数理解析研究所講究録 1322 巻 2003 年 46-65

Global existence of solutions for the water wave equation in higher space dimensions

東京工業大学大学院理工学研究科数学専攻
井口 達雄 (Tatsuo IGUCHI)

Department of Mathematics, Graduate School of Science and Engineering,
Tokyo Institute of Technology,
2-12-1 Oh-okayama, Megro-ku, Tokyo 152-8551, JAPAN

1 Introduction

In this communication we are concerned with the initial value problem for the water wave equation in general $(n+1)$-dimensional space, especially, the existence of solutions globally in time. The water wave equation is a model system for the motion of water with free surface. It is formulated as the irrotational flow of incompressible ideal fluid in the gravitational field.

We assume that the domain $\Omega(t)$ occupied by the fluid and the free surface $\Gamma(t)$ at time $t \geq 0$ are of the forms

$$\Omega(t) = \{z \in \mathbf{R}^{n+1};\, z_{n+1} < F(t, z_1, \ldots, z_n)\},$$
$$\Gamma(t) = \{z \in \mathbf{R}^{n+1};\, z_{n+1} = F(t, z_1, \ldots, z_n)\},$$

where $z = (z_1, \ldots, z_n, z_{n+1})$ and F is the unknown. The motion of the fluid is described by the velocity $v = (v_1, \ldots, v_n, v_{n+1})$ and the pressure p satisfying the equations

$$(1)\qquad \begin{cases} \rho\big(v_t + (v\cdot\nabla)v\big) + \nabla p = -\rho g \mathbf{e}_{n+1}, \\ \operatorname{div} v = 0, \quad \operatorname{rot} v = 0 \qquad\qquad \text{in} \quad \Omega(t),\ t>0, \end{cases}$$

$$(2)\qquad \begin{cases} \Big(\frac{\partial}{\partial t} + v\cdot\nabla\Big)\big(z_{n+1} - F(t, z_1, \ldots, z_n)\big) = 0, \\ p = p_0 \qquad\qquad \text{on} \quad \Gamma(t),\ t>0, \end{cases}$$

$$(3)\qquad F|_{t=0} = F_0 \quad \text{on} \quad \mathbf{R}^n, \qquad v|_{t=0} = v_0 \quad \text{in} \quad \Omega(0),$$

where ρ is a constant density and g is the gravitational constant assumed to be positive. p_0 is the atmospheric pressure assumed to be constant, and $\mathbf{e}_{n+1}$ is the unit vector in the vertical direction z_{n+1}. In multi-dimensional cases, the rotation is defined as the twice anti-symmetric part of the Jacobian matrix, that is, $\operatorname{rot} v = (\partial v_i/\partial z_j - \partial v_j/\partial z_i)$. The initial data F_0 and v_0 in (3) should satisfy the compatibility conditions

$$\operatorname{div} v_0 = 0, \quad \operatorname{rot} v_0 = 0 \quad \text{in} \quad \Omega(0).$$

Our purpose is to show the existence of solution to the initial value problem (1)–(3).

A rigorous existence theorem for the solutions of the initial value problem was initially investigated in the framework of analytic functions by using an abstract Cauchy-Kowalevski theorem in a scaled Banach space introduced by Ovsjannikov. Then, in connection with the well-posedness of the problem the solvability in a class of Sobolev spaces were studied. More precisely, in his pioneering work [11] V.I. Nalimov investigated the well-posedness of the problem in the case where the motion of the fluid is two-dimensional and the fluid has infinite depth. He showed that if the initial data are sufficiently small in a suitable Sobolev space, namely, if the initial surface is almost flat and the initial movement of the fluid is sufficiently small, then there exists a unique solution of the problem locally in time in a suitable Sobolev space. Then, H. Yosihara [14] extended the Nalimov's result to the case of presence of almost horizontal bottom. Moreover, H. Yosihara [15] gave an existence theorem for the initial value problem by taking the surface tension into account on the free surface. He also studied the convergence of solutions as the surface tension coefficient tends to zero. We remark that the water waves with surface tension on the free surface are also called capillary-gravity waves. The two-phose problem for water waves was investigated in [7], where we showed that the corresponding initial value problem is well-posed if we take the surface tension into account and the initial data are sufficiently small, while the problem is ill-posed if there is no surface tension on the free surface. All these results dealt with sufficiently small initial data. S. Wu [12] studied the problem in exactly the same situation as Nalimov's and gave the existence theorem locally in time without assuming the initial data to be small. In [6] the author studied the initial value problem for capillary-gravity waves with rough bottom and gave an existance theorem for large initial data, where we merely assume for the bottom that it is a graph of a Lipchitz continous function. The problem of well-posedness for the three-dimensional water waves was resolved by S. Wu in her nice paper [13]. Although she considered only the three-dimensional case, her analysis can be directly extended for multi-dimensinal surface waves without any difficulties. Until now no one gives the existence theorem globally in time even if it is assumed that the initial data are sufficiently small. However, thanks of Wu's result we can now treat the multi-dimensinal water waves and, as we will see later, we have nice decay properties in time of solutions for the linearized equation if the space dimension is appropriately large. By making use of such decay property as well as

Wu's analysis where she used Clifford analysis, we will give a partial result for the global existence in time of solutions. We should also mention that the well-posedness locally in time for capillary-gravity waves in the three-dimensional case is still open.

2 Reformulation of the problem

Following S. Wu [13], we will use the Clifford analysis (see, for example, [5]). Let $\mathfrak{A}_{n+1}$ be the Clifford algebra associated with the Euclidean space $\mathbf{R}^{n+1}$ and $\mathrm{e}_1, \dots, \mathrm{e}_n, \mathrm{e}_{n+1}$ the normalized basis, that is,

$$\mathrm{e}_i^2 = -1, \quad \mathrm{e}_i\mathrm{e}_j = -\mathrm{e}_j\mathrm{e}_i \quad (i \neq j).$$

The Euclidean space $\mathbf{R}^{n+1}$ can be included in the Clifford algebra $\mathfrak{A}_{n+1}$ under the embedding

$$v = (v_1, \dots, v_n, v_{n+1}) \longmapsto v_1\mathrm{e}_1 + \cdots v_n\mathrm{e}_n + v_{n+1}\mathrm{e}_{n+1}.$$

The Dirac operator $\mathfrak{D}$ associated with the Clifford algebra $\mathfrak{A}_{n+1}$ is the first order differential operator defined by

$$\mathfrak{D} = \sum_{i=1}^{n+1} \mathrm{e}_i \frac{\partial}{\partial x_i},$$

which acts on the functions with the value in $\mathfrak{A}_{n+1}$.

Let Ω be a domain in $\mathbf{R}^{n+1}$ with the boundary $\Gamma := \partial\Omega$ of C^2-class. For $f \in C^1(\Omega; \mathfrak{A}_{n+1})$ we say that f is Clifford analytic in Ω if f is a kernal of the Dirac operator, namely, $\mathfrak{D}f = 0$ in Ω. Since $\mathfrak{D}^2 = -\Delta$, where Δ is the Laplacian in $\mathbf{R}^{n+1}$, the components of any Clifford analytic function are always harmonic. In the case where f is a 1-vector, that is, f is in the form $f(x) = \sum_{i=1}^{n+1} f_i(x)\mathrm{e}_i$, where f_i is a real-valued function, then f is Clifford analytic in Ω if and only if $\operatorname{div} f = 0$ and $\operatorname{rot} f = 0$ in Ω. Therefore, the velocity v embedded in the Clifford algebra is Clifford analytic in $\Omega(t)$ due to the second and the third equations in (1). The following proposition is a Clifford version of well-known Cauchy's integral theorem.

Proposition 1. *Suppose that f is a Clifford analytic function in Ω. Then, for any subdomain $\omega \Subset \Omega$ with C^1-boundary $\partial\omega$ we have the identity*

$$\int_{\partial\omega} \mathbf{n}(\xi) f(\xi) dS_\xi = 0,$$

where $\mathbf{n} = \sum_{i=1}^{n+1} n_i\mathrm{e}_i$ is the outward unit normal to $\partial\omega$.

Although the Clifford analysis was originally developed for analysis of Dirac operator, we can regard it as one of extensions of complex variable theory to a multi-dimensional complex variables theory thanks to this proposition. To state the next proposition we introduce a linear transform H^Γ as a singular integral operator on the boundary Γ in the form

$$H^\Gamma f(\xi) := \text{p.v.}\int_\Gamma K(\xi' - \xi)\mathbf{n}(\xi')f(\xi')dS_{\xi'} \quad \text{for} \quad \xi \in \Gamma,$$

where

$$K(\xi) \equiv K_1(\xi)\mathrm{e}_1 + \cdots + K_{n+1}(\xi)\mathrm{e}_{n+1} = -\frac{2}{c_n}\frac{\xi}{|\xi|^{n+1}},$$

c_n is the surface area of the unit ball in $\mathbf{R}^{n+1}$ and $\mathbf{n}$ is the outward unit normal to Γ. This is called the Hilbert transform for Ω and defined for functions on $\Gamma = \partial\Omega$.

Proposition 2. *Suppose that f is a C^1-function in $\overline{\Omega}$ with the value in $\mathfrak{A}_{n+1}$. Then, f is Clifford analytic in Ω if and only if the trace of f on $\Gamma = \partial\Omega$ is a fixed point of the Hilbert Transform for Ω, that is, $f(\xi) = H^\Gamma f(\xi)$ for $\xi \in \Gamma$.*

We now reformulate the initial value problem (1)–(3) to that on the free surface by using the Lagrangian coordinates. In the following it is asssumed that the Euclidean space $\mathbf{R}^{n+1}$ is embedded in the Clifford algebra $\mathfrak{A}_{n+1}$. Let

$$\Gamma(t)\ :\ z = \xi(t,x) \equiv \big(\xi_1(t,x), \dots, \xi_n(t,x), \xi_{n+1}(t,x)\big), \quad x \in \mathbf{R}^n$$

be the parameter-representation of the free surface such that

$$\xi_t(t,x) = v(t, \xi(t,x)) \quad \text{for} \quad x \in \mathbf{R}^n,\ t > 0.$$

Then, the problem is reduced to find this unknown parametrization $\xi = \xi(t,x)$. In fact, once we find such a parametrization, we determine the unknown fluid domain $\Omega(t)$ together with the free surface $\Gamma(t)$. By regarding the above relation as the boundary condition for v, we solve the first order elliptic system $(1)_{2,3}$ for the velocity v. Then, by integrating the equation $(1)_1$ we obtain the pressure p, so that we can determine all the physical quantities. Therefore, it is sufficient to determine $\xi = \xi(t,x)$ from the initial data. To this end we have to derive the evolution equations for ξ.

Differentiating the above relation with respect to t and using the first equation in (1) we see that

$$\begin{aligned}\xi_{tt}(t,x) &= v_t(t,\xi(t,x)) + \big(\xi_t(t,x)\cdot\nabla\big)v(t,\xi(t,x)) \\ &= \big(v + (v\cdot\nabla)v\big)(t,\xi(t,x)) \\ &= -g\mathrm{e}_{n+1} - \rho^{-1}\nabla p.\end{aligned}$$

On the other hand, the pressure p is constant on the free surface $\Gamma(t)$ because of the second equation in (2) so that the gradient ∇p is parallel to the normal vector $\mathbf{n}$ on $\Gamma(t)$. Therefore, there should exists a scalar valued function a such that $\xi_{tt} + g\mathrm{e}_{n+1} = a\mathbf{n}$. As we mentioned before, the velocity v is Clifford analytic in $\Omega(t)$ because of the second and the third equations in (1). Hence, Proposition 2 says that the trace of the velocity v on the free surface, which is equal to ξ_t, should be a fixed point of the Hilbert transform for $\Omega(t)$, so that we have $\xi_t = H_{\Gamma(t)}\xi_t$, where $H_{\Gamma(t)}$ is the Hilbert transform under the Lagrangian coordinates. More precisely, it can be written in the form

$$(H_{\Gamma(t)}u)(x) = \frac{2}{c_n}\,\mathrm{p.v.}\int_{\mathbf{R}^n} K(\xi(t,y)-\xi(t,x))\big(\Pi(\xi_{y_1}(t,y_1)\wedge\cdots\wedge\xi_{y_n}(t,y))\big)u(y)dy,$$

where Π is a linear transform from the space $\mathfrak{A}^{(n)}_{n+1}$ of n-multivectors to the space $\mathfrak{A}^{(1)}_{n+1}$ of 1-vectors defined by the relation

$$\Pi(\mathrm{e}_1\cdots\mathrm{e}_{i-1}\mathrm{e}_{i+1}\cdots\mathrm{e}_{n+1}) = (-1)^{i-1}\mathrm{e}_i \qquad (1\le i\le n+1).$$

Now, the problem is reduced equivalently to the following initial value problem for new unknown $\xi = \xi(t,x)$.

$$(4)\qquad \begin{cases} \xi_{tt} + g\mathrm{e}_{n+1} = a\mathbf{n} & \text{for } x\in\mathbf{R}^n,\ t>0,\\ \xi_t = H_{\Gamma(t)}\xi_t & \text{for } x\in\mathbf{R}^n,\ t>0,\\ (\xi,\xi_t)|_{t=0} = (\xi^0,\xi^1) & \text{for } x\in\mathbf{R}^n, \end{cases}$$

where a is an unknown scalar function, $\mathbf{n}$ is the unit outward normal vector to $\Gamma(t)$ and $H_{\Gamma(t)}$ is the Hilbert transform under the Lagrangian coordinates. The initial data ξ^0 and ξ^1 should satisfy the compatibility condition $\xi^1 = H_{\Gamma(0)}\xi^1$. In the following we concentrate our attention on the analysis of the initial value problem (4).

3 Local existence

In order to understand basic propertities of the equations in (4), it would be better first to study a linearized problem. To this end we introduce new unknowns $X = (X_1, X_2, \dots, X_{n+1})$ by

$$\begin{aligned}\xi(t,x) &= x + X(t,x)\\ &= \big(x_1 + X_1(t,x)\big)\mathrm{e}_1 + \cdots + \big(x_n + X_n(t,x)\big)\mathrm{e}_n + X_{n+1}(t,x)\mathrm{e}_{n+1}.\end{aligned}$$

It is easy to check that $X\equiv 0$ is a solution of the equations (4). This solution corresponds to the trivial flow, which means that the free surface is flat and the fluid does not move. Linearizing the equations in (4) around this trivial solution $X\equiv 0$ we can obtain the following system of equations.

$$\begin{cases} X_{jt} + gX_{n+1\,x_j} = 0,\\ X_{n+1\,t} = R_1X_{1t} + \cdots + R_nX_{nt},\\ X_{jt} = -R_jX_{n+1\,t}, \quad R_iX_{jt} = R_jX_{it} \quad \text{for} \quad i,j = 1,2,\dots,n, \end{cases}$$

where $R = (R_1, R_2, \dots, R_n)$ is the Riesz transform. By setting $F := X_{n+1}$ and $u := X_t = (X_{1t}, \dots, X_{nt}, X_{n+1\,t})$, it follows from the above equations that

$$F_{tt} + g|D|F = 0, \qquad u_{tt} + g|D|u = 0,$$

where we used the notation of Fourier multiplier, that is, the operator $|D|$ is defined by

$$|D|F = \mathcal{F}^{-1}\big[|\xi|(\mathcal{F}F)(\xi)\big] \quad (= \sqrt{-\Delta}F)$$

with the Fourier transform $\mathcal{F}$. From the above equation for F we obtain the energy identity

$$\frac{d}{dt}\left(\|F_t(t)\|_s^2 + g\||D|^{1/2}F(t)\|_s^2\right) = 0,$$

which implies that if $g > 0$, then the corresponding initial value problem is well-posed in the class $C^j([0,T]; H^{s+1/2-j/2})$, where H^s is the usual Sobolev spaces of order s and $\|\cdot\|_s$ is a norm of H^s. Conversely, if $g < 0$, then the problem becomes ill-posed.

Remark 1. (i) $|D|$ coincides with the Dirichlet-to-Neumann mapping for the Laplacian Δ in the lower half-space $\mathbf{R}^{n+1}_-$.

(ii) X_j, $j = 1, 2, \dots, n$, does not satisfy the evolution equation $X_{jtt} + g|D|X_j = 0$ in general.

The following is a local existence theorem for the initial value problem (4) itself, which is essentially due to S. Wu [13].

Theorem 1. *Suppose that the gravitational constant g is positive and that the initial data ξ^0 and ξ^1 satisfy the conditions*

$$\begin{cases} \partial_{x_j}\xi^0 - \mathrm{e}_j \in H^m \ (1 \le j \le n), \quad \xi^1 \in H^{m+1/2}, \quad m \ge [n/2] + 4, \\ \|\partial_{x_1}\xi^0 - \mathrm{e}_1\|_m + \cdots + \|\partial_{x_n}\xi^0 - \mathrm{e}_n\|_m + \|\xi^1\|_{m+1/2} \le C_1 < +\infty, \\ |(\partial_{x_1}\xi^0)(x) \wedge \cdots \wedge (\partial_{x_n}\xi^0)(x)| \ge \mu > 0, \\ |\xi^0(x) - \xi^0(y)| \ge \mu|x - y| \quad \textit{for} \quad x, y \in \mathbf{R}^n, \\ \xi^1 = H_{\Gamma(0)}\xi^1. \end{cases}$$

Then, there exists $T = T(m, n, \mu, C_1, g) > 0$ such that the initial value problem (4) has a unique solution $\xi = \xi(t,x)$ satisfying

$$\xi_t \in C^j([0,T]; H^{m+1/2-j/2}), \quad j = 0, 1, 2.$$

The proof of this theorem is based on the derivation of a corresponding quasi-linear system of equations for (4) and suitable energy estimates. By setting $u = \xi_t$, the quasi-linear system has the form

$$(5) \qquad \begin{cases} u_{tt} + a\Lambda(\nabla\xi)u = f(\xi, u, u_t) & \text{for } x \in \mathbf{R}^n,\ t > 0, \\ \xi = \xi^0 + \displaystyle\int_0^t u(\tau)d\tau & \text{for } x \in \mathbf{R}^n,\ t > 0, \\ (u, u_t)|_{t=0} = (\xi^1, \xi^2) & \text{for } x \in \mathbf{R}^n, \end{cases}$$

where $a = |u_t + ge_{n+1}|$, $\Lambda(\nabla\xi)$ is the Dirichlet-to-Neumann mapping for $\triangle$ in $\Omega(t)$ and $f(\xi, u, u_t)$ is the collection of lower order terms. Introducing the singular integral operator $\mathcal{K}^*$ by

$$(\mathcal{K}^* f)(\xi) = \text{p.v.}\int_{\Gamma(t)} (K(\xi' - \xi) \cdot \mathbf{n}(\xi)) f(\xi') dS_{\xi'} \quad \text{for} \quad \xi \in \Gamma(t),$$

we can express explicitely the Dirichlet-to-Neumann mapping $\Lambda(\nabla\xi)$ as

$$\begin{aligned} \Lambda(\nabla\xi)f &= (I + \mathcal{K}^*)^{-1} \sum_{\substack{1\le i<j\le n+1 \\ 1\le k\le n}} (-1)^{i+j+k}\, \text{p.v.}\int_{\mathbf{R}^n} (n_i(\xi)K_j(\xi' - \xi) - n_j(\xi)K_i(\xi' - \xi)) \\ &\qquad \times \frac{\partial(\xi'_1, \ldots, \widehat{\xi'_i}, \ldots, \widehat{\xi'_j}, \ldots, \xi'_{n+1})}{\partial(y_1, \ldots, \widehat{y_k}, \ldots, y_n)} f'_{y_k} dy, \end{aligned}$$

where $\xi = \xi(t, x)$, $\xi' = \xi(t, y)$, $f' = f(y)$ and

$$(y_1, \ldots, \widehat{y_k}, \ldots, y_n) = (y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n), \quad \text{etc.},$$

and similar notations will be used in the following without any comments. Note that for scalar valued function f we have the identity

$$\mathcal{K}^* f = \mathfrak{R}\big(\mathbf{n} H_{\Gamma(t)}(\mathbf{n} f)\big),$$

where $\mathfrak{R}v$ is the scalar part of $v \in \mathfrak{A}_{n+1}$. Moreover, if $\xi = \xi(t, x)$ is a solution of the problem (4), then we have $\mathbf{n} = (\xi_{tt} + ge_{n+1})/|\xi_{tt} + ge_{n+1}|$. Taking these relations into account we define linear operators $\widetilde{\mathcal{K}}^*$ and $L = L(\xi, \xi_{tt})$ by

$$\widetilde{\mathcal{K}}^* f := \mathfrak{R}\big(\tilde{\mathbf{n}} H_{\Gamma(t)}(\tilde{\mathbf{n}} f)\big), \quad \tilde{\mathbf{n}} := \frac{\xi_{tt} + ge_{n+1}}{|\xi_{tt} + ge_{n+1}|},$$

and

$$\begin{aligned} L(\xi, \xi_{tt}) &= (1 + |D|)^{-1}\Big\{(I + \mathcal{K}^*)^{-1} - \sum_{j=1}^n R_j (I + \mathcal{K}^*)^{-1}(iD_j) \\ &\qquad + \sum_{j=1}^n R_j (I + \mathcal{K}^*)^{-1}[iD_j, \widetilde{\mathcal{K}}^*](I + \mathcal{K}^*)^{-1}\Big\}, \end{aligned}$$

respectively, where $[\cdot,\cdot]$ denotes the commutator. Then, we see that if ξ is a solution of (4), then it holds that $\widetilde{\mathcal{K}}^* = \mathcal{K}^*$ and $L(\xi, \xi_{tt}) = (I + \mathcal{K}^*)^{-1}$. Using this operator we can write the collection of lower order terms $f(\xi, \xi_t, \xi_{tt})$ in (5) as

$$f(\xi, \xi_t, \xi_{tt}) = L(\xi, \xi_{tt})h(\xi, \xi_t, \xi_{tt}),$$

where

$$\begin{aligned}
&h(\xi, \xi_t, \xi_{tt})\\
&= \sum_{j=1}^{n}(-1)^j\Big\{2\,\text{p.v.}\int_{\mathbf{R}^n} K(\xi'-\xi)\Pi\big((\xi'_t-\xi_t)\wedge\xi'_{y_1}\wedge\cdots\wedge\widehat{\xi'_{y_j}}\wedge\cdots\wedge\xi'_{y_n}\big)\xi'_{tty_j}dy\\
&\quad +2\,\text{p.v.}\int_{\mathbf{R}^n} K(\xi'-\xi)\Pi\big((\xi'_{tt}-\xi_{tt})\wedge\xi'_{y_1}\wedge\cdots\wedge\widehat{\xi'_{y_j}}\wedge\cdots\wedge\xi'_{y_n}\big)\xi'_{ty_j}dy\\
&\quad +\sum_{i=1}^{j-1}\text{p.v.}\int_{\mathbf{R}^n} K(\xi'-\xi)\Pi\big((\xi'_t-\xi_t)\wedge\xi'_{y_1}\wedge\cdots\wedge\xi'_{ty_i}\wedge\cdots\wedge\widehat{\xi'_{y_j}}\wedge\cdots\wedge\xi'_{y_n}\big)\xi'_{ty_j}dy\\
&\quad +\sum_{i=j+1}^{n}\text{p.v.}\int_{\mathbf{R}^n} K(\xi'-\xi)\Pi\big((\xi'_t-\xi_t)\wedge\xi'_{y_1}\wedge\cdots\wedge\widehat{\xi'_{y_j}}\wedge\cdots\wedge\xi'_{ty_i}\wedge\cdots\wedge\xi'_{y_n}\big)\xi'_{ty_j}dy\\
&\quad -\text{p.v.}\int_{\mathbf{R}^n} K(\xi'-\xi)\Pi\big(\xi'_{ty_j}\wedge\xi'_{y_1}\wedge\cdots\wedge\widehat{\xi'_{y_j}}\wedge\cdots\wedge\xi'_{y_n}\big)\big((\xi'_t-\xi_t)\cdot\nabla_{\xi'}\big)\xi'_t dy\\
&\quad -\text{p.v.}\int_{\mathbf{R}^n} K(\xi'-\xi)\Pi\big((\xi'_t-\xi_t)\wedge\xi'_{y_1}\wedge\cdots\wedge\widehat{\xi'_{y_j}}\wedge\cdots\wedge\xi'_{y_n}\big)(\xi'_{ty_j}\cdot\nabla_{\xi'})\xi'_t dy\\
&\quad -\text{p.v.}\int_{\mathbf{R}^n} K(\xi'-\xi)\Pi\big((\xi'_t-\xi_t)\wedge\xi'_{y_1}\wedge\cdots\wedge\widehat{\xi'_{y_j}}\wedge\cdots\wedge\xi'_{y_n}\big)\big((\xi'_t-\xi_t)\cdot\partial_{y_j}\nabla_{\xi'}\big)\xi'_t dy\Big\}.
\end{aligned}$$

The operator ∇_ξ in the above expression is defined as follows. For a scalar valued function $u = u(t,x)$ we let $U = U(t,\eta), \eta \in \Omega(t)$, be the harmonic function in $\Omega(t)$ satisfing the boundary condtion $U(t, \xi(t,x)) = u(t,x)$, namely,

$$\begin{cases} \Delta U(t,\cdot) = 0 & \text{in} \quad \Omega(t), \\ U(t,\cdot) = u(t,\cdot) & \text{on} \quad \Gamma(t). \end{cases}$$

Then, we set

$$\nabla_\xi u(t,x) := (\nabla U)(t, \xi(t,x)),$$

where ∇U is the gradient of U with respect to η. By making use of the Dirichlet-to-Neumann mapping $\Lambda(\nabla\xi)$ and the Hilbert transform $H_{\Gamma(t)}$, we can express this operator ∇_ξ as

$$\nabla_\xi u = -(I + H_{\Gamma(t)})\big(\mathbf{n}(-I + \mathcal{K}^*)^{-1}\Lambda(\nabla\xi)u\big).$$

Here, we also mention that the definition of the Dirichlet-to-Neumann mapping is as follows:

$$\Lambda(\nabla\xi)u(t,x) := \mathbf{n}(\xi(t,x)) \cdot \nabla_\xi u(t,x).$$

4 Global existence

We proceed to discuss the existence globally in time of solution for the initial value problem (4). In order to construct the solution we will use the decay property in time of solutions for the linearlized equation, so that let us consider the initial value problem for the linearized equation:

$$\begin{cases} F_{tt} + |D|F = 0 & \text{for} \quad x \in \mathbf{R}^n,\ t > 0, \\ (F, F_t)|_{t=0} = (F_0, F_1) & \text{for} \quad x \in \mathbf{R}^n, \end{cases}$$

where we have put $g = 1$ for simplicity. By using the Fourier multiplier the solution of this problem can be written explicitly as

$$F(t) = \cos(|D|^{1/2}t)F_0 + \frac{\sin(|D|^{1/2}t)}{|D|^{1/2}}F_1 \quad \text{for} \quad t > 0.$$

Therefore, to study the decay property of the solution it is sufficient to investigate the evolution operator $e^{it|D|^{1/2}}$, which can be written as

$$\text{(6)} \qquad e^{it|D|^{1/2}}u(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{it\Psi(\xi;y)}(\mathcal{F}u)(\xi)d\xi,$$

where

$$\Psi(\xi; y) = |\xi|^{1/2} + y \cdot \xi, \qquad y = x/t.$$

For any $y \in \mathbf{R}^n$ ($y \neq 0$) this phase function has only one stationary point $\xi_0 = -y/(4|y|^3)$ and the Hessian of this phase function at the stationary point ξ_0 can be caluculated as

$$\det\big(\text{Hess}\,\Psi(\xi_0; y)\big) = -\frac{1}{2^{n+1}|\xi_0|^{3n/2}} = -2^{2n-1}|y|^{3n}.$$

Therefore, by the stationary phase method yields that if $y \neq 0$, then the above integral (6) decays to the order $n/2$ as the time t tends to infty. Although the determinant vanishes when $y = 0$, the corresponding stationary point is infinity. Hence, it is natural to expect that the integral (6) decays to the order $n/2$ uniformly with respect to the space variables. In fact, we have the following proposition.

Proposition 3. *Let $2 \leq q \leq \infty$, $1/p + 1/q = 1$ and $N > (n+1)(1/p - 1/q)$. Then, there exists a constant $C = C(n, q, N) > 0$ such that for any $t \in \mathbf{R}$ and any $u \in C_0^\infty(\mathbf{R}^n)$ we have*

$$|e^{it|D|^{1/2}}u|_q \leq C(1 + |t|)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}|u|_{N,p},$$

where $|\cdot|_q$ and $|\cdot|_{N,p}$ are the norms of the Lebesgue space $L^q(\mathbf{R}^n)$ and the Sobolev space $W^{N,p}(\mathbf{R}^n)$, respectively.

Proof. It is sufficient to show the estimate

$$|e^{it|D|^{1/2}}u|_\infty \le C(1+|t|)^{-n/2}|u|_{n+1,1},$$

because by interpolating this inequality with $\|e^{it|D|^{1/2}}u\| = \|u\|$ we obtain the desired estimate. Now, let us recall the following lemma due to W. Littman [10].

Lemma 1. *Let Ω be a domain in $\mathbf{R}^n$ and suppose that*

$$\begin{cases} \phi \in C^\infty(\Omega),\ u \in C_0^\infty(\mathbf{R}^n),\ \operatorname{supp} u \subseteq \Omega, \\ |\det\big(\operatorname{Hess}\phi(\xi)\big)| \ge c_1 > 0 \quad \textit{for} \quad \forall \xi \in \operatorname{supp} u. \end{cases}$$

Then, there exists a constant $C > 0$ such that for any $t \in \mathbf{R}$ we have

$$|\mathcal{F}^{-1}[e^{it\phi}u]|_\infty \le C(1+|t|)^{-n/2}.$$

Take $\psi \in C_0^\infty(\mathbf{R}^n)$ such that

- $\operatorname{supp}\psi \subseteq \{\xi \in \mathbf{R}^n\,;\ 1/2 \le |\xi| \le 2\}$
- Setting $\varphi_j(\xi) := \psi(2^j\xi)$ for $j \in \mathbf{Z}$, we have

$$\sum_{j\in\mathbf{Z}} \varphi_j(\xi) \equiv 1 \quad \text{for} \quad \forall\xi \in \mathbf{R}^n\ (\xi \neq 0)$$

Then, we can decompose the integral as

$$\begin{aligned} \mathcal{F}^{-1}\big[e^{it|\cdot|^{1/2}}\mathcal{F}u\big] &= \sum_{j\in\mathbf{Z}} \mathcal{F}^{-1}\big[e^{it|\cdot|^{1/2}}\varphi_j\mathcal{F}u\big] \\ &= \sum_{j=0}^{\infty} \mathcal{F}^{-1}\big[e^{it|\cdot|^{1/2}}\varphi_j\big] * u + \sum_{j=1}^{\infty} \mathcal{F}^{-1}\left[e^{it|\cdot|^{1/2}}\frac{\varphi_{-j}}{|\cdot|^\alpha}\right] * \big(|D|^\alpha u\big), \end{aligned}$$

where $*$ denotes the convolution operator. Here, we have

$$\mathcal{F}^{-1}\big[e^{it|\cdot|^{1/2}}\varphi_j\big](x) = (2^j)^{-n}\mathcal{F}^{-1}\big[e^{i2^{-j/2}t|\cdot|^{1/2}}\psi\big](2^{-j}x)$$

and

$$\mathcal{F}^{-1}\left[e^{it|\cdot|^{1/2}}\frac{\varphi_{-j}}{|\cdot|^\alpha}\right](x) = (2^j)^{n-\alpha}\mathcal{F}^{-1}\left[e^{i2^{j/2}t|\cdot|^{1/2}}\frac{\psi}{|\cdot|^\alpha}\right](2^jx).$$

Therefore, by Lemma 1 we see that

$$\begin{aligned} \big|\mathcal{F}^{-1}\big[e^{it|\cdot|^{1/2}}\varphi_j\big](x)\big|_\infty &\le (2^j)^{-n}C(1+2^{-j/2}|t|)^{-n/2} \\ &\le C(2^{3n/4})^{-j}|t|^{-n/2} \end{aligned}$$

and that

$$\left|\mathcal{F}^{-1}\left[e^{it|\cdot|^{1/2}}\frac{\varphi_{-j}}{|\cdot|^{\alpha}}\right](x)\right|_{\infty} \le (2^j)^{n-\alpha}C(1+2^{j/2}|t|)^{-n/2}$$
$$\le C(2^{\alpha-3n/4})^{-j}|t|^{-n/2},$$

where the constant C does not depend on t nor j. Hence, if $\alpha > 3n/4$, then we get

$$\left|\mathcal{F}^{-1}\left[e^{it|\cdot|^{1/2}}\mathcal{F}u\right]\right|_{\infty} \le C|t|^{-n/2}\big(|u|_1 + ||D|^{\alpha}u|_1\big).$$

On the other hand, if $\beta > n$, then we see that

$$\left|\mathcal{F}^{-1}\left[e^{it|\cdot|^{1/2}}\frac{\varphi_{-j}}{|\cdot|^{\alpha}}\right](x)\right|_{\infty} \le \int_{\mathbf{R}^n}|(\mathcal{F}u)(\xi)|d\xi$$
$$\le |\mathcal{F}u|_{\infty}\int_{|\xi|<1}d\xi + ||\cdot|^{\beta}\mathcal{F}u|_{\infty}\int_{|\xi|>1}\frac{d\xi}{|\xi|^{\beta}}$$
$$\le C\big(|u|_1 + ||D|^{\beta}u|_1\big).$$

The above two inequalities yield the desired estimate. □

By using this decay property we can show the following main theorem.

Theorem 2. *Suppose that*

$$g > 0, \quad n \ge 5, \quad l \ge [n/4] + 3, \quad m \ge l + [(n+1)/2] + 2. \tag{7}$$

Then, there exist positive constants δ and C such that if the initial data satisfy the conditions

$$\begin{cases} E_0 \equiv \sum_{j=1}^{n}\|\partial_{x_j}\xi^0 - e_j\|_m + \||D|^{1/2}\xi^1\|_m \le \delta, \\ E_1 \equiv |F_1|_{m-1,4/3} + ||D|^{1/2}F_0|_{m-1,4/3} \le \delta, \end{cases}$$

then the initial value problem (4) *has a unique solution $\xi = \xi(t,x)$ globally in time satisfying*

$$\xi_t \in C^j([0,\infty); H^{m+1/2-j/2}), \quad j = 0,1,2.$$

Moreover, for any $t \ge 0$ we have

$$\begin{cases} \|\xi_{tt}(t)\|_m + \||D|^{1/2}\xi_t(t)\|_m \le CE_0, \\ |F_t(t)|_{l,4} + ||D|^{1/2}F(t)|_{l,4} \le CE_1(1+t)^{-n/4}, \end{cases}$$

where $|\cdot|_{l,p}$ is a norm of the Sobolev space $W^{l,p}(\mathbf{R}^n)$ and $\|\cdot\|_m = |\cdot|_{m,2}$.

The proof of this theorem is based on the energy estimates with the time-decay estimates in $L^4(\mathbf{R}^n)$. Here, we should note that the quentities appearing in the equations does not always decay as time tends to infinity. As we state in Remark 1, the quentitie X_j, $j=1,2,\ldots,n$, does not satisfy the evolution equation $X_{jtt}+g|D|X_j=0$ in the linerized case, so that we do not know any decay properties for quentities X_j, $j=1,2,\ldots,n$, themselves. Actually, such quentities represent the pertubation in the horizontal direction, and hence it is natural that the quentities do not decay to zero but decay to non-zero constants as time tends to infinity. Therefore, in order to use the decay properties stated in Proposition 3 effecitively, it would be better to use the Eulerian coordinates. However, in order to control the energy norms we are forced to use the Lagrangian coordinates. That is the reason why we will use the both coordinates in order to obtain a priori estimates.

Now, let us consider the relations between the Lagrangian coordinates and the Eulerian coordinates. Note that the free surface $\Gamma(t)$ was represented as

$$\Gamma(t)\ :\ z=\xi(t,x)\equiv\xi_1(t,x)\mathrm{e}_1+\cdots+\xi_n(t,x)\mathrm{e}_n+\xi_{n+1}(t,x)\mathrm{e}_{n+1},\qquad x\in\mathbf{R}^n.$$

Taking this into account we introduce a mapping $\Phi(\cdot;t)$ with a parameter t by

$$\Phi(x;t):=\big(\xi_1(t,x),\ldots,\xi_n(t,x)\big)\qquad\text{for}\qquad x\in\mathbf{R}^n.$$

It is easy to see that there exists a constant $\varepsilon>0$ such that if the function $\xi=\xi(t,x)$ satisfies the conditions

$$|\partial_{x_j}\xi(t,x)-\mathrm{e}_j|\le\varepsilon\quad\text{for}\quad x\in\mathbf{R}^n,\ j=1,2,\ldots,n,$$

then $\Phi(\cdot;t)\ :\ \mathbf{R}^n\to\mathbf{R}^n$ is a diffeomorphism. Now, we define the function $F=F(t,y)$ by

$$F(t,y):=\xi_{n+1}(t,\Phi^{-1}(y;t))\equiv(\xi_{n+1}\circ\Phi^{-1})(t,y). \tag{8}$$

Then, the free surface $\Gamma(t)$ can also be represented as

$$\Gamma(t)\ :\ z=\eta(t,y)\equiv y_1\mathrm{e}_1+\cdots+y_n\mathrm{e}_n+F(t,x)\mathrm{e}_{n+1},\qquad y\in\mathbf{R}^n.$$

Therefore, this parametrization y in (8) is the Eulerian coordinates and that the relation (8) represents the relation between the Lagrangian and Eulerian coordinates.

We proceed to derive an evolution equation for $F=F(t,x)$. Let us introduce new function $V=V(t,y)$ by

$$V(y,t):=v(t,\eta(t,y))\equiv\xi_t(t,\Phi^{-1}(y,t)),$$

which is the boundary value on the free surface $\Gamma(t)$ of the fluid velocity v. Since v is Clifford analytic in $\Omega(t)$, Proposition 2 says again that the trace of the velocity v on the free surface , which is equal to V, should be a fixed point of the Hilbert transform for $\Omega(t)$. Therefore, we have $V=H(\nabla F)V$, where $H(\nabla F)$ is the Hilbert transform under the Eulerian coordinates, that is,

$$\big(H(\nabla F)u\big)(x)=\frac{2}{c_n}\,\text{p.v.}\int_{\mathbf{R}^n}K(\eta(t,y)-\eta(t,x))\big(\Pi\big(\eta_{y_1}(t,y_1)\wedge\cdots\wedge\eta_{y_n}(t,y)\big)\big)u(y)dy,$$

Decomposing the operator $H(\nabla F)$ as

$$H(\nabla F) = H_0 + H_1(\nabla F) \qquad \big(H_1(0) = 0\big)$$

and writing V as

$$V = V_1 e_1 + \cdots + V_n e_n + V_{n+1} e_{n+1} = V_{\tan} + V_{n+1} e_{n+1},$$

we obtain the relations

$$V_{\tan} = -\sum_{j=1}^{n} e_j \mathfrak{R}\big(e_j H_1(\nabla F) V_{\tan}\big) - \sum_{j=1}^{n} e_j \big(R_j + \mathfrak{R}\big(e_j H_1(\nabla F) e_{n+1}\big)\big) V_{n+1}.$$

In view of this relation we introduce operators $A_1(\nabla F)$ and $A_2(\nabla F)$ depending on ∇F by

$$\begin{cases} A_1(\nabla F)u := \displaystyle\sum_{j=1}^{n} e_j \mathfrak{R}\big(e_j H_1(\nabla F) u\big), \\ A_2(\nabla F)f := \displaystyle\sum_{j=1}^{n} e_j \big(R_j f + \mathfrak{R}\big(e_j H_1(\nabla F) e_{n+1} f\big)\big). \end{cases}$$

Then, we get

$$V_{\tan} = K(\nabla F) V_{n+1}, \tag{9}$$

where

$$K(\nabla F) = K_1(\nabla F) e_1 + \cdots + K_n(\nabla F) e_n,$$
$$K_j(\nabla F) = \mathfrak{R}\big(e_j (1 + A_1(\nabla F))^{-1} A_2(\nabla F)\big), \quad 1 \le j \le n.$$

It follows from (8) that

$$F_t + (V_{\tan} \cdot \nabla) F = V_{n+1},$$

which together with (9) implies the relations

$$\begin{cases} V_{n+1} = \big(1 - (\nabla F) \cdot K(\nabla F)\big)^{-1} F_t, \\ V_{\tan} = K(\nabla F)\big(1 - (\nabla F) \cdot K(\nabla F)\big)^{-1} F_t. \end{cases} \tag{10}$$

Hence, the trace of the velocity v on the free surface can be expressed in temrs only of F. On the other hand, in the Eulerian coordinates the first equation in (4) can be written in the form

$$V_{\tan\, t} + (V_{\tan} \cdot \nabla) V_{\tan} + (\nabla F)\big(V_{n+1\, t} + (V_{\tan} \cdot \nabla) V_{n+1}\big) + g \nabla F = 0.$$

Putting (10) into the above equation we see after some caluculation that that the function $F = F(t, x)$ should be a solution of the initial value problem

$$\begin{cases} F_{tt} + g|D|F = N(F_t, \nabla F) & \text{in } \mathbf{R}^n,\ t > 0, \\ (F, F_t)|_{t=0} = (F_0, F_1) & \text{in } \mathbf{R}^n, \end{cases} \tag{11}$$

where

$$
\begin{aligned}
N(F_t,\nabla F) \;=\;& \big(1-(\nabla F)\cdot K(\nabla F)\big)\big(1+R\cdot\big(K_1(\nabla F)+\nabla F\big)\big)^{-1}N_1(F_t,\nabla F)\\
&+\big((\nabla F)\cdot K(\nabla F)\big)\big(1+R\cdot\big(K_1(\nabla F)+\nabla F\big)\big)^{-1}\big(g|D|F\big)\\
&+\big(R\cdot\big(K_1(\nabla F)+\nabla F\big)\big)\big(1+R\cdot\big(K_1(\nabla F)+\nabla F\big)\big)^{-1}\big(g|D|F\big)
\end{aligned}
$$

and

$$
\begin{aligned}
N_1(F_t,\nabla F) \;=\;& -R\cdot\big\{[\partial_t K(\nabla F)]V_{n+1}+(V_{\tan}\cdot\nabla)V_{\tan}+(\nabla F)(V_{\tan}\cdot\nabla)V_{n+1}\\
&+\big(K(\nabla F)+\nabla F\big)\big(1-(\nabla F)\cdot K(\nabla F)\big)^{-1}[\partial_t,(\nabla F)\cdot K(\nabla F)]V_{\tan}\big\}.
\end{aligned}
$$

In the above equation V should be replaced by a suitable function of F according to (10). We note that $N(F_t,\nabla F)$ is the collection of non-linear terms, however, it contains the highest order terms. Therefore, we can not obtain a nice energy estimates from the equation in (11).

5 A priori estimates

We will sketch the outline of the proof of Theorem 2, which is, of course, based on suitable a priori estimates. We first note the following lemma.

Lemma 2. *For any $u,v\in C_0^\infty(\mathbf{R}^n)$ we have*

$$
(u,N\Lambda(\nabla\xi)v)=(N\Lambda(\nabla\xi)u,v),
$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\mathbf{R}^n)$ and $N=|\xi_{x_1}\wedge\xi_{x_2}\wedge\cdots\wedge\xi_{x_n}|$.

Taking this identity into account we define the energy norm $E_m(t)$ by

$$
\begin{aligned}
E_m(t) \;:=\;& (J\xi_{tt},\xi_{tt})+g\||D|^{1/2}(\xi_t\circ\Phi^{-1})\|^2\\
&+(N\Lambda(\nabla\xi)^m\xi_{tt},\Lambda(\nabla\xi)^m\xi_{tt})+(Na\Lambda(\nabla\xi)^{m+1}\xi_t,\Lambda(\nabla\xi)^m\xi_t),
\end{aligned}
$$

where $J=\det\big((\partial\xi_j/\partial x_i)_{1\le i,j\le n}\big)$. Since the Dirichlet-to-Neumann mapping $\Lambda(\nabla\xi)$ is an operator of order 1, this energy norm is almost equivalent to the norm

$$
\varphi_m(t)^2:=\|\xi_{tt}(t)\|_m^2+\||D|^{1/2}\xi_t(t)\|_m^2.
$$

Taking the derivative with respect to t for the energy norm we see that

$$
\begin{aligned}
\frac{d}{dt}E_m(t) \;=\;& \frac{d}{dt}\big((J\xi_{tt},\xi_{tt})+g\||D|^{1/2}(\xi_t\circ\Phi^{-1})\|^2\big)\\
&+(Na\Lambda(\nabla\xi)^{m+1}\xi_{tt},\Lambda(\nabla\xi)^m\xi_t)+(Na\Lambda(\nabla\xi)^{m+1}\xi_t,\Lambda(\nabla\xi)^m\xi_{tt})\\
&+2(N\Lambda(\nabla\xi)^m\xi_{ttt},\Lambda(\nabla\xi)^m\xi_{tt})+I_1(t),
\end{aligned}
$$

where $I_1(t)$ is the collection of lower order terms and written as

$$\begin{aligned} I_1(t) &= ([\partial_t, N\Lambda(\nabla\xi)^m]\xi_{tt}, \Lambda(\nabla\xi)^m\xi_{tt}) + (N\Lambda(\nabla\xi)^m\xi_{tt}, [\partial_t, \Lambda(\nabla\xi)^m]\xi_{tt}) \\ &\quad +([\partial_t, Na\Lambda(\nabla\xi)^{m+1}]\xi_t, \Lambda(\nabla\xi)^m\xi_t) + (Na\Lambda(\nabla\xi)^m\xi_t, [\partial_t, \Lambda(\nabla\xi)^m]\xi_t). \end{aligned}$$

By using the equation in (5) we have

$$\begin{aligned} &\frac{d}{dt}\big((J\xi_{tt}, \xi_{tt}) + g\||D|^{1/2}(\xi_t \circ \Phi^{-1})\|^2\big) \\ &= -2(J\xi_{tt}, g\Lambda(\nabla\xi)\xi_t) + 2(J\xi_{tt}, (g-a)\Lambda(\nabla\xi)\xi_t) + 2(J\xi_{tt}, f(\xi, \xi_t, \xi_{tt})) \\ &\quad +(J_t\xi_{tt}, \xi_{tt}) + 2((\xi_t \circ \Phi^{-1})_t, g|D|(\xi_t \circ \Phi^{-1})). \end{aligned}$$

Lemma 3. *For any $u \in C_0^\infty(\mathbf{R}^n)$ we have the identity*

$$\big(\Lambda(\nabla\xi)u\big) \circ \Phi^{-1} = \Lambda(\nabla\eta)(u \circ \Phi^{-1}).$$

Proof. Let U be the harmonic extension of u in $\Omega(t)$, that is,

$$\begin{cases} \Delta U(t,\cdot) = 0 & \text{in} \quad \Omega(t), \\ U(t, \xi(t,x)) = u(x) & \text{for} \quad x \in \mathbf{R}^n. \end{cases}$$

Then, by definition we have

$$\big(\Lambda(\nabla\xi)u\big)(t,x) = \mathbf{n}(\xi(t,x)) \cdot (\nabla U)(t, \xi(t,x)).$$

Since $\xi(t, \Phi^{-1}(y;t)) = \eta(t,y)$, we see that

$$U(t, \eta(t,y)) = u(\Phi^{-1}(y;t)) = (u \circ \Phi^{-1})(t,y).$$

Therefore, we obtain

$$\begin{aligned} \big(\Lambda(\nabla\xi)u\big)(t, \Phi^{-1}(y;t)) &= \mathbf{n}(\eta(t,y)) \cdot (\nabla U)(t, \eta(t,y)) \\ &= \big(\Lambda(\nabla\eta)(u \circ \Phi^{-1})\big)(t,y), \end{aligned}$$

which is the desired identity. □

By this lemma we have

$$(J\xi_{tt}, \Lambda(\nabla\xi)\xi_t) = (\xi_{tt} \circ \Phi^{-1}, \Lambda(\nabla\eta)(\xi_t \circ \Phi^{-1})).$$

Moreover, it is easy to see that

$$(\xi_t \circ \Phi^{-1})_t = \xi_{tt} \circ \Phi^{-1} + (D\xi_t \circ \Phi^{-1})(\Phi^{-1})_t.$$

Hence, we obtain

$$
\begin{aligned}
&\frac{d}{dt}((J\xi_{tt},\xi_{tt})+g\||D|^{1/2}(\xi_t\circ\Phi^{-1})\|^2)\\
&= 2(\xi_{tt}\circ\Phi^{-1},(\Lambda(\nabla\eta)-|D|)(\xi_t\circ\Phi^{-1}))+2((D\xi_t\circ\Phi^{-1})(\Phi^{-1})_t,|D|(\xi_t\circ\Phi^{-1}))\\
&\quad +2(J\xi_{tt},f(\xi,\xi_t,\xi_{tt}))+(J\xi_{tt},(g-a)\Lambda(\nabla\xi)\xi_t)+(J_t\xi_{tt},\xi_{tt})\\
&=: I_2(t).
\end{aligned}
$$

By using the equation in (5) again we see that

$$
\begin{aligned}
&2(N\Lambda(\nabla\xi)^m\xi_{ttt},\Lambda(\nabla\xi)^m\xi_{tt})\\
&= -2(N\Lambda(\nabla\xi)^m(a\Lambda(\nabla\xi)\xi_t),\Lambda(\nabla\xi)^m\xi_{tt})+2(N\Lambda(\nabla\xi)^m f(\xi,\xi_t,\xi_{tt}),\Lambda(\nabla\xi)^m\xi_{tt})\\
&= -(Na\Lambda(\nabla\xi)^{m+1}\xi_{tt},\Lambda(\nabla\xi)^m\xi_t)-(Na\Lambda(\nabla\xi)^{m+1}\xi_t,\Lambda(\nabla\xi)^m\xi_{tt})+I_3(t),
\end{aligned}
$$

where $I_3(t)$ is the collection of lower order terms and written as

$$
\begin{aligned}
I_3(t) &= -(N\Lambda(\nabla\xi)[\Lambda(\nabla\xi)^{m-1},a]\Lambda(\nabla\xi)\xi_t,\Lambda(\nabla\xi)^m\xi_{tt})\\
&\quad -(N[\Lambda(\nabla\xi)^m,a]\Lambda(\nabla\xi)\xi_t,\Lambda(\nabla\xi)^m\xi_{tt})\\
&\quad +2(N\Lambda(\nabla\xi)^m f(\xi,\xi_t,\xi_{tt}),\Lambda(\nabla\xi)^m\xi_{tt}).
\end{aligned}
$$

Summarizing the above calculations yields that

$$\frac{d}{dt}E_m(t)=I_1(t)+I_2(t)+I_3(t)=:I(t),$$

so that we have

$$E_m(t)\le E_m(0)+\int_0^t|I(\tau)|d\tau. \tag{12}$$

On the other hand, it follows from (11) that

$$
\begin{aligned}
F(t)&=\cos\big((g|D|)^{1/2}t\big)F_0+\frac{\sin\big((g|D|)^{1/2}t\big)}{(g|D|)^{1/2}}F_1\\
&\quad+\int_0^t\frac{\sin\big((g|D|)^{1/2}(t-\tau)\big)}{(g|D|)^{1/2}}N(F_t,\nabla F)(\tau)d\tau.
\end{aligned}
$$

This and Proposition 3 imply that

$$
\begin{aligned}
&|F_t(t)|_{l,4}+||D|^{1/2}F(t)|_{l,4}\\
&\le C(1+t)^{-n/4}\big(|F_1|_{l+\beta,4/3}+||D|^{1/2}F_0|_{l+\beta,4/3}\big)\\
&\quad+C\int_0^t(1+t-\tau)^{-n/4}|N(F_t,\nabla F)(\tau)|_{l+\beta,4/3}d\tau
\end{aligned}
\tag{13}
$$

for any $t\ge 0$, where $l\ge 0$ and $\beta=[(n+1)/2]+1$.

Now, we have to estimate the non-linear terms $I(t)$ and $N(F_t, \nabla F)$. However, it is not so easy to do that because the nonlinearities come from the kernel of singular integral operators. Therefore, we proceed to investigate the singular integral operator of the form

$$Tf(x) = \text{p.v.}\int_{\mathbf{R}^n} K(x,y)f(y)dy$$

with the kernal

$$K(x,y) = \frac{1}{|x-y|^n}\left(\prod_{k=1}^{M}\frac{A_k(x)-A_k(y)}{|x-y|}\right)G\left(\frac{B_1(x)-B_1(y)}{|x-y|},\dots,\frac{B_N(x)-B_N(y)}{|x-y|}\right).$$

For any $\varepsilon > 0$ we define an (absolutely conergente) integral operator T_ε by

$$T_\varepsilon f(x) := \int_{|x-y|>\varepsilon} K(x,y)f(y)dy$$

and put

$$T_* f(x) := \sup_{\varepsilon>0}|T_\varepsilon f(x)|,$$

which is called the maximal operator corresponding the singular integral operator T.

Proposition 4. *Let $R > 0$ and suppose that $G \in C^\infty(B_{2R})$ and $G(-z) = (-1)^{M+1}G(z)$ for $z \in B_{2R}$, where B_{2R} is a ball in $\mathbf{R}^N$ with center 0 and radius $2R$. Then, there exists a positive constant C such that under the conditions*

$$\begin{cases} \nabla A_k \in L^k(\mathbf{R}^n), \quad k = 1,2,\dots,M, \\ \nabla B_l \in L^\infty(\mathbf{R}^n),\ |\nabla B_l|_\infty \le R, \quad l = 1,2,\dots,N, \\ 1 < p < \infty,\ 1 < p_k \le \infty,\ 1 < q \le \infty, \\ 1/p = 1/p_1 + 1/p_2 + \dots + 1/p_M + 1/q, \end{cases}$$

we have

$$|T_* f|_p \le C|\nabla A_1|_{p_1}|\nabla A_2|_{p_2}\cdots|\nabla A_M|_{p_M}|f|_q.$$

Roughly speaking, this is a singular integral version of Hölder's inequality.

The above estimate in the case $n = 1$, $p = q = 2$ and $p_1 = p_2 = \dots = p_M = \infty$ is the result due to R.R. Coifman, A. MaIntosh and Y. Meyer [3]. Their result together with the well-known Calderón and Zygumond theory generalizes it to the case $1 < p = q < \infty$. The generalization of the results to the case $n \ge 2$ is due to the method of rotation. (See also [2] and [4].) In [1] R.R. Coifman and Y. Meyer derived the similar estimate as above for the commutator singular integral of Calderón, which corresponds to the case $n = 1$, $M = 2$ and $G(z) \equiv 1$. However, their thechnique can be applicable to the above operator with slight modifications.

Now, we turn to a priori estimates. In what follows, for simplicity, we will use the abbreviation

$$|\partial\xi - \mathrm{e}| = |\partial_{x_1}\xi - \mathrm{e}_1| + \cdots + |\partial_{x_n}\xi - \mathrm{e}_n|,$$

and similar notation will be used without any comments. Based on this Proposition 4 we can show the following lemma.

Lemma 4. *There exists a small positive constant δ_1 such that under the conditions*

$$\begin{cases} |\partial\xi - \mathrm{e}|_{\alpha+1,4} + |\xi_t|_{\alpha+1,4} + |\xi_{tt}|_{\alpha+1,4} \le \delta_1, \\ |F_t|_{\alpha,4} + |\nabla F|_{\alpha,4} \le \delta_1, \qquad \alpha = [n/4] + 1, \end{cases}$$

we have

$$E_m(t) \le C\big(\varphi_m(t)^2 + \|\partial\xi(t) - \mathrm{e}\|^2_{m-1/2}\big),$$

$$\varphi_m(t)^2 \le C\big(E_m(t) + (|F_t(t)|_{\alpha+2,4} + |\nabla F(t)|_{\alpha+1,4})^2\|\partial\xi - \mathrm{e}\|^2_{m-1/2}\big),$$

$$\begin{aligned} |I(t)| \ \le\ & C(|F_t(t)|_{\alpha+2,4} + |\nabla F(t)|_{\alpha+1,4})\varphi_m(t)^2 \\ & + C(|F_t(t)|_{\alpha+2,4} + |\nabla F(t)|_{\alpha+1,4})^3\|\partial\xi(t) - \mathrm{e}\|^2_{m-1/2} \end{aligned}$$

and

$$\begin{aligned} |N(F_t, \nabla F)(t)|_{s,4/3} \ \le\ & C(|F_t(t)|_{\alpha,4} + |\nabla F(t)|_{\alpha,4})\{\varphi_{s+1}(t) \\ & + (|F_t(t)|_{\alpha+1,4} + |\nabla F(t)|_{\alpha+1,4})\|\partial\xi(t) - \mathrm{e}\|_s\}, \end{aligned}$$

where C is a positive constant independent of the solution ξ.

Put

$$\psi_l(t) := (1+t)^{n/4}\big(|F_t(t)|_{l,4} + |\nabla F(t)|_{l,4}\big)$$

and

$$\widetilde{\varphi}_m(t) := \sup_{0\le\tau\le t} \varphi_m(\tau), \qquad \widetilde{\psi}_l(t) := \sup_{0\le\tau\le t} \psi_l(\tau).$$

In the following we assume that n, l and m satisfy the conditions in (7). In view of the estimate

$$\begin{aligned} \|\partial\xi(t) - \mathrm{e}\|_s \ &\le\ \|\partial\xi^0 - \mathrm{e}\|_s + \int_0^t \|\nabla\xi_t(\tau)\|_s d\tau \\ &\le\ \|\partial\xi^0 - \mathrm{e}\|_s + t\widetilde{\varphi}_{s+1/2}(t), \end{aligned}$$

we see in turn that

$$|I(t)| \le C(1+t)^{-n/4}\psi_{\alpha+2}(t)\big\{\varphi_m(t)^2 + \psi_{\alpha+2}(t)^2\big(\|\partial\xi^0 - \mathrm{e}\|_{m-1/2} + \widetilde{\varphi}_m(t)\big)^2\big\},$$

$$E_m(t) \le E_m(0) + C\widetilde{\psi}_{\alpha+2}(t)\big\{\widetilde{\varphi}_m(t)^2 + \widetilde{\psi}_{\alpha+2}(t)^2\big(\|\partial\xi^0 - \mathrm{e}\|_{m-1/2}\widetilde{\varphi}_m(t)^2\big)\big\}$$

and that

$$\widetilde{\varphi}_m(t)^2 \le C\big(E_m(0) + \widetilde{\psi}_l(t)\widetilde{\varphi}_m(t)^2 + \widetilde{\psi}_l(t)^2(\|\partial\xi^0 - \mathrm{e}\|_{m-1/2} + \widetilde{\varphi}_m(t)^2)\big).$$

Moreover, it follows that

$$\begin{aligned}&|N(F_t, \nabla F)(t)|_{l+\beta,4/3}\\ &\le \ C(1+t)^{-n/4}\psi_l(t)\{\varphi_m(t) + \psi_l(t)\big(\|\partial\xi^0 - \mathrm{e}\|_{m-1} + \widetilde{\varphi}_m(t)\big)\}\end{aligned}$$

and that

$$\begin{aligned}\widetilde{\psi}_l(t) \le\ & C\big(|F_1|_{m-1,4/3} + ||D|^{1/2}F_0|_{m-1,4/3}\big)\\ &+ C\widetilde{\psi}_l(t)\{\widetilde{\varphi}_m(t) + \widetilde{\psi}_l(t)\big(\|\partial\xi^0 - \mathrm{e}\|_{m-1} + \widetilde{\varphi}_m(t)\big)\}.\end{aligned}$$

Therefore, we obtain the following lemma.

Lemma 5. *There exists a small positive constant δ_2 such that if*

$$\widetilde{\varphi}_m(t) + \widetilde{\psi}_l(t) \le \delta_2,$$

then we have

$$\widetilde{\varphi}_m(t)^2 \le C\big(E_m(0) + \|\partial\xi^0 - \mathrm{e}\|_{m-1/2}\big),$$

$$\widetilde{\psi}_l(t) \le C\big(|F_1|_{m-1,4/3} + ||D|^{1/2}F_0|_{m-1,4/3} + \|\partial\xi^0 - \mathrm{e}\|_{m-1}\big).$$

Since

$$E_m(0) \le C\big(\||D|^{1/2}\xi^1\|_m^2 + \|\partial\xi^0 - \mathrm{e}\|_m^2\big),$$

we can prove our main theorem by standard arguments.

The details will be published elsewhere.

References

[1] R.R. Coifman & Y. Meyer, On commutators of singular integrals and bilinear singular integrals, Trans. Amer. Math. Soc., **212** (1975), 315–331.

[2] R.R. Coifman, G. David & Y. Meyer, La solution des conjecture de Calderón, Adv. in Math., **48** (1983), 144–148 [French].

[3] R.R. Coifman, A. McIntosh & Y. Meyer, L'integrale de Cauchy definit un operateur borne sur L^2 pour les courbes lipschitziennes, Ann. of Math., **116** (1982), 361–387 [French].

[4] G. David, Operateurs integraux singuliers sur certaines courbes du plan complexe, Ann. Sci. Ecole Norm. Sup., (4) **17** (1984), 157–189 [French].

[5] J. E. Gilbert & M. A. M. Murray, Clifford algebras and Dirac operators in harmonic analysis, Cambridge Studies in Advanced Mathematics, 26. Cambridge University Press, Cambridge, 1991.

[6] T. Iguchi, Well-posedness of the initial value problem for capillary-gravity waves, Funkcial. Ekvac., **44** (2001), 219–241.

[7] T. Iguchi, N. Tanaka & A. Tani, On the two-phase free boundary problem for two-dimensional water waves, Math. Ann., **309** (1997), 199–223.

[8] C. E. Kenig, Recent progress on boundary value problems on Lipschitz domains. Pseudodifferential operators and applications (Notre Dame, Ind., 1984), 175–205, Proc. Sympos. Pure Math., **43**, Amer. Math. Soc., Providence, RI, 1985.

[9] C. E. Kenig, Elliptic boundary value problems on Lipschitz domains. Beijing lectures in harmonic analysis (Beijing, 1984), 131–183, Ann. of Math. Stud., **112**, Princeton Univ. Press, Princeton, NJ, 1986.

[10] W. Littman, Fourier transforms of surface-carried measures and differentiability of surface averages, Bull. Amer. Math. Soc., **69** (1963), 766–770.

[11] V. I. Nalimov, The Cauchy-Poisson problem, Dinamika Splošn. Sredy, **18** (1974), 104–210 [Russian].

[12] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 2-D, Invent. Math., **130** (1997), 39–72.

[13] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 3-D, J. Amer. Math. Soc., **12** (1999), 445–495.

[14] H. Yosihara, Gravity waves on the free surface of an incompressible perfect fluid of finite depth, Publ. RIMS Kyoto Univ., **18** (1982), 49–96.

[15] H. Yosihara, Capillary-gravity waves for an incompressible ideal fluid, J. Math. Kyoto Univ., **23** (1983), 649–694.