \zeta^n Z(\zeta)\exp(-\zeta^2)\mathrm{d}\zeta. \tag{35b}$$

For example,

$$\hat{\gamma}_2(\hat{T}_{SB0}) = 1.922284066 \quad \text{(a hard-sphere gas)}, \quad \hat{\gamma}_2(\hat{T}_{SB0}) = \hat{T}_{SB0}^{1/2} \quad \text{(the BKW model)}.$$

The collision integral $\hat{J}(\hat{f}_{SB1},\hat{f}_{SB1})$ in the inhomogeneous term in Eq. (14) for $m=2$ is arranged with the aid of formulas in Ref. 1, and then the whole inhomogeneous term is further arranged with the aid of the solvability conditions (23) and (32)–(34). Thus, we obtain the equation for $\phi_2(x_i,\tilde{\zeta}_i)$ [or $\phi_2(\tilde{\zeta}_i)$ for short] in the following form:

$$\begin{aligned}
&\mathcal{L}_{\hat{T}_{SB0}}(\phi_2(\tilde{\zeta}_i)) \\
&= -\frac{1}{2}\left(\frac{\hat{T}_{SB1}}{\hat{T}_{SB0}}\right)^2 \mathcal{L}_{\hat{T}_{SB0}}(\tilde{\zeta}^2(1-\tilde{\zeta}^2)) - \frac{2\hat{T}_{SB0}^{1/2}\hat{p}_{SB0}\hat{v}_{iSB1}}{\hat{p}_{SB0}}\frac{\hat{T}_{SB1}}{\hat{T}_{SB0}}\mathcal{L}_{\hat{T}_{SB0}}(\tilde{\zeta}_i(1-\tilde{\zeta}^2)) \\
&- \left(\frac{\hat{T}_{SB0}^{1/2}\hat{\rho}_{SB0}}{\hat{p}_{SB0}}\right)^2 \hat{v}_{iSB1}\hat{v}_{jSB1}\mathcal{L}_{\hat{T}_{SB0}}(\delta_{ij}-2\tilde{\zeta}_i\tilde{\zeta}_j) \\
&+ \frac{\hat{\rho}_{SB0}\hat{v}_{iSB1}\hat{T}_{SB0}^{1/2}}{\hat{p}_{SB0}^2}\frac{\partial \hat{T}_{SB0}}{\partial x_j}\mathcal{L}_{\hat{T}_{SB0}}\left(\frac{\tilde{\zeta}_i\tilde{\zeta}_j}{\tilde{\zeta}}\frac{\partial \mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})}{\partial\tilde{\zeta}} - (2\tilde{\zeta}_i\tilde{\zeta}_j-\delta_{ij})\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})\right) \\
&+ \frac{1}{\hat{p}_{SB0}}\frac{\hat{T}_{SB1}}{\hat{T}_{SB0}}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\mathcal{L}_{\hat{T}_{SB0}}\left(\tilde{\zeta}_i\left[\frac{1}{2}\tilde{\zeta}\frac{\partial \mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})}{\partial\tilde{\zeta}} - \left(\tilde{\zeta}^2-3\right)\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0}) - \hat{T}_{SB0}\frac{\partial \mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})}{\partial \hat{T}_{SB0}}\right]\right) \\
&+ \frac{1}{\hat{p}_{SB0}}\frac{\partial \hat{T}_{SB1}}{\partial x_i}\underbrace{\tilde{\zeta}_i\left(\tilde{\zeta}^2-\frac{5}{2}\right)} + 2\left(\frac{\hat{T}_{SB0}^{1/2}}{\hat{p}_{SB0}}\frac{\partial \hat{v}_{jSB1}}{\partial x_i}\right)\underbrace{\left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)} \\
&- \frac{1}{3\hat{p}_{SB0}^2}\left(\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right)\left(\frac{\partial \hat{T}_{SB0}}{\partial x_j}\right)\left[\frac{3}{2}\underbrace{\mathrm{Iha1}} + \delta_{ij}\underbrace{\mathrm{Iha2}} + \delta_{ij}\underbrace{\mathrm{Iha3}} + \delta_{ij}\underbrace{\mathrm{Iha4}} + 3\mathcal{J}_{\hat{T}_{SB0}}(\tilde{\zeta}_i\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0}),\tilde{\zeta}_j\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0}))\right] \\
&- \frac{\hat{T}_{SB0}}{3\hat{p}_{SB0}^2}\frac{\partial^2 \hat{T}_{SB0}}{\partial x_i \partial x_j}(3\underbrace{\mathrm{Ihb1}} + \delta_{ij}\underbrace{\mathrm{Ihb2}}),
\end{aligned} \tag{36}$$

where

$$\mathcal{J}_{\hat{T}_{SB0}}(\phi(\tilde{\zeta}_i),\psi(\tilde{\zeta}_i)) = \frac{1}{2}\int E(\tilde{\zeta}_*)(\phi'_*\psi' + \phi'\psi'_* - \phi_*\psi - \phi\psi_*)\hat{B}_{\hat{T}_{SB0}}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathrm{d}\tilde{\boldsymbol{\zeta}}_*,$$

with ϕ'_*, ϕ', ϕ_*, ϕ, and $\hat{B}_{\hat{T}_{SB0}}$ defined by Eqs. (27b) and (27c), and

$$\mathrm{Iha1} = \left(\tilde{\zeta}_j\tilde{\zeta}_i - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)\left(2\left(\tilde{\zeta}^2-3\right)\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0}) - \tilde{\zeta}\frac{\partial \mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})}{\partial\tilde{\zeta}} + 2\hat{T}_{SB0}\frac{\partial \mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})}{\partial \hat{T}_{SB0}}\right),$$

$$\mathrm{Iha2} = \tilde{\zeta}^2\left(\tilde{\zeta}^2-\frac{7}{2}\right)\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0}) - \frac{1}{2}\tilde{\zeta}^3\frac{\partial \mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})}{\partial\tilde{\zeta}},$$

$$\mathrm{Iha3} = \frac{1}{2}\left(\tilde{\zeta}^2\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0}) - \frac{5}{2}\hat{\gamma}_2(\hat{T}_{SB0})\left(\tilde{\zeta}^2-\frac{3}{2}\right)\right),$$

$$\mathrm{Iha4} = \hat{T}_{SB0}\tilde{\zeta}^2\frac{\partial \mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})}{\partial \hat{T}_{SB0}} - \frac{5}{2}\hat{T}_{SB0}\frac{\mathrm{d}\hat{\gamma}_2(\hat{T}_{SB0})}{\mathrm{d}\hat{T}_{SB0}}\left(\tilde{\zeta}^2-\frac{3}{2}\right),$$

and

$$\mathrm{Ihb1} = \left(\tilde{\zeta}_j\tilde{\zeta}_i - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0}), \quad \mathrm{Ihb2} = \tilde{\zeta}^2\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0}) - \frac{5}{2}\hat{\gamma}_2(\hat{T}_{SB0})\left(\tilde{\zeta}^2-\frac{3}{2}\right) = 2\mathrm{Iha3}.$$

Here, each of the inhomogeneous terms marked by $\underbrace{***}$, as well as the terms expressed by the operator $\mathcal{L}_{\hat{T}_{SB0}}$, satisfies the solvability condition (18).

The solution of the integral equation (36) is expressed in the following form:

$$
\begin{aligned}
\phi_2(\tilde{\zeta}_i) = {} & \frac{\hat{\rho}_{SB2}}{\hat{\rho}_{SB0}} + \frac{2(\hat{\rho}_{SB1}\hat{v}_{iSB1} + \hat{\rho}_{SB0}\hat{v}_{iSB2})\hat{T}_{SB0}^{1/2}}{\hat{p}_{SB0}}\tilde{\zeta}_i \\
& + \left[\frac{\hat{p}_{SB2}}{\hat{p}_{SB0}} - \frac{\hat{\rho}_{SB2}}{\hat{\rho}_{SB0}} + \frac{2\hat{\rho}_{SB0}\hat{v}_{iSB1}^2}{3\hat{p}_{SB0}}\right]\left(\tilde{\zeta}^2 - \frac{3}{2}\right) \\
& + \frac{1}{2}\left(\frac{\hat{T}_{SB1}}{\hat{T}_{SB0}}\right)^2 \underleftrightarrow{\left(\tilde{\zeta}^4 - 5\tilde{\zeta}^2 + \frac{15}{4}\right)} + \frac{2\hat{T}_{SB0}^{1/2}\hat{\rho}_{SB0}\hat{v}_{iSB1}}{\hat{p}_{SB0}}\frac{\hat{T}_{SB1}}{\hat{T}_{SB0}}\underleftrightarrow{\tilde{\zeta}_i\left(\tilde{\zeta}^2 - \frac{5}{2}\right)} \\
& + 2\left(\frac{\hat{T}_{SB0}^{1/2}\hat{\rho}_{SB0}}{\hat{p}_{SB0}}\right)^2 \hat{v}_{iSB1}\hat{v}_{jSB1}\underleftrightarrow{\left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)} \\
& + \frac{\hat{\rho}_{SB0}\hat{v}_{iSB1}\hat{T}_{SB0}^{1/2}}{\hat{p}_{SB0}^2}\frac{\partial \hat{T}_{SB0}}{\partial x_j}\left(\frac{\tilde{\zeta}_i\tilde{\zeta}_j}{\tilde{\zeta}}\frac{\partial \mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})}{\partial \tilde{\zeta}} - (2\tilde{\zeta}_i\tilde{\zeta}_j - \delta_{ij})\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})\right) \\
& + \frac{1}{\hat{p}_{SB0}}\frac{\hat{T}_{SB1}}{\hat{T}_{SB0}}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\left(\tilde{\zeta}_i\left[\frac{1}{2}\tilde{\zeta}\frac{\partial \mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})}{\partial \tilde{\zeta}} - \left(\tilde{\zeta}^2 - 3\right)\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0}) - \hat{T}_{SB0}\frac{\partial \mathcal{A}(\tilde{\zeta},\hat{T}_{SB0})}{\partial \hat{T}_{SB0}}\right]\right) \\
& - \frac{1}{\hat{p}_{SB0}}\frac{\partial \hat{T}_{SB1}}{\partial x_i}\tilde{\zeta}_i\mathcal{A}(\tilde{\zeta},\hat{T}_{SB0}) - \left(\frac{\hat{T}_{SB0}^{1/2}}{\hat{p}_{SB0}}\frac{\partial \hat{v}_{jSB1}}{\partial x_i}\right)\left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)\mathcal{B}(\tilde{\zeta},\hat{T}_{SB0}) \\
& + \frac{1}{2\hat{p}_{SB0}^2}\left(\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right)\left(\frac{\partial \hat{T}_{SB0}}{\partial x_j}\right)\left[\left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)\mathcal{B}_2 + \delta_{ij}\mathcal{N}^A\right] \\
& + \frac{\hat{T}_{SB0}}{\hat{p}_{SB0}^2}\frac{\partial^2 \hat{T}_{SB0}}{\partial x_i \partial x_j}\left[\left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)\mathcal{B}_1 + \delta_{ij}\mathcal{N}^B\right],
\end{aligned} \tag{37}
$$

where $\mathcal{B}(\tilde{\zeta},\hat{T}_{SB0})$, $\mathcal{B}_1(\tilde{\zeta},\hat{T}_{SB0})$, $\mathcal{B}_2(\tilde{\zeta},\hat{T}_{SB0})$, $\mathcal{N}^A(\tilde{\zeta},\hat{T}_{SB0})$, and $\mathcal{N}^B(\tilde{\zeta},\hat{T}_{SB0})$ are defined in Appendix A. The first six terms on the right-hand side are the second-order terms of the local Maxwellian. The terms marked by $\underleftrightarrow{***}$ are obtained by modifying the obvious solutions known from the form of their inhomogeneous terms expressed by $\mathcal{L}_{\hat{T}_{SB0}}$ operator with the solutions of the corresponding homogeneous equation in order for the orthogonal condition to be satisfied.

We proceed with the analysis in a similar way. Then, from the solvability condition (18) for $m = 3$, we obtain the following equations:

$$
\frac{\partial \hat{\rho}_{SB0}\hat{v}_{iSB2}}{\partial x_i} + \frac{\partial \hat{\rho}_{SB1}\hat{v}_{iSB1}}{\partial x_i} = 0, \tag{38}
$$

$$
\begin{aligned}
\hat{\rho}_{SB0}\hat{v}_{jSB1}\frac{\partial \hat{v}_{iSB1}}{\partial x_j} = {} & -\frac{1}{2}\frac{\partial \hat{p}_{SB2}}{\partial x_i} + \hat{\rho}_{SB0}\hat{g}_{i2} \\
& + \frac{1}{2}\frac{\partial}{\partial x_j}\left[\hat{\gamma}_1(\hat{T}_{SB0})\hat{T}_{SB0}^{1/2}\left(\frac{\partial \hat{v}_{iSB1}}{\partial x_j} + \frac{\partial \hat{v}_{jSB1}}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{v}_{kSB1}}{\partial x_k}\delta_{ij}\right)\right] \\
& - \frac{1}{2\hat{p}_{SB0}}\frac{\partial}{\partial x_j}\left\{\hat{\gamma}_7(\hat{T}_{SB0})\left[\frac{\partial \hat{T}_{SB0}}{\partial x_i}\frac{\partial \hat{T}_{SB0}}{\partial x_j} - \frac{1}{3}\left(\frac{\partial \hat{T}_{SB0}}{\partial x_k}\right)^2\delta_{ij}\right]\right\} \\
& - \frac{1}{2\hat{p}_{SB0}}\frac{\partial}{\partial x_j}\left[\hat{\gamma}_3(\hat{T}_{SB0})\hat{T}_{SB0}\left(\frac{\partial^2 \hat{T}_{SB0}}{\partial x_i \partial x_j} - \frac{1}{3}\frac{\partial^2 \hat{T}_{SB0}}{\partial x_k^2}\delta_{ij}\right)\right],
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
& \hat{\rho}_{SB0}\hat{v}_{iSB1}\frac{\partial \hat{T}_{SB1}}{\partial x_i} + (\hat{\rho}_{SB0}\hat{v}_{iSB2} + \hat{\rho}_{SB1}\hat{v}_{iSB1})\frac{\partial \hat{T}_{SB0}}{\partial x_i} \\
& \quad = \frac{1}{2}\frac{\partial}{\partial x_i}\left(\hat{\gamma}_2(\hat{T}_{SB0})\hat{T}_{SB0}^{1/2}\frac{\partial \hat{T}_{SB1}}{\partial x_i} + \hat{T}_{SB1}\frac{\mathrm{d}\hat{\gamma}_2(\hat{T}_{SB0})\hat{T}_{SB0}^{1/2}}{\mathrm{d}\hat{T}_{SB0}}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right),
\end{aligned} \tag{40}
$$

where $\hat{\gamma}_1(\hat{T}_{SB0})$, $\hat{\gamma}_3(\hat{T}_{SB0})$, and $\hat{\gamma}_7(\hat{T}_{SB0})$, related to transport coefficients, are defined by the following integrals [see Eq. (35b]:

$$\hat{\gamma}_1(a) = I_6(\mathcal{B}(\zeta, a)),\ \hat{\gamma}_3(a) = 2I_6(\mathcal{B}_1(\zeta, a)),\ \hat{\gamma}_7(a) = I_6(\mathcal{B}_2(\zeta, a)).$$

For a hard-sphere gas,

$$\hat{\gamma}_1(\hat{T}_{SB0}) = 1.270042427, \quad \hat{\gamma}_3(\hat{T}_{SB0}) = 1.947906335, \quad \hat{\gamma}_7(\hat{T}_{SB0}) = 0.189201,$$

and for the BKW model,

$$\hat{\gamma}_1(\hat{T}_{SB0}) = \hat{T}_{SB0}^{1/2}, \quad \hat{\gamma}_3(\hat{T}_{SB0}) = \hat{T}_{SB0}, \quad \hat{\gamma}_7(\hat{T}_{SB0}) = \hat{T}_{SB0}.$$

Now, at the stage of the solvability condition (18) for $m = 3$, the equations that determine the component functions of the macroscopic variables at the leading order are lined up. From Eqs. (23) and (33), which are required for the flow velocity $\hat{v}_i$ to be a small quantity of the order of k, $\hat{p}_{SB0}$ and $\hat{p}_{SB1}$ are constants (say, $\hat{p}_0$ and $\hat{p}_1$):

$$\hat{p}_{SB0} = \hat{p}_0, \quad \hat{p}_{SB1} = \hat{p}_1, \tag{41}$$

from which

$$\hat{\rho}_{SB0} = \frac{\hat{p}_0}{\hat{T}_{SB0}}, \quad \hat{\rho}_{SB1} = \frac{\hat{p}_1 - \hat{\rho}_{SB0}\hat{T}_{SB1}}{\hat{T}_{SB0}}, \tag{42}$$

with the aid of the equations of state (11d) and (12d). Equations (32), (34), and (39), which are derived from the solvability condition (18) for ($m = 2$, $\psi = 1$ and ζ_i^2) and ($m = 3$, $\psi = \zeta_i$), contain the component functions $\hat{\rho}_{SB0}, \hat{T}_{SB0}, \hat{v}_{iSB1}$, and $\hat{p}_{SB2}$, but from Eq. (42), they are the equations for $\hat{T}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{p}_{SB2}$. Generally, the set of equations derived from the solvability condition (18) for ($m = s+2$, $\psi = 1$ and ζ_i^2) and ($m = s+3$, $\psi = \zeta_i$) contains the functions $\hat{\rho}_{SBs}$, $\hat{T}_{SBs}$, $\hat{v}_{iSBs+1}$, and $\hat{p}_{SBs+2}$ as well as functions appeared in the equations at the previous stages [or the functions $\hat{\rho}_{SBr}$, $\hat{T}_{SBr}$, $\hat{v}_{iSBr+1}$, and $\hat{p}_{SBr+2}$ ($r \leq s-1$)]. Thus, with the aid of the expanded form of the equation of state (8d), the staggered combination of functions $\hat{\rho}_{SBs}$, $\hat{T}_{SBs}$, $\hat{v}_{iSBs+1}$, and $\hat{p}_{SBs+2}$ is determined consistently and successively from the lowest order by the rearranged sets of equations given by the solvability condition (18).

The set of equations for $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{p}_{SB2}$ has a striking feature. That is, the leading temperature field $\hat{T}_{SB0}$ is determined together with the next-order velocity component $\hat{v}_{iSB1}$. This is an important result related to the incompleteness of the classical gas dynamics (ghost effect), which is discussed in detail in Ref. 2. Furthermore, the gravity, which vanishes in the continuum limit, enters Eq. (39) or the set of equations for $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{p}_{SB2}$. This is another ghost effect and its example will be presented in Section 3. The presentation of this ghost effect and its combination of the first one is the purpose of the present study. The component function $\hat{f}_{SBm}$ of the velocity distribution function is determined by the macroscopic variables $\hat{\rho}_{SBs}$, $\hat{T}_{SBs}$, $\hat{v}_{iSBs}$, and $\hat{p}_{SBs}$ ($s \leq m$). The leading component function $\hat{f}_{SB0}$ is the Maxwellian at rest with parameters $\hat{\rho}_{SB0}$ and $\hat{T}_{SB0}$, i.e.,

$$\hat{f}_{SB0} = \frac{\hat{\rho}_{SB0}}{(\pi\hat{T}_{SB0})^{3/2}} \exp\left(-\frac{\zeta_i^2}{\hat{T}_{SB0}}\right). \tag{43}$$

However, the parameter $\hat{T}_{SB0}$ is not determined by the Euler set of equations. We have already seen this type of example in Refs. 4 and 5. Furthermore, in the present case it is determined together with the higher-order variable $\hat{v}_{iSB1}$ and parameter $\hat{g}_{i2}$.

From $\hat{f}_{SBm}$ obtained [Eqs. (15) and (25) with (29) and (37)], the component functions $\hat{p}_{ijSBm}$ and $\hat{q}_{iSBm}$ of the stress tensor and heat-flow vector are easily obtained as follows:

$$\hat{p}_{ijSB0} = \hat{p}_{SB0}\delta_{ij}, \tag{44a}$$

$$\hat{p}_{ijSB1} = \hat{p}_{SB1}\delta_{ij}, \tag{44b}$$

$$\begin{aligned}\hat{p}_{ijSB2} = \hat{p}_{SB2}\delta_{ij} &- \hat{\gamma}_1\hat{T}_{SB0}^{1/2}\left(\frac{\partial\hat{v}_{iSB1}}{\partial x_j} + \frac{\partial\hat{v}_{jSB1}}{\partial x_i} - \frac{2}{3}\frac{\partial\hat{v}_{kSB1}}{\partial x_k}\delta_{ij}\right)\\ &+ \frac{\hat{\gamma}_7}{\hat{p}_{SB0}}\left[\frac{\partial\hat{T}_{SB0}}{\partial x_i}\frac{\partial\hat{T}_{SB0}}{\partial x_j} - \frac{1}{3}\left(\frac{\partial\hat{T}_{SB0}}{\partial x_k}\right)^2\delta_{ij}\right] + \frac{\hat{\gamma}_3\hat{T}_{SB0}}{\hat{p}_{SB0}}\left(\frac{\partial^2\hat{T}_{SB0}}{\partial x_i\partial x_j} - \frac{1}{3}\frac{\partial^2\hat{T}_{SB0}}{\partial x_k^2}\delta_{ij}\right),\end{aligned} \tag{44c}$$

$$\hat{q}_{iSB0} = 0, \tag{45a}$$

$$\hat{q}_{iSB1} = -\frac{5}{4}\hat{\gamma}_2 \hat{T}_{SB0}^{1/2} \frac{\partial \hat{T}_{SB0}}{\partial x_i}, \tag{45b}$$

$$\hat{q}_{iSB2} = -\frac{5}{4}\left(\hat{\gamma}_2 \hat{T}_{SB0}^{1/2} \frac{\partial \hat{T}_{SB1}}{\partial x_i} + \hat{T}_{SB1} \frac{\mathrm{d}\hat{\gamma}_2 \hat{T}_{SB0}^{1/2}}{\mathrm{d}\hat{T}_{SB0}} \frac{\partial \hat{T}_{SB0}}{\partial x_i}\right). \tag{45c}$$

The term with the factor $\hat{\gamma}_1$ in $\hat{p}_{ijSB2}$ is the viscous stress, due to the first-order velocity field $\hat{v}_{iSB1}$, given by the Newton law, and the terms with factor $\hat{\gamma}_2$ in $\hat{q}_{iSBm}$ are the heat flow by the Fourier law. The $\hat{\gamma}_1 \hat{T}_{SB0}^{1/2}$ and $\hat{\gamma}_2 \hat{T}_{SB0}^{1/2}$ are, respectively, the (nondimensional) viscosity and thermal conductivity of the gas, and $\hat{T}_{SB1}\mathrm{d}\hat{\gamma}_2 \hat{T}_{SB0}^{1/2}/\mathrm{d}\hat{T}_{SB0}$ in $\hat{q}_{iSB2}$ is due to the temperature dependence of the thermal conductivity. The third and fourth terms in $\hat{p}_{ijSB2}$, as a whole, are called thermal stress, and are the source of Kogan's flow.[6]

2.3 Knudsen-Layer Analysis and Boundary Condition for SB Solution

In the previous section, we have derived the set of fluid-dynamic-type equations describing the behavior of the gas in the continuum limit, putting aside the boundary condition. The problem is discussed here.

The leading term of the SB solution $\hat{f}_{SB0}$ is Maxwellian without flow [Eq. (43)]. This distribution satisfy the diffuse reflection condition (6a) if the boundary value of T_{SB0} is taken as T_{w0}:

$$T_{SB0} = T_{w0} \quad \text{on a boundary.} \tag{46}$$

The next-order distribution $\hat{f}_{SB1}$, which is not Maxwellian, cannot be made to satisfy the diffuse reflection boundary condition, which is the corresponding part of Maxwellian. Thus, we introduce the correction in a neighborhood of the boundary, i.e., a Knudsen-layer correction, to the SB solution. That is, we put the solution $\hat{f}$ in the form

$$\hat{f} = \hat{f}_{SB} + \hat{f}_K, \tag{47}$$

where $\hat{f}_K$ is the Knudsen-layer solution, for which the condition on the SB solution is loosened. That is, the length scale of variation of $\hat{f}_K$ in the direction normal to the boundary is of the order of the mean free path [i.e., $n_i \partial \hat{f}_K/\partial x_i = O(\hat{f}_K)$], and $\hat{f}_K$ is assumed to be appreciable only in a thin layer, with thickness of the order of the mean free path, adjacent to the boundary.

Here, the following Knudsen-layer coordinates are introduced:

$$x_i = k\eta n_i(\chi_1, \chi_2) + x_{wi}(\chi_1, \chi_2), \tag{48}$$

where x_{wi} is the boundary surface, η is a stretched coordinate normal to the boundary, χ_1 and χ_2 are (unstretched) coordinates within a parallel surface $\eta = $ const, and the normal vector n_i is a function of χ_1 and χ_2. The Knudsen-layer correction $\hat{f}_K$ is expanded in a power series of k:

$$\hat{f}_K = \hat{f}_{K1} k + \cdots, \tag{49}$$

where the series starts from the order of k, since the diffuse reflection condition is satisfied by $\hat{f}_{SB0}$ at the order of unity. The expansion of $\hat{f}_{SB}$ in Eq. (9) is reshuffled here, since the following power-series expansion in $k\eta$ can be applied in the Knudsen layer, where $\eta = O(1)$:

$$\hat{f}_{SB} = (\hat{f}_{SB0})_0 + \left[(\hat{f}_{SB1})_0 + \left(n_i \frac{\partial \hat{f}_{SB0}}{\partial x_i}\right)_0 \eta\right] k + \cdots, \tag{50}$$

where the quantities in the parentheses with subscript 0, $(\cdots)_0$, are evaluated on the boundary.

Substituting the split form (47) with the series (50) and (49) into the Boltzmann equation (4a) and rewriting in the Knudsen-layer variables (48), we obtain the series of equations for $\hat{f}_{Km}$

$$\zeta_i n_i \frac{\partial \hat{f}_{K1}}{\partial \eta} = 2\hat{J}((\hat{f}_{SB0})_0, \hat{f}_{K1}), \tag{51}$$

$$\cdots\cdots\cdots.$$

The sum $\hat{f}_{SB} + \hat{f}_K$ being substituted into the diffuse reflection condition (6a) and the result being expanded in k, the boundary condition for $\hat{f}_{Km}$ on the boundary is obtained. That is, at $\eta = 0$,

$$\hat{f}_{K1} = \hat{f}_{SB0}\left[\frac{\hat{\sigma}_{w1} - \hat{\rho}_{SB1}}{\hat{\rho}_{SB0}} + \frac{2\zeta_i(\hat{v}_{wi1} - \hat{v}_{iSB1})}{\hat{T}_{w0}} + \left(\frac{\zeta_i^2}{\hat{T}_{w0}} - \frac{3}{2}\right)\frac{\hat{T}_{w1} - \hat{T}_{SB1}}{\hat{T}_{w0}} + \frac{\zeta_i \mathcal{A}(\zeta/\hat{T}_{w0}^{1/2}, \hat{T}_{w0})}{\hat{T}_{w0}^{1/2}\hat{p}_0}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right] \quad (\zeta_i n_i > 0), \tag{52}$$

where

$$\frac{\hat{\sigma}_{w1}}{\hat{\rho}_{SB0}} = \frac{\hat{\rho}_{SB1}}{\hat{\rho}_{SB0}} - \frac{\hat{T}_{w1} - \hat{T}_{SB1}}{2\hat{T}_{w0}} - \frac{\sqrt{\pi}\hat{v}_{iSB1}n_i}{\hat{T}_{w0}^{1/2}} - \frac{2\sqrt{\pi}\hat{T}_{w0}^{1/2}}{\hat{p}_0}\int_{\zeta_i n_i < 0} \zeta_i n_i \hat{f}_{K1} \mathbf{d}\boldsymbol{\zeta},$$

$$\cdots\cdots\cdots.$$

The Knudsen-layer correction $\hat{f}_K$ being introduced as the correction to $\hat{f}_{SB}$ in the neighborhood of the boundary, it should vanish as $\eta \to \infty$:

$$\hat{f}_{K1} \to 0 \quad \text{as } \eta \to \infty.$$

Thus, $\hat{f}_{K1}$ is determined by the half-space boundary-value problem of the linearized Boltzmann equation with one-space variable η. The boundary-value problem is considered for more general situation for the BKW equation in Refs. 4 and 5, and the undetermined boundary values $\hat{v}_{iSB1}$ and $\hat{T}_{SB1}$ are related to $\partial\hat{T}_{SB0}/\partial x_i$ for the solution to exist. This is confirmed by mathematical studies of the existence and uniqueness of the solution of the boundary-value problem (e.g., Ref. 7; see also Ref. 1).

The relations are given in the following form:

$$\frac{(\hat{v}_{jSB1} - \hat{v}_{wj1})(\delta_{ij} - n_j n_i)}{\hat{T}_{w0}^{1/2}} = -\frac{\hat{K}_1}{\hat{p}_0}\frac{\partial \hat{T}_{SB0}}{\partial x_j}(\delta_{ij} - n_j n_i), \tag{53a}$$

$$\hat{v}_{jSB1}n_j = 0, \tag{53b}$$

$$\frac{\hat{T}_{SB1} - \hat{T}_{w1}}{\hat{T}_{w0}} = \frac{\hat{d}_1}{\hat{p}_0}\frac{\partial \hat{T}_{SB0}}{\partial x_j}n_j, \tag{53c}$$

where $\hat{K}_1$ and $\hat{d}_1$, which are called, respectively, thermal-creep and temperature-jump coefficients,[8-12] are functions of $\hat{T}_{w0}$ depending on molecular models. For example,

$$\hat{K}_1 = -0.6463, \quad \hat{d}_1 = 2.4001 \quad \text{(a hard-sphere gas)},$$

$$\hat{K}_1/\hat{T}_{w0}^{1/2} = -0.38316, \quad \hat{d}_1/\hat{T}_{w0}^{1/2} = 1.30272 \quad \text{(BKW)}.$$

The relations (53a)–(53c) give the boundary conditions for $\hat{v}_{jSB1}$ and $\hat{T}_{SB1}$.

At this stage, the equations and their associated boundary conditions that determine the behavior of the gas in the continuum limit are lined up. That is, the equations are Eqs. (32), (39), and (34) and the boundary conditions are Eqs. (46), (53a), and (53b).

2.4 Asymptotic Fluid-Dynamic-type Equations and their Boundary Conditions

For the convenience, we summarize the fluid-dynamic-type equations and their associated boundary conditions that describe the behavior of a gas in the continuum limit under the assumptions introduced at the beginning of Section 2.1. The fluid-dynamic-type equations are

$$\frac{\partial \hat{\rho}_{SB0}\hat{v}_{iSB1}}{\partial x_i} = 0, \tag{54}$$

$$\hat{\rho}_{SB0}\hat{v}_{jSB1}\frac{\partial \hat{v}_{iSB1}}{\partial x_j} = -\frac{1}{2}\frac{\partial \hat{p}_{SB2}^*}{\partial x_i} + \hat{\rho}_{SB0}\hat{g}_{i2} + \frac{1}{2}\frac{\partial}{\partial x_j}\left[\Gamma_1(\hat{T}_{SB0})\left(\frac{\partial \hat{v}_{iSB1}}{\partial x_j} + \frac{\partial \hat{v}_{jSB1}}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{v}_{kSB1}}{\partial x_k}\delta_{ij}\right)\right] + \frac{1}{2\hat{p}_0}\frac{\partial}{\partial x_j}\left(\Gamma_7(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_i}\frac{\partial \hat{T}_{SB0}}{\partial x_j}\right), \tag{55}$$

$$\hat{\rho}_{SB0}\hat{v}_{iSB1}\frac{\partial \hat{T}_{SB0}}{\partial x_i} = \frac{1}{2}\frac{\partial}{\partial x_i}\left(\Gamma_2(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right), \tag{56}$$

where

$$\hat{\rho}_{SB0} = \frac{\hat{p}_0}{\hat{T}_{SB0}}, \quad \hat{p}^*_{SB2} = \hat{p}_{SB2} + \frac{2\hat{\gamma}_3\hat{T}_{SB0}}{3\hat{p}_0}\frac{\partial^2 \hat{T}_{SB0}}{\partial x_k^2} + \frac{\bar{\Gamma}_7}{\hat{p}_0}\left(\frac{\partial \hat{T}_{SB0}}{\partial x_k}\right)^2, \tag{57a}$$

$$\Gamma_1(\hat{T}_{SB0}) = \hat{\gamma}_1(\hat{T}_{SB0})\hat{T}^{1/2}_{SB0}, \qquad \Gamma_2(\hat{T}_{SB0}) = \hat{\gamma}_2(\hat{T}_{SB0})\hat{T}^{1/2}_{SB0}, \tag{57b}$$

$$\Gamma_7(\hat{T}_{SB0}) = \frac{\mathrm{d}\hat{\gamma}_3\hat{T}_{SB0}}{\mathrm{d}\hat{T}_{SB0}} - \hat{\gamma}_7, \qquad \bar{\Gamma}_7(\hat{T}_{SB0}) = \frac{\mathrm{d}\hat{\gamma}_3\hat{T}_{SB0}}{\mathrm{d}\hat{T}_{SB0}} - \frac{1}{3}\hat{\gamma}_7. \tag{57c}$$

By the introduction of the quasi-pressure $\hat{p}^*_{SB2}$, Eq. (39) of the third order is reduced to Eq. (55) of the second order. That is, Eq. (39) is a third-order equation only in its appearance. The thermal-stress term (or the third term on the right-hand side) in Eq. (55) can be further reduced to the first order with the aid of Eq. (56). With the new modified pressure $\hat{p}^\dagger_{SB2}$ defined by

$$\begin{aligned}\hat{p}^\dagger_{SB2} &= \hat{p}_{SB2} + \frac{2}{3\hat{p}_0}\frac{\partial}{\partial x_k}\left(\hat{\gamma}_3(\hat{T}_{SB0})\hat{T}_{SB0}\frac{\partial \hat{T}_{SB0}}{\partial x_k}\right) - \frac{\Gamma_7(\hat{T}_{SB0})}{6\hat{p}_0}\left(\frac{\partial \hat{T}_{SB0}}{\partial x_k}\right)^2 \\ &= \hat{p}^*_{SB2} - \frac{\Gamma_7(\hat{T}_{SB0})}{6\hat{p}_0}\left(\frac{\partial \hat{T}_{SB0}}{\partial x_k}\right)^2,\end{aligned} \tag{58}$$

Eq. (55) is rewritten in the following form with the first-order thermal-stress term:

$$\begin{aligned}\hat{\rho}_{SB0}\hat{v}_{jSB1}\frac{\partial \hat{v}_{iSB1}}{\partial x_j} = &-\frac{1}{2}\frac{\partial \hat{p}^\dagger_{SB2}}{\partial x_i} + \hat{\rho}_{SB0}\hat{g}_{i2} + \frac{1}{2}\frac{\partial}{\partial x_j}\left[\Gamma_1\left(\frac{\partial \hat{v}_{iSB1}}{\partial x_j} + \frac{\partial \hat{v}_{jSB1}}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{v}_{kSB1}}{\partial x_k}\delta_{ij}\right)\right] \\ &+ \left[\frac{\Gamma_7}{\Gamma_2}\frac{\hat{v}_{jSB1}}{\hat{T}_{SB0}}\frac{\partial \hat{T}_{SB0}}{\partial x_j} + \frac{\Gamma_2^2}{4\hat{p}_0}\frac{\mathrm{d}\Gamma_7/\Gamma_2^2}{\mathrm{d}\hat{T}_{SB0}}\left(\frac{\partial \hat{T}_{SB0}}{\partial x_j}\right)^2\right]\frac{\partial \hat{T}_{SB0}}{\partial x_i},\end{aligned} \tag{59}$$

where $\Gamma_1 = \Gamma_1(\hat{T}_{SB0})$, $\Gamma_2 = \Gamma_2(\hat{T}_{SB0})$, and $\Gamma_7 = \Gamma_7(\hat{T}_{SB0})$. Incidentally,

$$\Gamma_7 = 1.758705, \quad \bar{\Gamma}_7 = 1.884839 \quad \text{(a hard-sphere gas)},$$

$$\Gamma_7 = \hat{T}_{SB0}, \qquad \bar{\Gamma}_7 = \frac{5}{3}\hat{T}_{SB0} \quad \text{(BKW)}.$$

The boundary conditions are

$$\hat{T}_{SB0} = \hat{T}_{w0}, \tag{60a}$$

$$\frac{(\hat{v}_{jSB1} - \hat{v}_{wj1})(\delta_{ij} - n_j n_i)}{\hat{T}^{1/2}_{w0}} = -\frac{\hat{K}_1}{\hat{p}_0}\frac{\partial \hat{T}_{SB0}}{\partial x_j}(\delta_{ij} - n_j n_i), \quad \hat{v}_{jSB1}n_j = 0. \tag{60b}$$

The effect of molecular property enters the above system only through the transport coefficients $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{\gamma}_3$, and $\hat{\gamma}_7$ (or Γ_1, Γ_2, Γ_7 and $\bar{\Gamma}_7$) and the slip coefficient $\hat{K}_1$. Thus, the fundamental structure of the equations and boundary conditions is generally common to molecular models.

3 Bénard Problem

Consider a gas in a time-independent (or steady) state under the uniform gravity between two parallel plane walls with different temperatures. The gravity is in the direction normal to the wall, that is $\hat{g}_{22} = -\hat{g}$ ($\hat{g} \geq 0$) and $\hat{g}_{12} = \hat{g}_{32} = 0$. Let L, T_A, and T_B be, respectively, the distance between the wall, the temperature of the lower wall, and that of the upper. The coordinate system is taken in such a way that the lower wall is at $x_2 = 0$ and the upper wall is at $x_2 = 1$. The parameters being taken to satisfy the assumptions at the beginning of Section 2.1, and the behavior of the gas is analyzed on the basis of the fluid-dynamic-type equations (54), (55) [or (59)], and (56) and the boundary conditions (60a) and

(60b). The analysis is limited to a two-dimensional case where the variables are independent of x_3 (or $\partial/\partial x_3 = 0$) and $v_3 = 0$. The behavior in the limit that $\mathrm{Kn} \to 0$ being interested in, the variables $\hat{T}$, $\hat{\rho}$, u_i, and $\mathcal{P}$ (or $\mathcal{P}^*$) and the parameter $\hat{T}_B$ are, respectively, used for $\hat{T}_{SB0}$, $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{p}_{SB2}$ (or $\hat{p}^*_{SB2}$) and T_B/T_A. Thus, $\hat{\rho} = \hat{p}_0/\hat{T}$. Here, the temperature T_A is taken as the reference temperature T_0 in the definition of $\hat{T}$. The parameters included in Eqs. (54)–(56) and (60a) and (60b) are $\hat{T}_B$, $\hat{g}$, and $\hat{p}_0$. It may be better to add some comment on the parameter $\hat{p}_0$. At present, $\hat{p}_0$ is not specified in the problem stated above. Let the average density of the gas in the domain be taken as the reference density ρ_0 in the definition of the nondimensional variables. Then the constant $\hat{p}_0$ is specified with the other parameters $\hat{T}_B$ and $\hat{g}$, but the explicit relation is given only after the solution is obtained. That is,

$$\hat{p}_0 = \left(\overline{1/\hat{T}}\right)^{-1}, \tag{61}$$

where the bar — over $1/\hat{T}$ indicates its average over the domain.

3.1 One-Dimensional Solution

First consider the case where the behavior of the gas is uniform in the direction parallel to the walls (or $\partial/\partial x_1 = \partial/\partial x_3 = 0$). Then, the solution of Eqs. (54)–(56) under the boundary conditions (60a) and (60b) are expressed in the following form:

$$\hat{T} = \hat{T}_U, \quad \hat{\rho} = \hat{\rho}_U = \hat{p}_0/\hat{T}_U, \tag{62a}$$

$$u_1 = u_2 = u_3 = 0, \tag{62b}$$

$$\mathcal{P} = \mathcal{P}_U = -2\hat{g}\hat{p}_0 \int^{\hat{T}_U} \frac{\Gamma_2(t)}{t}\mathrm{d}t \Big/ \int_1^{\hat{T}_B} \Gamma_2(t)\mathrm{d}t\,, \tag{62c}$$

where $\hat{T}_U$ is given by the implicit function

$$x_2 = \int_1^{\hat{T}_U} \Gamma_2(t)\mathrm{d}t \Big/ \int_1^{\hat{T}_B} \Gamma_2(t)\mathrm{d}t\,. \tag{63}$$

The function Γ_2 is related to the nondimensional thermal conductivity $\hat{\gamma}_2$ by

$$\Gamma_2(t) = \hat{\gamma}_2(t)t^{1/2}. \tag{64}$$

When $\Gamma_2(t) = c_0 t^n$ [$n = 1/2$ (hard-sphere), $n = 1$ (BKW); c_0: a constant], the relation (63) can be made explicit:

$$\hat{T}_U = [1 + (\hat{T}_B^{n+1} - 1)x_2]^{1/(n+1)}. \tag{65}$$

The undetermined constant $\hat{p}_0$ is related to the average density (say ρ_0) in the domain in the following way. By definition,

$$\rho_0 = \frac{1}{L}\int_0^L \rho \mathrm{d}X_2 = \rho_0 \int_0^1 \frac{\hat{p}_0}{\hat{T}_U}\mathrm{d}x_2.$$

Thus,

$$\hat{p}_0 = \int_1^{\hat{T}_B} \Gamma_2(t)\mathrm{d}t \Big/ \int_1^{\hat{T}_B} t^{-1}\Gamma_2(t)\mathrm{d}t. \tag{66}$$

We will investigate the possibility of bifurcation from this one-dimensional solution, which will be called 1D solution and denoted by $\hat{h}_U$ for simplicity. In the following analysis, we consider only the case where the quantities are independent of x_3 ($\partial/\partial x_3 = 0$).

3.2 Bifurcation from One-Dimensional Solution

Consider a solution that is periodic, with period $2\pi/\alpha$, with respect to the x_1 direction. We examine whether the periodic solution bifurcates from the 1D solution [Eqs. (62a) and (62b)] and clarify the behavior of the solution in the neighborhood of the bifurcation point, if any. Let the values of the parameters $\hat{T}$ and $\hat{g}$ at a bifurcation point be $\hat{T}_{Bb}$ and $\hat{g}_b$. The value of $\hat{p}_0$, given by Eq. (66) for the 1D solution, is denoted by $\hat{p}_{0b}$. That is,

$$\hat{p}_{0b} = \int_1^{\hat{T}_{Bb}} \Gamma_2(t)\mathrm{d}t \Big/ \int_1^{\hat{T}_{Bb}} t^{-1}\Gamma_2(t)\mathrm{d}t. \tag{67}$$

For the solution periodic with respect to x_1 to be considered hereafter, $\hat{p}_0$ is given by

$$\hat{p}_0 = \left(\frac{\alpha}{2\pi}\int_0^1\int_0^{2\pi/\alpha}\frac{1}{\hat{T}}\mathrm{d}x_1\mathrm{d}x_2\right)^{-1}. \tag{68}$$

For this purpose, we try to find the solution (say, $\hat{h}$) as a perturbation to the 1D solution (say, $\hat{h}_{Ub}$) at the bifurcation point in the following form:

$$\begin{aligned}\hat{T} = \hat{T}_{Ub}(x_2) + \delta\hat{T}_{11}(x_2)\cos\alpha x_1 + \delta^2[\hat{T}_{20}(x_2) + \hat{T}_{21}(x_2)\cos\alpha x_1 + \hat{T}_{22}(x_2)\cos 2\alpha x_1]\\ + \delta^3[\hat{T}_{30}(x_2) + \hat{T}_{31}(x_2)\cos\alpha x_1 + \cdots + \hat{T}_{33}(x_2)\cos 3\alpha x_1] + \cdots,\end{aligned} \tag{69a}$$

$$\begin{aligned}\hat{\rho} = \hat{\rho}_{Ub}(x_2) + \delta\hat{\rho}_{11}(x_2)\cos\alpha x_1 + \delta^2[\hat{\rho}_{20}(x_2) + \hat{\rho}_{21}(x_2)\cos\alpha x_1 + \hat{\rho}_{22}(x_2)\cos 2\alpha x_1]\\ + \delta^3[\hat{\rho}_{30}(x_2) + \hat{\rho}_{31}(x_2)\cos\alpha x_1 + \cdots + \hat{\rho}_{33}(x_2)\cos 3\alpha x_1] + \cdots,\end{aligned} \tag{69b}$$

$$\begin{aligned}u_1 = \delta U_{11}(x_2)\sin\alpha x_1 + \delta^2[U_{21}(x_2)\sin\alpha x_1 + U_{22}(x_2)\sin 2\alpha x_1]\\ + \delta^3[U_{31}(x_2)\sin\alpha x_1 + \cdots + U_{33}(x_2)\sin 3\alpha x_1] + \cdots,\end{aligned} \tag{69c}$$

$$\begin{aligned}u_2 = \delta V_{11}(x_2)\cos\alpha x_1 + \delta^2[V_{20}(x_2) + V_{21}(x_2)\cos\alpha x_1 + V_{22}(x_2)\cos 2\alpha x_1]\\ + \delta^3[V_{30}(x_2) + V_{31}(x_2)\cos\alpha x_1 + \cdots + V_{33}(x_2)\cos 3\alpha x_1] + \cdots,\end{aligned} \tag{69d}$$

$$u_3 = 0, \tag{69e}$$

$$\begin{aligned}\mathcal{P}^* = \mathcal{P}^*_{Ub}(x_2) + \delta\mathcal{P}^*_{11}(x_2)\cos\alpha x_1 + \delta^2[\mathcal{P}^*_{20}(x_2) + \mathcal{P}^*_{21}(x_2)\cos\alpha x_1 + \mathcal{P}^*_{22}(x_2)\cos 2\alpha x_1]\\ + \delta^3[\mathcal{P}^*_{30}(x_2) + \mathcal{P}^*_{31}(x_2)\cos\alpha x_1 + \cdots + \mathcal{P}^*_{33}(x_2)\cos 3\alpha x_1] + \cdots,\end{aligned} \tag{69f}$$

where δ^2 indicates the deviation from the bifurcation point, for example, $\delta^2 = [(\hat{T}_B - \hat{T}_{Bb})^2 + (\hat{g} - \hat{g}_b)^2]^{1/2}$, but it is not necessary to be explicit here. Corresponding to the expansion using δ, the parameters $\hat{T}_B$ and $\hat{g}$ away from the bifurcation point $(\hat{T}_{Bb}, \hat{g}_b)$ are expressed as

$$\hat{T}_B = \hat{T}_{Bb} + \delta^2\frac{(\hat{T}_B - \hat{T}_{Bb})}{\delta^2}, \quad \hat{g} = \hat{g}_b + \delta^2\frac{(\hat{g} - \hat{g}_b)}{\delta^2}, \tag{70}$$

where $(\hat{T}_B - \hat{T}_{Bb})/\delta^2$ and $(\hat{g} - \hat{g}_b)/\delta^2$ are quantities of the order of unity.

The basic equations are the conservation equations (54)–(56) and the equation of state (57a), with the new notations. It is, however, convenient here to eliminate the $\hat{p}^*_{SB2}$ (or $\mathcal{P}^*$) by taking the curl of Eq. (55), since $\hat{p}^*_{SB2}$ (or $\mathcal{P}^*$) does not appear in the boundary conditions and it is not a quantity of physical interest here. Substituting the series (69a)–(69e), and (70) into the basic equations (54), the curl of Eq. (55)–(56), and (57a) and arranging the same order terms of δ, we obtain a series of linear ordinary differential equations that determine the component functions $\hat{T}_{mn}$, $\hat{\rho}_{mn}$, U_{mn}, and V_{mn}. The $\mathcal{P}^*_{mn}$ is obtained from these quantities from Eq. (55). In the series of equations, the component functions appear in such a way that they can be formally determined successively from the lowest order (or in the order of m). The leading-order component functions $\hat{T}_{11}$, $\hat{\rho}_{11}$, U_{11}, and V_{11} are governed by the following equations:

$$L_1(U_{11}, V_{11}, \alpha) = 0, \tag{71a}$$

$$L_2(U_{11}, V_{11}, \hat{T}_{11}, \alpha) = 0, \tag{71b}$$

$$L_3(V_{11}, \hat{T}_{11}, \alpha) = 0, \tag{71c}$$

and

$$\hat{\rho}_{11} = -\frac{\hat{p}_{0b}}{\hat{T}_{Ub}}\frac{\hat{T}_{11}}{\hat{T}_{Ub}}, \tag{72}$$

$$\alpha\mathcal{P}_{11}^{*} = \frac{4\alpha^2\Gamma_{1b}}{3}U_{11} - \dot{\Gamma}_{1b}\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}\frac{\mathrm{d}U_{11}}{\mathrm{d}x_2} - \Gamma_{1b}\frac{\mathrm{d}^2U_{11}}{\mathrm{d}x_2^2} + \alpha\dot{\Gamma}_{1b}\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}V_{11} + \frac{\alpha\Gamma_{1b}}{3}\frac{\mathrm{d}V_{11}}{\mathrm{d}x_2}$$
$$+ \frac{\alpha}{p_{0b}}\left\{\left[\dot{\Gamma}_{7b}\left(\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}\right)^2 + \Gamma_{7b}\frac{\mathrm{d}^2\hat{T}_{Ub}}{\mathrm{d}x_2^2}\right]\hat{T}_{11} + \Gamma_{7b}\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}\frac{\mathrm{d}\hat{T}_{11}}{\mathrm{d}x_2}\right\}.$$

Here, L_1, L_2, and L_3 are the operators defined as follows:

$$L_1(U,V,\alpha) = \alpha U + \frac{\mathrm{d}V}{\mathrm{d}x_2} + \frac{1}{\hat{\rho}_{Ub}}\frac{\mathrm{d}\hat{\rho}_{Ub}}{\mathrm{d}x_2}V, \tag{73}$$

$$L_2(U,V,\hat{T},\alpha) = \frac{\mathrm{d}^3U}{\mathrm{d}x_2^3} + 2\frac{\dot{\Gamma}_{1b}}{\Gamma_{1b}}\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}\frac{\mathrm{d}^2U}{\mathrm{d}x_2^2} + \left[\frac{\ddot{\Gamma}_{1b}}{\Gamma_{1b}}\left(\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}\right)^2 + \frac{\dot{\Gamma}_{1b}}{\Gamma_{1b}}\frac{\mathrm{d}^2\hat{T}_{Ub}}{\mathrm{d}x_2^2} - \alpha^2\right]\frac{\mathrm{d}U}{\mathrm{d}x_2}$$
$$- 2\alpha^2\frac{\dot{\Gamma}_{1b}}{\Gamma_{1b}}\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}U + \alpha\frac{\mathrm{d}^2V}{\mathrm{d}x_2^2} - \alpha\left[\frac{\ddot{\Gamma}_{1b}}{\Gamma_{1b}}\left(\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}\right)^2 + \frac{\dot{\Gamma}_{1b}}{\Gamma_{1b}}\frac{\mathrm{d}^2\hat{T}_{Ub}}{\mathrm{d}x_2^2} + \alpha^2\right]V$$
$$+ \frac{\alpha}{\hat{p}_{0b}}\left\{\frac{\Gamma_{7b}}{\Gamma_{1b}}\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}\frac{\mathrm{d}^2\hat{T}}{\mathrm{d}x_2^2} + \frac{\dot{\Gamma}_{7b}}{\Gamma_{1b}}\left(\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}\right)^2\frac{\mathrm{d}\hat{T}}{\mathrm{d}x_2}\right.$$
$$\left. + \left[-\alpha^2\frac{\Gamma_{7b}}{\Gamma_{1b}}\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2} - \frac{\dot{\Gamma}_{7b}}{\Gamma_{1b}}\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}\frac{\mathrm{d}^2\hat{T}_{Ub}}{\mathrm{d}x_2^2} - \frac{\Gamma_{7b}}{\Gamma_{1b}}\frac{\mathrm{d}^3\hat{T}_{Ub}}{\mathrm{d}x_2^3} + \frac{2\hat{g}_b}{\Gamma_{1b}}\left(\frac{\hat{p}_{0b}}{\hat{T}_{Ub}}\right)^2\right]\hat{T}\right\}, \tag{74}$$

$$L_3(V,\hat{T},\alpha) = -\frac{2\hat{\rho}_{Ub}}{\Gamma_{2b}}\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}V + \frac{\mathrm{d}^2\hat{T}}{\mathrm{d}x_2^2} + 2\left(\frac{\dot{\Gamma}_{2b}}{\Gamma_{2b}}\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}\right)\frac{\mathrm{d}\hat{T}}{\mathrm{d}x_2} + \left[\frac{\ddot{\Gamma}_{2b}}{\Gamma_{2b}}\left(\frac{\mathrm{d}\hat{T}_{Ub}}{\mathrm{d}x_2}\right)^2 + \frac{\dot{\Gamma}_{2b}}{\Gamma_{2b}}\frac{\mathrm{d}^2\hat{T}_{Ub}}{\mathrm{d}x_2^2} - \alpha^2\right]\hat{T}, \tag{75}$$

where

$$\Gamma_{1b} = \Gamma_1(\hat{T}_{Bb}),\ \Gamma_{2b} = \Gamma_2(\hat{T}_{Bb}),\ \Gamma_{7b} = \Gamma_7(\hat{T}_{Bb}),$$
$$\dot{\Gamma}_{mb} = \left(\mathrm{d}\Gamma_m/\mathrm{d}\hat{T}\right)_{\hat{T}=\hat{T}_{Bb}},\quad \ddot{\Gamma}_{mb} = \left(\mathrm{d}^2\Gamma_m/\mathrm{d}\hat{T}^2\right)_{\hat{T}=\hat{T}_{Bb}}.$$

From Eqs. (60a) and (60b), the boundary conditions for these equations are

$$\hat{T}_{11} = U_{11} = V_{11} = 0 \quad \text{at } x_2 = 0 \text{ and } x_2 = 1. \tag{76}$$

The boundary-value problem [(71a)–(71c), and (76)] is homogeneous. Thus, the problem can, generally, have a nontrivial solution only when the parameters $\hat{T}_{Bb}$, $\hat{g}_b$, and α satisfy some relation, say,

$$F_b(\hat{T}_{Bb}, \hat{g}_b, \alpha) = 0. \tag{77}$$

This is the relation among the parameters $\hat{T}_{Bb}$, $\hat{g}_b$, and α for which the solution (69a)–(69f) bifurcates from the one-dimensional solution (62a)–(62c). The curve $\hat{g}_b$ versus $\hat{T}_{Bb}$ for a given α, which is obtained numerically for a hard-sphere gas, is shown in Fig. 1, where the corresponding curve when the thermal stress terms [the terms containing Γ_{7b} and $\dot{\Gamma}_{7b}$ in the operator L_2 defined by Eq. (74)] are neglected in Eq. (71b) is shown in dashed lines for comparison. There is appreciable difference for small $\hat{T}_{Bb}$. The relation being expressed as $\hat{g}_b = \hat{g}_b(\hat{T}_{Bb}, \alpha)$, consider the minimum value of $\hat{g}_b$ with respect to α with $\hat{T}_{Bb}$ being fixed and denote it by $(\hat{g}_b)_{\mathrm{m}}$ and the minimum point by α_{m}. The curves $(\hat{g}_b)_{\mathrm{m}}$ and α_{m} versus $\hat{T}_{Bb}$ are shown in Fig. 2.

When the condition (77) is satisfied, the solution is determined except for a constant factor. This factor is determined by the higher-order analysis, which is only touched on owing to limited space. The

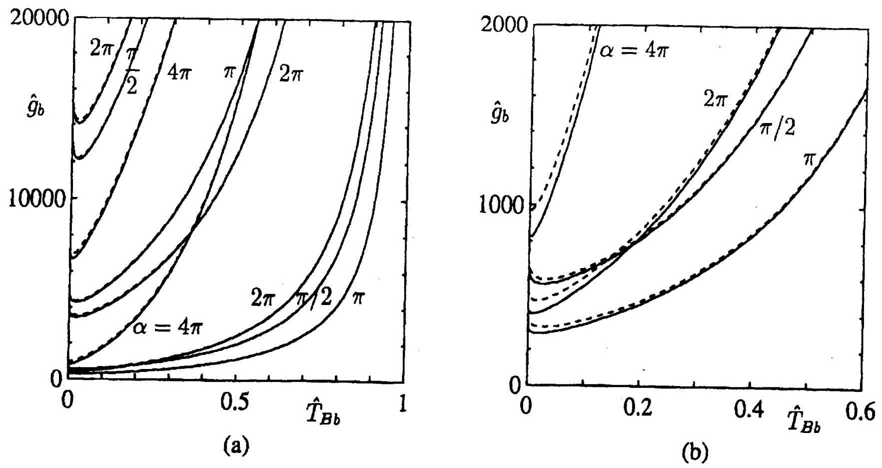

Figure 1: Bifurcation curves I: $\hat{g}_b$ versus $\hat{T}_{Bb}$ for various α. (a) Wider range of $\hat{g}_b$ showing several branches and (b) magnified figure of the the first branch. The solid lines —— indicate the bifurcation curve for a hard-sphere gas; the dashed lines --- indicate the corresponding curve when the thermal stress terms are neglected.

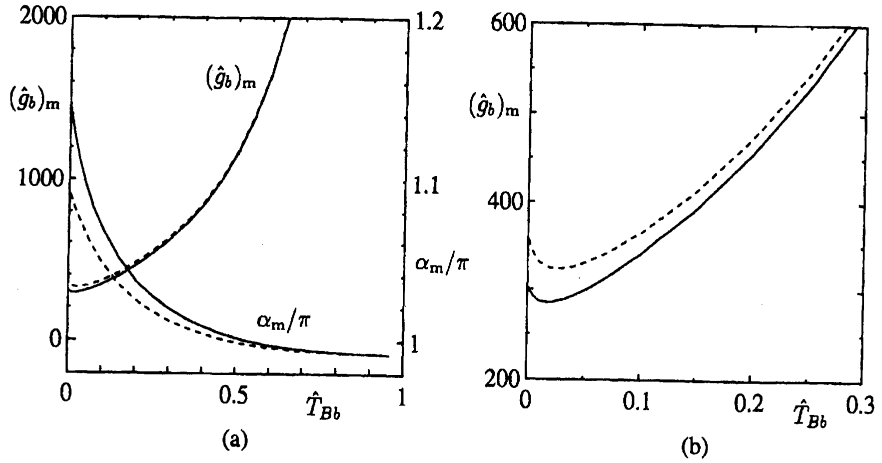

Figure 2: Bifurcation curves II: The curves $(\hat{g}_b)_m$ and α_m versus $\hat{T}_{Bb}$. (a) the two curves $(\hat{g}_b)_m$ and α_m versus $\hat{T}_{Bb}$ and (b) a magnified figure of the curve $(\hat{g}_b)_m$ versus $\hat{T}_{Bb}$. The solid lines —— indicate the bifurcation curve for a hard-sphere gas; the dashed lines --- indicate the corresponding curve when the thermal stress terms are neglected.

boundary-value problem for U_{21}, V_{21}, and $\hat{T}_{21}$ is homogeneous and of the same form as that for U_{11}, V_{11}, and $\hat{T}_{11}$. The problem for U_{m1}, V_{m1}, and $\hat{T}_{m1}$ ($m \geq 3$) is inhomogeneous, and its homogeneous part is of the same form as that for U_{11}, V_{11}, and $\hat{T}_{11}$. Thus, its inhomogeneous part must satisfy some relation (solvability condition) for the solution U_{m1}, V_{m1}, and $\hat{T}_{m1}$ to exist. The homogeneous part of the boundary-value problem for U_{mn}, V_{mn}, and $\hat{T}_{mn}$ ($n \neq 1$) has no nontrivial solution unless an additional condition among $\hat{T}_{Bb}$, $\hat{g}_b$, and α is satisfied.

Let the undetermined constant factor (or the norm) of the set $\delta(U_{11}, V_{11}, \hat{T}_{11})$ be δA, where the norm may be defined, for example, as $A = [\int_0^1 (U_{11}^2 + V_{11}^2 + \hat{T}_{11}^2)\mathrm{d}x_2]^{1/2}$. Then, the solvability condition of the boundary-value problem for U_{31}, V_{31}, and $\hat{T}_{31}$, is expressed in the following form:

$$A[a_T(\hat{T}_B - \hat{T}_{Bb})/\delta^2 + a_g(\hat{g} - \hat{g}_b)/\delta^2 - a_O A^2] = 0. \tag{78}$$

Thus,

$$A^2 = \frac{a_T}{a_O}\frac{(\hat{T}_B - \hat{T}_{Bb})}{\delta^2} + \frac{a_g}{a_O}\frac{(\hat{g} - \hat{g}_b)}{\delta^2}, \text{ or } A = 0,$$

where a_T/a_O and a_g/a_O are determined by $\hat{T}_{Bb}, \hat{g}_b$, and α. The first equation gives the amplitude of the bifurcated solution, and the second is the one-dimensional solution. The bifurcated solution extends to the range

$$\frac{a_T}{a_O}(\hat{T}_B - \hat{T}_{Bb}) + \frac{a_g}{a_O}(\hat{g} - \hat{g}_b) > 0, \tag{79}$$

in the parameter plane $(\hat{T}_B, \hat{g})$, and the amplitude A remains zero along the direction $(\hat{T}_B - \hat{T}_{Bb},\ \hat{g} - \hat{g}_b)$ given by

$$\frac{a_T}{a_O}(\hat{T}_B - \hat{T}_{Bb}) + \frac{a_g}{a_O}(\hat{g} - \hat{g}_b) = 0. \tag{80}$$

That is, this is the direction of the bifurcation curve $F_b(\hat{T}_{Bb}, \hat{g}_b, \alpha) = 0$ in the $\hat{T}_{Bb}$-$\hat{g}_b$ plane, which is shown in Fig. 1.

When $a_O = 0$, the coefficients a_T/a_O and a_g/a_O are infinite. This indicates that the amplitude δA is much larger than δ (the square root of the deviation from the bifurcation point), and thus the preceding analysis should be reconsidered. The solution bifurcating from the bifurcation point $(\hat{T}_{Bb}, \hat{g}_b, \alpha)$ where the condition $a_O = 0$ is satisfied can be obtained in a similar way to the preceding analysis by modifying the power series (69a)–(69f) of δ to a power series of $\delta^{1/2}$. That is,

$$\begin{aligned} f = {} & f_{Ub}(x_2) + \delta^{1/2} f_{11}(x_2)\cos\alpha x_1 + \delta[f_{20}(x_2) + f_{21}(x_2)\cos\alpha x_1 + f_{22}(x_2)\cos 2\alpha x_1] \\ & + \delta^{3/2}[f_{30}(x_2) + f_{31}(x_2)\cos\alpha x_1 + \cdots + f_{33}(x_2)\cos 3\alpha x_1] \\ & + \delta^2[f_{40}(x_2) + f_{41}(x_2)\cos\alpha x_1 + \cdots + f_{44}(x_2)\cos 4\alpha x_1] + \cdots, \end{aligned} \tag{81a}$$

$$\begin{aligned} u_1 = {} & \delta^{1/2} U_{11}(x_2)\sin\alpha x_1 + \delta[U_{21}(x_2)\sin\alpha x_1 + U_{22}(x_2)\sin 2\alpha x_1] \\ & + \delta^{3/2}[U_{31}(x_2)\sin\alpha x_1 + \cdots + U_{33}(x_2)\sin 3\alpha x_1] \\ & + \delta^2[U_{41}(x_2)\sin\alpha x_1 + \cdots + U_{44}(x_2)\sin 4\alpha x_1] + \cdots, \end{aligned} \tag{81b}$$

where $f = \hat{T}$, u_2, $\hat{p}$, or $\mathcal{P}^*$. The boundary-value problem for $(U_{11}, V_{11}, \hat{T}_{11})$ is the same as that for $(U_{11}, V_{11}, \hat{T}_{11})$ in the preceding analysis, as should be. In the higher-order analysis, the homogeneous part is the same as before but some inhomogeneous terms degenerate because of the condition on a_O, and the amplitude of the solution $(U_{11}, V_{11}, \hat{T}_{11})$ is determined by the solvability condition of the equations for $(U_{51}, V_{51}, \hat{T}_{51})$. As the result, the fourth power A^4, instead of A^2 in the general case, of the amplitude of $(U_{11}, V_{11}, \hat{T}_{11})$ is expressed by a linear combination of $\hat{T}_B - \hat{T}_{Bb}$ and $\hat{g} - \hat{g}_b$.

When $n\alpha$, as well as α, satisfies the bifurcation relation (77) for some set of integer n (n^*), some comments are in order. Then the leading terms (the terms of the order δ) of the perturbation should be the sum of corresponding Fourier components, that is,

$$\delta \sum_{n=(n^*)} f_{1n}(x_2)\cos\alpha n(x_1 - c_n) \text{ or } \delta \sum_{n=(n^*)} u_{1n}(x_2)\sin\alpha n(x_1 - c_n),$$

where c_n is some constant, and the following terms correspondingly consist of more terms than before. The analysis can be carried out in a similar way to that of the preceding analysis. Incidentally, if the neighboring integers (say m and $m+1$) belong to the set (n^*), the corresponding amplitudes, A_m and A_{m+1}, vanish.

3.3 Two-Dimensional Temperature Field under Infinitesimal Flow Velocity

In the previous section we have found that there is a bifurcation of temperature field under infinitesimal flow velocity (the first-order infinitesimal) and gravity (the second-order infinitesimal) and that the nonlinear thermal stress, which is the second-order infinitesimal, affects the bifurcation of the temperature field. In this section, we will study the temperature field away from bifurcation point by numerical analysis of the system summarized in Section 2.4.

The numerical computation is carried out in the following way. Consider a gas in the finite domain $(0 < x_1 < \pi/\alpha,\ 0 < x_2 < 1)$ and take the following conditions on the side boundaries:

$$\frac{\partial \hat{T}}{\partial x_1} = 0, \quad u_1 = 0, \quad \frac{\partial u_2}{\partial x_1} = 0 \ \text{ at } x_1 = 0 \text{ and } x_1 = \frac{\pi}{\alpha}, \tag{82}$$

in addition to the conditions

$$\hat{T}=1,\quad u_1=u_2=0 \quad \text{at} \quad x_2=0, \quad \text{and} \quad \hat{T}=\hat{T}_B,\ u_1=u_2=0 \quad \text{at} \quad x_2=1, \tag{83}$$

and

$$\hat{p}_0=\left(\frac{\alpha}{\pi}\int_0^1\int_0^{\pi/\alpha}\frac{1}{\hat{T}}\mathrm{d}x_1\mathrm{d}x_2\right)^{-1}. \tag{84}$$

Incidentally, from the basic equations (54)–(56) with (57a) and the boundary condition (82), it is found that $\partial\mathcal{P}/\partial x_1=0$ at $x_1=0$ and $x_1=\pi/\alpha$.

Let a solution of the above problem in a rectangular domain be S1. Then its mirror image with respect to the vertical boundary is also a solution of the problem (say S2). The two kinds of solutions S1 and S2 being alternately arranged laterally, the resulting function is found to be two times continuously differentiable across the vertical connection lines $x_1=n\pi/\alpha$ $(n=0,1,2,\cdots)$, because it satisfies Eqs. (54)–(56) except on the connection lines and satisfies the condition (82) at the connection point. That is, the function thus constructed is a periodic solution with period $2\pi/\alpha$ with respect to x_1 in the infinite domain between the two plane walls at $x_2=0$ and $x_2=1$.

The boundary-value problem, i.e., Eqs. (54)–(56), (82)–(84), is solved numerically by a finite difference method. The solution of the boundary-value problem for the finite-difference equations is obtained by the method of iteration. The outline of the process is as follows: (i) First rewrite Eqs. (54)–(56) in the following form, where $\partial p^*_{SB2}/\partial x_i$ term is eliminated from Eq. (55) by taking the curl of it, and the vorticity ω and the stream function Ψ, in place of the continuity equation (54), are introduced, and the superscript with parentheses showing the step of iteration is attached for convenience of explanation.

$$\frac{\partial}{\partial x_i}\left(\Gamma_2^{(n)}\frac{\partial\hat{T}^{(n+1)}}{\partial x_i}\right)=2\hat{p}_0^{(n)}\frac{u_i^{(n)}}{\hat{T}^{(n)}}\frac{\partial\hat{T}^{(n)}}{\partial x_i}, \tag{85}$$

$$\hat{p}_0^{(n+1)}=\left(\frac{\alpha}{\pi}\int_0^1\int_0^{\pi/\alpha}\frac{1}{\hat{T}^{(n+1)}}\mathrm{d}x_1\mathrm{d}x_2\right)^{-1}, \tag{86}$$

$$\begin{aligned}\Gamma_1^{(n+1)}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\omega^{(n+1)}=&-\frac{2\hat{g}\hat{p}_0^{(n+1)}}{(\hat{T}^{(n+1)})^2}\frac{\partial\hat{T}^{(n+1)}}{\partial x_1}-\dot{\Gamma}_1^{(n+1)}\frac{\partial\hat{T}^{(n+1)}}{\partial x_i}\frac{\partial\omega^{(n)}}{\partial x_i}\\&+\frac{\partial}{\partial x_i}\left\{\dot{\Gamma}_1^{(n+1)}\left[\frac{\partial\hat{T}^{(n+1)}}{\partial x_2}\left(\frac{\partial u_1^{(n)}}{\partial x_i}+\frac{\partial u_i^{(n)}}{\partial x_1}\right)-\frac{\partial\hat{T}^{(n+1)}}{\partial x_1}\left(\frac{\partial u_2^{(n)}}{\partial x_i}+\frac{\partial u_i^{(n)}}{\partial x_2}\right)\right]\right\}\\&+\frac{1}{\hat{p}_0^{(n+1)}}\frac{\partial}{\partial x_i}\left[-\frac{\partial}{\partial x_1}\left(\Gamma_7^{(n+1)}\frac{\partial\hat{T}^{(n+1)}}{\partial x_2}\frac{\partial\hat{T}^{(n+1)}}{\partial x_i}\right)+\frac{\partial}{\partial x_2}\left(\Gamma_7^{(n+1)}\frac{\partial\hat{T}^{(n+1)}}{\partial x_1}\frac{\partial\hat{T}^{(n+1)}}{\partial x_i}\right)\right]\\&+2\hat{p}_0^{(n+1)}\frac{u_i^{(n)}}{(\hat{T}^{(n+1)})^2}\left(-\frac{\partial u_2^{(n)}}{\partial x_i}\frac{\partial\hat{T}^{(n+1)}}{\partial x_1}+\frac{\partial u_1^{(n)}}{\partial x_i}\frac{\partial\hat{T}^{(n+1)}}{\partial x_2}\right)+\frac{2\hat{p}_0^{(n+1)}}{\hat{T}^{(n+1)}}\frac{\partial u_i^{(n)}\omega^{(n)}}{\partial x_i},\end{aligned} \tag{87}$$

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\Psi^{(n+1)}=-\frac{1}{\hat{T}^{(n+1)}}\omega^{(n+1)}+\frac{1}{(\hat{T}^{(n+1)})^2}\left(u_2^{(n)}\frac{\partial}{\partial x_1}-u_1^{(n)}\frac{\partial}{\partial x_2}\right)\hat{T}^{(n+1)}, \tag{88}$$

$$u_1^{(n+1)}=\hat{T}^{(n+1)}\frac{\partial\Psi^{(n+1)}}{\partial x_2},\qquad u_2^{(n+1)}=-\hat{T}^{(n+1)}\frac{\partial\Psi^{(n+1)}}{\partial x_1}, \tag{89}$$

where

$$\Gamma_2^{(n)}=\Gamma_2(\hat{T}^{(n)}),\quad \Gamma_1^{(n+1)}=\Gamma_1(\hat{T}^{(n+1)}),\quad \Gamma_7^{(n+1)}=\Gamma_7(\hat{T}^{(n+1)}), \tag{90}$$

and Eq. (88) corresponds to the relation $\omega=\partial u_2/\partial x_1-\partial u_1/\partial x_2$. Then, the finite difference form of these equations is prepared. (ii) Choose an initial set of $(u_1^{(0)}, u_2^{(0)}, \hat{T}^{(0)}, \hat{p}_0^{(0)}, \omega^{(0)})$. (iii) Obtain $\hat{T}^{(n+1)}$, $\hat{p}_0^{(n+1)}$, $\omega^{(n+1)}$, $\Psi^{(n+1)}$, $u_1^{(n+1)}$, and $u_2^{(n+1)}$ successively using Eqs. (85)–(89) according to their order with the set $(\hat{T}^{(n)}, \hat{p}_0^{(n)}, \omega^{(n)}, \Psi^{(n)}, u_1^{(n)}, u_2^{(n)})$ obtained at the previous stage (or given as the initial set). That is, $\hat{T}^{(n+1)}$ from Eq. (85), $\hat{p}_0^{(n+1)}$ from Eq. (86), $\omega^{(n+1)}$ from Eq. (87), $\Psi^{(n+1)}$ from Eq. (88), $u_1^{(n+1)}$ and $u_2^{(n+1)}$ from Eq. (89) using the data obtained at the previous stage. The boundary condition in the process of solution is obvious except that for $\omega^{(n+1)}$ on $x_2=0$ and 1, for which $\omega^{(n+1)}=\omega^{(n)}\mp\vartheta\, u_1^{(n)}$,

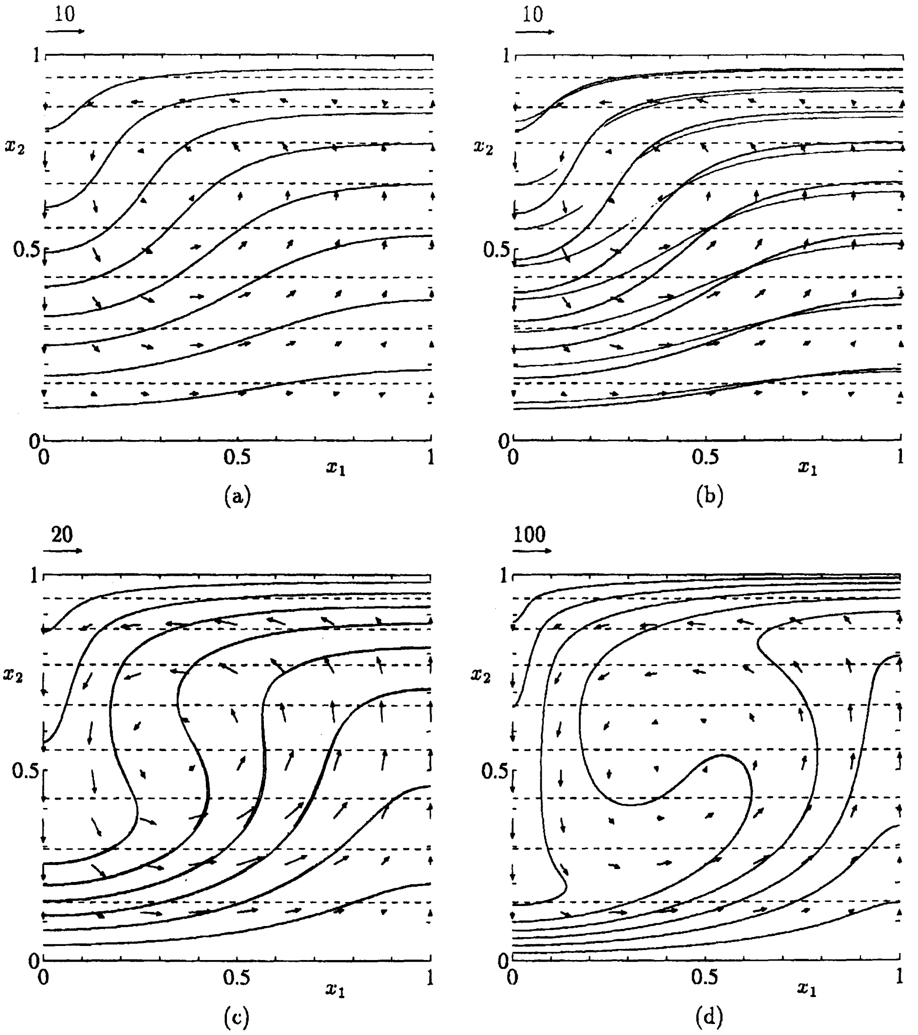

Figure 3: The bifurcated temperature field for a hard-sphere gas I: $\hat{T}_B = 0.1$. (a) $\hat{g} = 320$, (b) $\hat{g} = 328$, (c) $\hat{g} = 1000$, (d) $\hat{g} = 7000$. The solid lines indicate the isothermal lines [$\hat{T} = 0.1n$ ($n = 1, 2, \ldots, 10$) from the upper wall to the lower]; the arrows indicate u_i at their starting point and its scale is shown on the left shoulder of the figure. The thin lines indicate the corresponding results with the thermal stress effect neglected, and the dashed lines ----- indicate the 1D solution.

where ϑ is a constant properly chosen so that the iteration converges, is used. (iv) Return to step (iii) and continue the process with newly obtained ($\hat{T}^{(n+1)}$, $\hat{p}_0^{(n+1)}$, $\omega^{(n+1)}$, $\Psi^{(n+1)}$, $u_1^{(n+1)}$, $u_2^{(n+1)}$) as ($\hat{T}^{(n)}$, $\hat{p}_0^{(n)}$, $\omega^{(n)}$, $\Psi^{(n)}$, $u_1^{(n)}$, $u_2^{(n)}$). The essential problem at each step is to solve the Poisson equation.[13]

Some of the results of computation are shown in Figs. 3 and 4. When $\hat{T}_B = 0.1$ (Fig. 3), the bifurcated solution first extends to the direction of smaller $\hat{g}$ from the bifurcation point at $\hat{g} = 341.28$ and then to larger $\hat{g}$ after its amplitude grows to some size. At $\hat{g} = 320$, there is no bifurcated solution for the system with the thermal stress terms neglected, for which the bifurcation point is at $\hat{g} = 364.96$ [panel (a) of Fig. 3]; at $\hat{g} = 328$, the maximum difference of the temperature of the system without thermal stress amounts to 20% of the correct solution [panel (b) of Fig. 3]; and for $\hat{g} = 1000$ and 7000, slight differences of isothermal lines are seen in the central region in panels (c) and (d) of Fig. 3. When $\hat{T}_B = 0.5$ (Fig. 4), the bifurcated solution extends to the direction of larger $\hat{g}$ from the bifurcation point at $\hat{g} = 1162.28$. At $\hat{g} = 1170$ [panel (a) of Fig. 4], there is no bifurcated solution for the system without thermal stress; at $\hat{g} = 1180$ [panel (b) of Fig. 4], there is clearly a difference between the two solutions with and without thermal stress. The results clearly show that the Navier–Stokes system fails to describe the temperature field in the continuum limit. The strongly deformed temperature field in the absence of gas motion is the ghost effect.

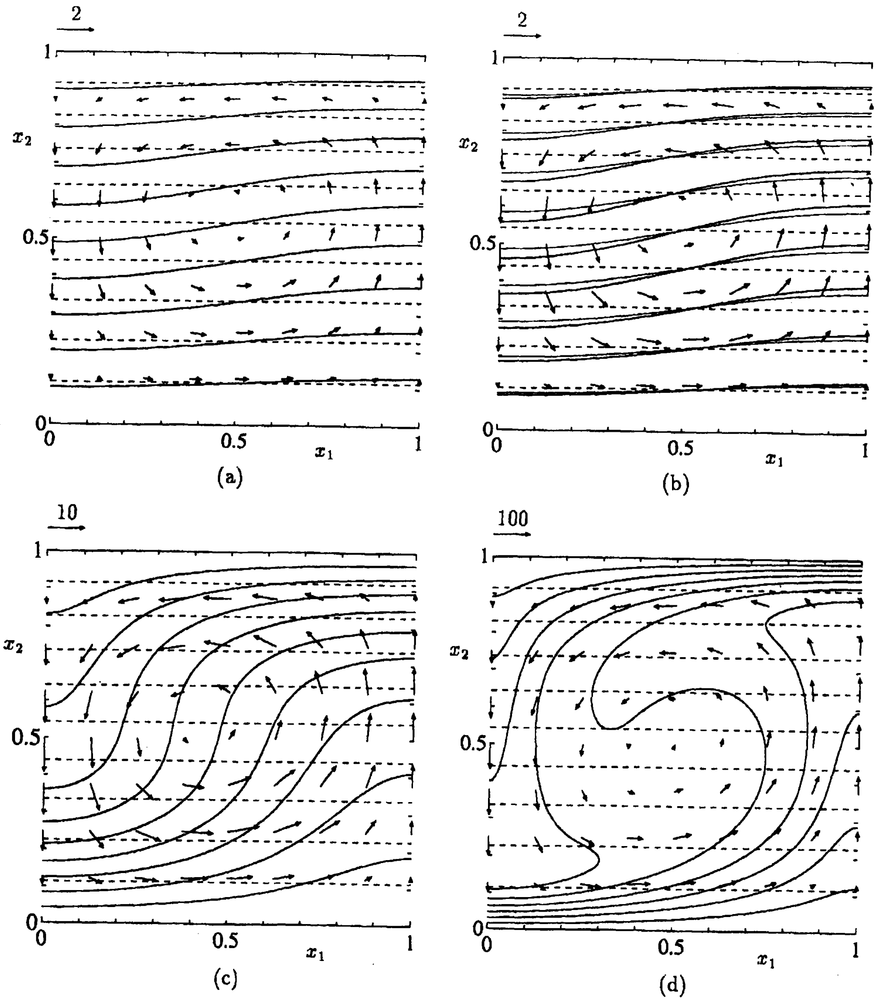

Figure 4: The bifurcated temperature field for a hard-sphere gas II: $\hat{T}_B = 0.5$. (a) $\hat{g} = 1170$, (b) $\hat{g} = 1180$, (c) $\hat{g} = 2000$, (d) $\hat{g} = 30000$. The solid lines indicate the isothermal lines [$\hat{T} = 0.05n+0.5$ ($n = 0, 1, \ldots, 10$) from the upper wall to the lower]; the arrows indicate u_i at their starting point and its scale is shown on the left shoulder of each panel. The thin lines indicate the corresponding results with the thermal stress effect neglected, and the dashed lines ----- indicate the 1D solution.

3.4 Discussions

In this section, the Bénard problem of a gas in the continuum limit between two parallel plane walls with different temperatures is studied on the basis of the asymptotic fluid-dynamic-type equations and their associated boundary conditions. The two-dimensional problem discussed in this work is, apparently, a plain problem which has already been studied sufficiently, but the result is not the one that is given by the classical gas dynamics. In the problem the temperature field is determined together with the infinitesimal velocity field. The infinitesimal velocity is not perceived in the continuum world (or in the world of the continuum limit). Thus, there is a bifurcation of the temperature field and it is strongly distorted even when there is no flow at all. In other words, the correct behavior of a gas in the continuum limit can not be obtained only by the quantities perceptible in its world.

A bifurcated and distorted temperature field is also obtained with the aid of the Navier–Stokes equations if the vanishing flow velocity is just retained. However, it does not give the correct answer. In the asymptotic fluid-dynamic-type equations, there is another contribution. It is the thermal stress. The thermal stress is of the second order in the Knudsen number and the viscous stress is generally of the first order. In the present case, the velocity is of the first order and therefore the viscous stress degenerates to the second order. Thus the thermal stress should be retained together with the viscous stress. Here we show the difference between the two results. The dotted lines in Fig. 1 are the corresponding bifurcation curves for the sets of equations (54)–(56) where the thermal stress terms [or the terms containing Γ_7 in Eq. (55)] are eliminated. Some examples of isothermal lines for the two results are compared in Figs. 3 and 4, where the results for the thermal stress neglected is shown in thin lines. These results clearly show the ghost effect and inappropriateness of the Navier–Stokes system for the description of the behavior of a gas in the continuum limit.

In a real gas, the mean free path may be very small but is not exactly zero. Then, the flow velocity is nonzero for the bifurcated temperature field. As an example, consider the following case: The distance L between the two walls is 10 m; the temperature T_B of the upper wall is 300 K; the gas between the channel is air (or nitrogen gas) and under atmospheric pressure, although it is not a monatomic gas and does not correspond exactly to the present asymptotic equations. Then, the mean free path near the upper wall is roughly 6×10^{-8} m. (i) When $T_B/T_A = 0.1$, the mean free path near the lower wall is 6×10^{-7} m and thus the Knudsen number is 6×10^{-8}. The gravity $|g_i|$ at the bifurcation point is $|g_i| = 2 \times 10^{-7}$ m/sec^2, which is 2×10^{-8} of the gravity on the earth. According to the numerical computation, $|u_i| \leq 5$ for $\hat{g} = 320$ or $|u_i| \leq 17$ for $\hat{g} = 1000$; that is, the flow velocity is, respectively, less than 0.4 or 2 mm/sec. The corresponding temperature field is given in panels (a) and (c) of Fig. 3. In the case of Fig. 3 (a), there is no distortion of the temperature field if the thermal stress terms [or the terms containing Γ_7 in Eq. (55)] are neglected. (ii) When $T_B/T_A = 0.5$, the mean free path near the lower wall is 10^{-7} m and thus the Knudsen number is 10^{-8}. The gravity $|g_i|$ at the bifurcation point for $T_B/T_A = 0.5$ is $|g_i| = 5 \times 10^{-9}$ m/sec^2, which is 5×10^{-10} of the gravity on the earth. According to the numerical computation, $|u_i| \leq 1$ for $\hat{g} = 1170$ or $|u_i| \leq 10$ for $\hat{g} = 2000$; that is, the flow velocity is, respectively, less than 5×10^{-3} or 5×10^{-2} mm/sec. The corresponding temperature field is given in panels (a) and (c) of Fig.4. In the case of Fig. 4 (a), there is no distortion of the temperature field if the thermal stress terms [or the terms containing Γ_7 in Eq. (55)] are neglected. In view of the temperature field and the scale of the system, the velocity is practically a vanishingly small quantity. In the analysis, we considered the case where the plane walls were at rest. Unless the motion of the walls is kept at rest with accuracy much less than the above speed, which is difficult to control, the analysis taking into account of this small motion ($\hat{v}_{wi1}$) into the boundary condition (60b) is required for the correct description of the behavior of the temperature field. We have considered a perfectly time-independent problem. Infinitesimal time-dependent quantities (e.g., $\hat{v}_{wi1}n_i$) may induce time-dependent or time-independent effect on the behavior of a gas in the continuum limit.

The effect of infinitesimal velocity is more striking when we consider the Bénard problem with the diffuse reflecting side walls. That is, a one-dimensional temperature field is impossible owing to the boundary condition (60b). In case of classical fluid dynamics (the Navier–Stokes equations under nonslip condition), the one-dimensional temperature field given in Section 3.1 is possible when the temperature of the side walls is given in harmony with 1D solution. Thus, the results of the two system disagree at the starting point of the study of the Bénard problem.

The present study shows that the behavior of a gas in the continuum limit cannot be described by the Navier–Stokes equations for an important class of problems and that infinitesimal quantities play an important role for its description (ghost effect). The ghost effect is also discussed in Refs. 2, 14, 15, and 16.

4 Concluding Remarks

In the present work, we considered a gas in a time-independent state in a weak gravity field in a general domain. Its asymptotic behavior for small Knudsen numbers was investigated on the basis of the Boltzmann system for the situation where the flow velocity and gravity were very small quantities (or more precisely, infinitesimals, respectively, of the first and second orders of the Knudsen number in its vanishing limit). A system of fluid-dynamic-type equations and their associated boundary conditions that describes the behavior of the gas in the limit that the Knudsen number tends to zero (or in the continuum limit) is derived from the Boltzmann system by the asymptotic analysis. Both infinitesimal quantities, the flow velocity and gravity, influence the behavior of the gas (or the temperature field) in the continuum limit. That is, the temperature field in the continuum limit is determined by the equations coupled with the two infinitesimal quantities amplified by infinite quantities (i.e., the inverse of the Knudsen number or its square). The asymptotic system of equations and boundary conditions was applied to the bifurcation analysis of the Bénard problem of a gas in a weak gravity field between two parallel plane walls with different temperatures.

The Bénard problem was studied analytically and numerically. Bifurcation from the temperature field uniform in the direction parallel to the plane walls (1D solution) was analyzed, and the bifurcation curve was obtained. The bifurcated temperature field away from the bifurcation point was studied numerically by a finite-difference method. In the continuum world (or to those who are living in the world where the mean free path of the gas molecules is vanishingly small), the (infinitesimal) flow velocity is not perceptible, or the gas is at rest. In spite of this, there is a bifurcation of the temperature field. Strongly distorted temperature field as well as the 1D temperature field exists in a gas at rest. This bifurcated temperature field is not correctly obtained by the Navier–Stokes system by retaining the infinitesimal velocity. Additional thermal stress terms are required to obtain the correct solution. Infinitesimal nonlinear-thermal-stress flow has the same-order effect on the bifurcation and the bifurcated temperature field. What is noted is that a one-dimensional temperature field cannot, in general, be possible in the Bénard problem in a domain with a finite lateral length.

The classical fluid dynamics is inappropriate to describe even the well-known Bénard problem for a gas in the continuum limit (unless the temperature ratio of the two walls is close to unity). The inappropriateness can be understood in the framework of the classical fluid dynamics if the order of the magnitude of the transport coefficients is taken into account. However, we have to resort to kinetic theory to obtain the correct system of equations and their associated boundary conditions.

A Function $\mathcal{B}(\tilde{\zeta}, \hat{T}_{SB0})$, $\mathcal{N}^A(\tilde{\zeta}, \hat{T}_{SB0})$, etc.

The functions $\mathcal{B}(\tilde{\zeta}, \hat{T}_{SB0})$, $\mathcal{N}^A(\tilde{\zeta}, \hat{T}_{SB0})$, etc. appeared in Eqs. (37) are expressed by linear combinations of solutions of the following integral equations related the linearized collision operator $\mathcal{L}_a(*)$:

$$\mathcal{L}_a\left[\left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{1}{3}\tilde{\zeta}^2\delta_{ij}\right)\mathcal{B}^{(m)}(\tilde{\zeta}, a)\right] = IB_{ij}^{(m)}; \tag{91}$$

$$\mathcal{L}_a[\mathcal{N}^{(m)}(\tilde{\zeta}, a)] = IN^{(m)} \quad \text{with the subsidiary conditions: } \int_0^\infty (1,\ \tilde{\zeta}^2)\tilde{\zeta}^2\mathcal{N}^{(m)}E(\tilde{\zeta})\mathrm{d}\tilde{\zeta} = 0. \tag{92}$$

The inhomogeneous terms $IB_{ij}^{(m)}$ and $IN^{(m)}$ in Eqs. (91)–(92) are as follows:

$$IB_{ij}^{(0)} = -2\left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right), \qquad IB_{ij}^{(1)} = \left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)\mathcal{A}(\tilde{\zeta}, a),$$

$$IB_{ij}^{(2)} = \left(\tilde{\zeta}_i\tilde{\zeta}_j - \frac{\tilde{\zeta}^2}{3}\delta_{ij}\right)\left(2(\tilde{\zeta}^2 - 3)\mathcal{A}(\tilde{\zeta}, a) - \tilde{\zeta}\frac{\partial\mathcal{A}(\tilde{\zeta}, a)}{\partial\tilde{\zeta}} + 2a\frac{\partial\mathcal{A}(\tilde{\zeta}, a)}{\partial a}\right),$$

$$IB_{ij}^{(3)} = \mathcal{J}_a(\tilde{\zeta}_i\mathcal{A}(\tilde{\zeta}, a), \tilde{\zeta}_j\mathcal{A}(\tilde{\zeta}, a)) - \frac{\delta_{ij}}{3}\sum_{k=1}^{3}\mathcal{J}_a(\tilde{\zeta}_k\mathcal{A}(\tilde{\zeta}, a), \tilde{\zeta}_k\mathcal{A}(\tilde{\zeta}, a)),$$

$$IN^{(0)} = 2\tilde{\zeta}^2\left(\tilde{\zeta}^2 - \frac{7}{2}\right)\mathcal{A}(\tilde{\zeta}, a) - \tilde{\zeta}^3\frac{\partial\mathcal{A}(\tilde{\zeta}, a)}{\partial\tilde{\zeta}}, \quad IN^{(1)} = 2a\frac{\partial\mathcal{A}(\tilde{\zeta}, a)}{\partial a}\tilde{\zeta}^2 - 5a\frac{\mathrm{d}\hat{\gamma}_2(a)}{\mathrm{d}a}\left(\tilde{\zeta}^2 - \frac{3}{2}\right),$$

$$IN^{(2)} = \tilde{\zeta}^2\mathcal{A}(\tilde{\zeta}, a) - \frac{5}{2}\hat{\gamma}_2(a)\left(\tilde{\zeta}^2 - \frac{3}{2}\right), \quad IN^{(3)} = \sum_{k=1}^{3}\mathcal{J}_a(\tilde{\zeta}_k\mathcal{A}(\tilde{\zeta}, a),\ \tilde{\zeta}_k\mathcal{A}(\tilde{\zeta}, a)).$$

The functions $\mathcal{B}(\tilde{\zeta}, a)$, $\mathcal{B}_1(\tilde{\zeta}, a)$, $\mathcal{B}_2(\tilde{\zeta}, a)$, $\mathcal{N}^A(\tilde{\zeta}, a)$, and $\mathcal{N}^B(\tilde{\zeta}, a)$ are expressed by the functions defined above as follows:

$$\mathcal{B}(\tilde{\zeta}, a) = \mathcal{B}^{(0)}(\tilde{\zeta}, a), \quad \mathcal{B}_1 = -\mathcal{B}^{(1)}, \quad \mathcal{B}_2 = -\mathcal{B}^{(2)} - 2\mathcal{B}^{(3)}, \tag{95a}$$

$$\mathcal{N}^A = -\tfrac{1}{3}(\mathcal{N}^{(0)} + \mathcal{N}^{(1)} + \mathcal{N}^{(2)} + 2\mathcal{N}^{(3)}), \quad \mathcal{N}^B = -\tfrac{1}{3}\mathcal{N}^{(2)}. \tag{95b}$$

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