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On large time behavior of solutions to the Compressible Navier-Stokes Equations in the half space in $\mathbf{R}^3$

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We consider large time behavior of solutions of the compressible Navier-Stokes equation in the half space $\mathbf{R}^3_+ = \{x = (x', x_3)\,;\, x' \in \mathbf{R}^2,\ x_3 > 0\}$:

$$
(1)\qquad
\begin{aligned}
&\partial_t \rho + \operatorname{div} m = 0,\\
&\partial_t m + \operatorname{div}\left(\frac{m\otimes m}{\rho}\right) + \nabla P(\rho) = \nu\Delta\left(\frac{m}{\rho}\right) + (\nu+\tilde{\nu})\nabla \operatorname{div}\left(\frac{m}{\rho}\right),\\
&m|_{x_3=0} = 0, \quad \rho(0,x) = \rho_0(x), \quad m(0,x) = m_0(x).
\end{aligned}
$$

Here $\rho = \rho(t,x)$ and $m = (m_1(t,x), m_2(t,x), m_3(t,x))$ denote the unknown density and momentum at time $t \geq 0$ and position $x \in \mathbf{R}^3_+$, respectively; $P = P(\rho)$ is the pressure; ν and $\tilde{\nu}$ are the viscosity coefficients that satisfy $\nu > 0, \frac{2}{3}\nu + \tilde{\nu} \geq 0$; $\operatorname{div} m$ denotes the usual divergence in x of m; and ∇f denotes the usual gradient in x of a scalar function f. The notation $\operatorname{div}\left(\frac{m\otimes m}{\rho}\right)$ means that its j-th component is given by $\operatorname{div}\left(\frac{m_j m}{\rho}\right)$. In this article we are interested in large time behavior of solutions to (1) for initial data (ρ_0, m_0) near a *constant equilibrium* $(\rho, m) = (\rho^*, 0)$, where ρ^* is a given positive number.

Large time behavior of solutions to the compressible Navier-Stokes equation has been widely studied. Concerning the Cauchy problem on the whole space, Matsumura and Nishida proved in [13] the existence of solutions globally in time for all (ρ_0, m_0) with $u_0 \equiv (\rho_0 - \rho^*, m_0)$ sufficiently small in $H^3 \cap L^1$ and the decay rate of the perturbation $\|U(t)\|_{L^2} \equiv \|(\rho(t) - \rho^*, m(t))\|_{L^2} =$

$O(t^{-3/4})$ as $t \to \infty$. Also, they proved in [14] the global existence of solutions for all (ρ_0, m_0) with u_0 sufficiently small in H^3 and the decay property $\|U(t)\|_{L^\infty} \to 0$ as $t \to \infty$. Kawashima, Matsumura and Nishida [7] proved that

$$\|U(t) - \overline{U}(t)u_0\|_{L^2} = O(t^{-5/4}),$$

where $\overline{U}(t)u_0$ denotes the solution of the linearized problem at $(\rho^*, 0)$ with the initial value u_0, namely, they proved that the solution of problem (1) is time asymptotic to the one of the linearized problem. These results were extended by Kawashima [6] to a general class of quasilinear hyperbolic-parabolic systems. Hoff and Zumbrun [3, 4] studied large time behavior in L^p spaces. They showed that $\|m(t)\|_{L^\infty} = O(t^{-3/2})$ and $\|\rho(t) - \rho^*\|_{L^\infty} = O(t^{-2})$, namely, the perturbation of the density decays faster than the momentum in the L^∞ norm. This is due to some interaction of hyperbolic and parabolic aspects of the problem. They also showed that due to some hyperbolic aspect of the problem the solution may grow in L^p norm for p near 1. These properties were also proved by Kobayashi and Shibata [10] in a different manner.

Concerning the problem on unbounded domains with the presence of boundary, Matsumura and Nishida [15] proved the global existence of solutions for all (ρ_0, m_0) with u_0 sufficiently small in H^3 and the decay property $\|U(t)\|_{L^\infty} \to 0$ as $t \to \infty$ for the half space and exterior domains. Decay rate of the perturbation $U(t)$ was obtained by Deckelnick [1, 2] for the half space and exterior problems; it was shown in [1, 2] that

$$\begin{aligned} &\|\partial_t U(t)\|_{L^2} = O(t^{-1/2}), \quad \|\partial_x U(t)\|_{L^2} = O(t^{-1/4}), \\ &\|m(t)\|_{L^\infty} = O(t^{-1/4}), \quad \|\rho(t) - \rho^*\|_{L^\infty} = O(t^{-1/8}) \end{aligned}$$

as $t \to \infty$ for (ρ_0, m_0) with u_0 sufficiently small in H^3. Furthermore, in the case of the exterior problem, Kobayashi and Shibata [9] proved that $\|U(t)\|_{L^2} = O(t^{-3/4})$ and $\|U(t)\|_{L^\infty} = O(t^{-3/2})$ under the additional assumption $u_0 \in H^4 \cap L^1$. We have recently obtained the corresponding decay results for the half space problem.

Theorem 1. (i) *Let $u_0 = (\rho_0 - \rho^*, m_0) \in (H^3(\mathbf{R}^3_+) \times H^3(\mathbf{R}^3_+)) \cap (L^1(\mathbf{R}^3_+) \times L^1(\mathbf{R}^3_+))$ and satisfy the compatibility condition:*

$$\begin{aligned} &m_0|_{x_3=0} = 0, \\ &-\mathrm{div}\left(\tfrac{m_0 \otimes m_0}{\rho_0}\right) - \nabla P(\rho_0) + \nu\Delta\left(\tfrac{m_0}{\rho_0}\right) + (\nu + \tilde{\nu})\nabla \mathrm{div}\left(\tfrac{m_0}{\rho_0}\right)\Big|_{x_3=0} = 0. \end{aligned}$$

Assume that $\partial_\rho P(\rho^*) > 0$ and that u_0 is sufficiently small in $H^3 \times H^3$. Then there exists a unique global solution $(\rho(t), m(t))$ of problem (1) with

$U(t) = (\rho(t) - \rho^*, m(t)) \in C([0,\infty), H^3 \times H^3)$ *; and* $U(t)$ *satisfies*

$$\|U(t)\|_{L^2\times L^2} = O(t^{-3/4}) \quad \textit{and} \quad \|U(t)\|_{L^\infty\times L^\infty} = O(t^{-3/2})$$

as $t \to \infty$. *Also,*

$$\|\partial_x U(t)\|_{L^2\times L^2} = O(t^{-9/8})$$

as $t \to \infty$.

(ii) *For* $u_0 = (\overline{\rho_0}, \overline{m_0})$ *with* $\overline{\rho_0} \in H^1$ *and* $\overline{m_0} = (\overline{m_{0,1}}, \overline{m_{0,2}}, \overline{m_{0,3}}) \in L^2$ *let* $\overline{U}(t)u_0(x) = (\overline{\rho}(t,x), \overline{m}(t,x))$ *denote the solution of the linearized problem at* $(\rho^*, 0)$*:*

$$\begin{aligned} &\partial_t\overline{\rho} + \operatorname{div}\overline{m} = 0 \\ &\partial_t\overline{m} - \hat{\nu}\Delta\overline{m} - (\hat{\nu} + \hat{\tilde{\nu}})\nabla\operatorname{div}\overline{m} + p_1\nabla\overline{\rho} = 0, \qquad (2) \\ &\overline{m}|_{x_3=0} = 0, \quad (\overline{\rho}(0,x), \overline{m}(0,x)) = u_0(x), \end{aligned}$$

where $\hat{\nu} = \nu/\rho^*, \hat{\tilde{\nu}} = \tilde{\nu}/\rho^*, p_1 = \partial_\rho P(\rho^*)$. *Then, under the same assumptions on* $(\rho_0 - \rho^*, m_0)$ *in* (i), *we have*

$$\|U(t) - \overline{U}(t)u_0\|_{L^2\times L^2} = O(t^{-1})$$

as $t \to \infty$, *where* $u_0 = (\rho_0 - \rho^*, m_0)$.

(iii) *In addition to the same assumption on* $u_0 = (\rho_0 - \rho^*, m_0)$, *if we assume that* $\int_{\mathbf{R}^3_+} (\rho_0(x) - \rho^*)\, dx \neq 0$, *then*

$$\|U(t)u_0\|_{L^2\times L^2} \geq Ct^{-3/4}$$

as $t \to \infty$.

Decay rates for $\|U(t)\|_{L^p\times L^p}$ $(p = 2, \infty)$ in Theorem 1 are the same as in the case of the Cauchy and exterior problems ([3, 9, 13]). As for the decay rate for $\|\partial_x U(t)\|_{L^2\times L^2}$ we have obtained the rate $t^{-9/8}$ which is slower than the rate $t^{-5/4}$ for the Cauchy and exterior problems ([3, 9, 13]). This difference of decay rate is due to the analysis for the linearized problem (2), where we have obtained only $\|\partial_x\overline{U}(t)u_0\|_{L^2\times L^2} = O(t^{-9/8})$, see Theorem 2 below.

The property $\|U(t) - \overline{U}(t)u_0\|_{L^2\times L^2} = O(t^{-1})$ in Theorem 1 is also different from the one in the case of the Cauchy problem, where $\|U(t) - \overline{U}(t)u_0\|_{L^2\times L^2} = O(t^{-5/4})$ holds ([3, 6, 7]). To prove this in the case of the Cauchy problem, the property $\|\overline{U}(t)\partial_x u_0\|_{L^2\times L^2} = O(t^{-5/4})$ for the linearized problem is used; while in the case of the problem (1) on the half space we have only $\|\overline{U}(t)\partial_x u_0\|_{L^2\times L^2} = O(t^{-1})$, which is, however, optimal (see Theorem 3 below). This difference is due to some interaction of hyperbolic and parabolic aspects of the problem not appearing in the Cauchy problem.

Theorem 1 is proved by combining the global H^3-energy bounds obtained by Matsumura and Nishida ([15]) and the following decay estimates for solutions to the linearized problem (2).

We write the solution $(\overline{\rho},\overline{m})$ of the linearized problem (2) as

$$\overline{U}(t)u_0 = (\overline{\Psi}(t)u_0, \overline{V}(t)u_0), \quad \overline{\Psi}(t)u_0 = \overline{\rho}(t,\cdot), \quad \overline{V}(t)u_0 = \overline{m}(t,\cdot),$$
$$\overline{V}(t)u_0 = (\overline{V_1}(t)u_0, \overline{V_2}(t)u_0, \overline{V_3}(t)u_0)$$

Theorem 2. *There exists a positive constant C such that the following estimates hold for all $t \geq 1$:*

(i)
$$\|\overline{U}(t)u_0\|_{L^2\times L^2} \leq Ct^{-3/4}(\|u_0\|_{L^1\times L^1} + \|u_0\|_{L^2\times L^2}),$$
$$\|\overline{U}(t)u_0\|_{L^\infty\times L^\infty} \leq Ct^{-3/2}(\|u_0\|_{L^1\times L^1} + \|u_0\|_{H^2\times H^1}),$$

(ii)
$$\|\partial_{x'}\overline{V}(t)u_0\|_{L^2\times L^2} \leq Ct^{-5/4}(\|u_0\|_{L^1\times L^1} + \|u_0\|_{L^2\times L^2}),$$
$$\|\partial_x\overline{\Psi}(t)u_0\|_{L^2\times L^2} \leq Ct^{-5/4}(\|u_0\|_{L^1\times L^1} + \|u_0\|_{H^1\times L^2}),$$

(iii)
$$\|\partial_{x_3}\overline{V_j}(t)u_0\|_{L^2\times L^2} \leq Ct^{-9/8}(\|u_0\|_{L^1\times L^1} + \|u_0\|_{L^2\times L^2}) \quad (j=1,2),$$
$$\|\partial_{x_3}\overline{V_3}(t)u_0\|_{L^2\times L^2} \leq Ct^{-5/4}(\|u_0\|_{L^1\times L^1} + \|u_0\|_{L^2\times L^2}),$$

(iv)
$$\|\overline{U}(t)\partial_{x'}u_0\|_{L^2\times L^2} \leq Ct^{-5/4}(\|u_0\|_{L^1\times L^1} + \|u_0\|_{H^1\times L^2}).$$

(v) *If $u_0 = (0, m_0)$, then*
$$\|\overline{U}(t)\partial_{x_3}u_0\|_{L^2\times L^2} \leq Ct^{-1}(\|m_0\|_{L^1} + \|m_0\|_{L^2}).$$

(vi) *Also, for $u_0 = (\rho_0, m_0)$,*
$$\|\partial_t\overline{U}(t)u_0\|_{L^2\times L^2} \leq Ct^{-5/4}(\|u_0\|_{L^1\times L^1} + \|u_0\|_{L^2\times L^2}).$$

The estimates in Theorem 2 (i) and (v) are optimal. In fact, we have the following lower bounds.

Theorem 3. *Let* $u_0 = (\rho_0, m_0) \in (H^1 \times L^2) \cap (L^1 \times L^1)$.

(i) *If* $\int_{\mathbf{R}^3_+} \rho_0(x)\,dx \neq 0$, *then*

$$\|\overline{U}(t)u_0\|_{L^2\times L^2} \geq Ct^{-3/4}$$

as $t \to \infty$.

(ii) *Assume that* $u_0 = (0, m_0)$ *with* $m_0 = (m_{0,1}, m_{0,2}, m_{0,3}) \in H^1 \cap L^1$ *and* $\int_{\mathbf{R}^3_+} m_{0,j}(x)\,dx \neq 0$ *for* $j = 1$ *or* 2. *Then*

$$\|\overline{U}(t)\partial_{x_3}u_0\|_{L^2\times L^2} \geq Ct^{-1}$$

as $t \to \infty$.

Although the optimal decay rate of $\|\partial_x\overline{U}(t)u_0\|_{L^2\times L^2}$ for general $u_0 = (\rho_0, m_0) \in (H^1 \times L^2) \cap (L^1 \times L^1)$ is unclear, we have the following decay rate under some additional assumption on u_0.

Theorem 4. (i) *Assume that* $u_0 = (\rho_0, m_0) \in (H^1 \times L^2) \cap (L^1 \times L^1)$. *Assume also that* $x_3u_0 \in L^1 \times L^1$. *Then*

$$\|\partial_x\overline{U}(t)u_0\|_{L^2\times L^2} \leq Ct^{-5/4}(\|(1+x_3)u_0\|_{L^1\times L^1} + \|u_0\|_{H^1\times L^2}).$$

(ii) *Furthermore, in addition to the assumption of* (i), *if* $\int_{\mathbf{R}^3_+} \rho_0(x)\,dx \neq 0$, *then*

$$\|\partial_x\overline{U}(t)u_0\|_{L^2\times L^2} \geq Ct^{-5/4}$$

as $t \to \infty$.

The proof of L^2 decay estimates in the above Theorems is given in [5]. The L^∞ decay estimates are obtained in a similar manner by using the fact that the Fourier transform is bounded from L^1 to L^∞.

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