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WELL-POSEDNESS FOR THE FOURTH ORDER NONLINEAR SCHRÖDINGER TYPE EQUATION RELATED TO THE VORTEX FILAMENT

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1. Introduction and Main Result

We call "the vortex filament" the slender vortex tube. In the present paper, we consider the initial value problem for the equation which describes three-dimensional motion of an isolated vortex filament embedded in an inviscid incompressible fluid fulfilled in infinite region. Especially, we are concerned with the initial value problem for the fourth order nonlinear Schrödiger type equation:

$$
(1.1)\qquad \begin{cases} i\partial_t u + \partial_x^2 u + \nu \partial_x^4 u = F(u, \overline{u}, \partial_x u, \partial_x \overline{u}, \partial_x^2 u, \partial_x^2 \overline{u}), & x, t \in \mathbf{R}, \\ u(x,0) = u_0(x), & x \in \mathbf{R}, \end{cases}
$$

where $u(x,t) : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ is unknown function, nonlinear term F is given by

$$
\begin{aligned} F(u, \overline{u}, \partial_x u, \partial_x \overline{u}, \partial_x^2 u, \partial_x^2 \overline{u}) = &-\frac{1}{2}|u|^2 u + \lambda_1 |u|^4 u + \lambda_2 (\partial_x u)^2 \overline{u} + \lambda_3 |\partial_x u|^2 u \\ &+ \lambda_4 u^2 \partial_x^2 \overline{u} + \lambda_5 |u|^2 \partial_x^2 u, \end{aligned}
$$

with $\lambda_1 = -3\mu/4, \lambda_2 = -2\mu + \nu/2, \lambda_3 = -4\mu - \nu, \lambda_4 = -\mu, \lambda_5 = -2\mu + \nu$ and real constants ν, μ.

To motivate our problem, let us review the history of the model equation of vortex motion. In [5], Da Rios introduced the following model of the motion of the vortex filament called "localized induction approximation" from Biot-Savart law: We denote the centerline of the vortex filament by $\mathbf{X} = \mathbf{X}(x,t)$, represented as functions of arclength x and time t. Let (κ, τ) be curcature and torsion, $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ be the Frenet-serret frame of the centerline, respectively. Then $\mathbf{X}$ satisfies following equation:

$$
\partial_t \mathbf{X} = \frac{\Gamma}{4\pi} \log\left(\frac{L}{\sigma}\right) \kappa \mathbf{b},
$$

where Γ is the circulation of the vortex motion, σ the core radius and remaining parameter L is the length of the segment whose contribution to the induction velocity is taken account of above. We see that a vortex goes in the direction of the binomial vector and it's velocity is in propotion to the curvature. By rescaling the time variable, we have

$$
(1.2)\qquad \partial_t \mathbf{X} = \kappa \mathbf{b}.
$$

To describe the motion of actual vortex filament precisely, much more detaileded models of equation have been introduced by several authors.

Fukumoto-Miyazaki [7] derived the following equation for the motion of a vortex filament with axial flow correct to the second order in time ratio of the core radius to that of curvature:

$$\partial_t \mathbf{X} = \kappa \mathbf{b} + \nu \left[\frac{1}{2}\kappa^2 \mathbf{t} + \partial_x \kappa \mathbf{n} + \kappa\tau \mathbf{b} \right], \tag{1.3}$$

where ν is real constants.

In further, the local self-induced flow around the core comprises not only a uniform flow but also a straining field which deforms the core into an ellipse. So, Fukumoto-Moffatt [8] proposed the following model:

$$\partial_t \mathbf{X} = \kappa \mathbf{b} + \nu[(2(\partial_x\kappa)\tau + \kappa(\partial_x\tau))\mathbf{n} + (\kappa\tau^2 - \partial_x^2\kappa)\mathbf{b} + \kappa^2\tau\mathbf{t}] + \mu\kappa^3\mathbf{b}, \tag{1.4}$$

where ν and μ are real constants.

By introducing the "Hasimoto transform" (see [10]) defined by

$$u(x,t) = \kappa e^{i\int^x \tau dx}, \tag{1.5}$$

equations (1.2) is transformed to the well-known cubic nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u + \frac{1}{2}|u|^2 u + A(t)u = 0. \tag{1.6}$$

Similarly, equation (1.3) is transformed to "Hirota equation" that is the third order nonlinear Schrödinger type equation which is including both the cubic nonlinear Schrödinger equaiton and the modified Korteweg-de Vries eqaution:

$$i\partial_t u + \partial_x^2 u + \frac{1}{2}|u|^2 u + A(t)u - i\nu\left\{\partial_x^3 u + \frac{3}{2}|u|^2\partial_x u\right\} = 0. \tag{1.7}$$

Furthermore, equation (1.4) is transformed to the fourth order nonlinear Schrödinger type equation:

$$\begin{aligned} i\partial_t u + \partial_x^2 u + \frac{1}{2}|u|^2 u + A(t)u - \nu\Big\{\partial_x^4 u + \frac{3}{2}(|u|^2\partial_x^2 u + (\partial_x u)^2\overline{u}) \\ + \left(\frac{3}{8}|u|^4 + \frac{1}{2}\partial_x^2|u|^2\right)u\Big\} + \left(\mu + \frac{\nu}{2}\right)\left\{\partial_x^2(|u|^2 u) + \frac{3}{4}|u|^4 u\right\} = 0. \end{aligned} \tag{1.8}$$

Here $A(t)$ is arbitrary function of t. We note that the localized induction equations (1.2) and (1.3) are completely integrable equations equivalent to cubic nonlinear Schrödinger equation and the Hirota equation, respectively. A brief summary above is given by Fukumoto [6].

We are interested in the solvability and well-posedness of those problems. Here, the well- posedness stands for the existence, uniqueness of the solutions and continuous dependence upon the initial data. The solvability and the well-posedness of those problem are studied by several authors. Nishiyama-Tani [20] and Tani-Nishiyama [26] showed the existence of weak solution for initial and initial-boundary value problems for the localized induction equation (1.2) and (1.3), respectively. More precisely, in [20], they have considered the following four situations:

(a) the curvature $|\partial_x^2 \mathbf{X}| \to 0$, as $|x| \to \infty$.
(b) $\mathbf{X}(x,t)$ approaches an exact solution $\mathbf{Y}(x,t)$ with $\partial_x^2 \mathbf{Y}(\pm\infty, t) \neq 0$, such as a herix or an elastica, as $|x| \to \infty$.
(c) $\partial_x^2 \mathbf{X}(\pm 1, t) = 0$ is satisfied when the domain of x is restricted to $(-1,1)$.
(d) the filament is closed, that is, $\mathbf{X}(x-1,t) = \mathbf{X}(x+1,t)$, for $x \in \mathbf{R}$.

In [26], they showed the existence and uniqueness of solution to (1.3) with the initial condition $\mathbf{X}(x,0) = \mathbf{X}_0$ and $\partial_x \mathbf{X}_0 = 1$. They also have considered the spatially periodic case $\partial_x \mathbf{X}(x,t) = \partial_x \mathbf{X}(x+1,t)$.

Tsutsumi [27] showed the global well-posedness of the initial value problem for (1.6) in L^2 (see also Ginibre-Velo [9], Cazenave-Weissler [4] and references therein). For initial value problem (1.7), Staffilani [22] proved the time local well-posedness in Sobolev space $H^s(\mathbf{R})$ with $s \geq 1/4$ (see also Laurey [19] and Takaoka [25]). In [25], Takaoka also showed the local well-posedness of (1.7) in $H^{1/2}$ for the spatially periodic case. Here, Sobolev space is defined by $H^s(\mathbf{R}) = \{f \in \mathcal{S}'(\mathbf{R}); \langle \xi \rangle^s \mathcal{F}_x f \in L^2(\mathbf{R})\}$,where$\langle x \rangle = (1+|x|^2)^{1/2}$ and $\mathcal{F}_x f$ is Fourier Transform of f with respect to x. The purpose of the paper is to show the local well-posedness for (1.1) for larger class of the initial data in the Sobolev space $H^s(\mathbf{R})$ when $\nu < 0$ and $\lambda_5 = -2\mu + \nu = 0$.

Before we state our results, we introduce several notations and function spaces. For a function $u(x,t)$, we denote by $\mathcal{F}_x u(\xi, t), \mathcal{F}_t u(x, \tau)$ the Fourier transforms with respect to x, t variable and by $\mathcal{F}_x^{-1} u(\xi,t), \mathcal{F}_t^{-1} u(x,\tau)$ the their inverse transforms in x, t variables respectively. We denote by $\widehat{u}(\xi,\tau) = \mathcal{F}_x \mathcal{F}_t u(\xi, \tau)$ the transform in both x and t. Let $L_x^p L_T^q$ and $L_T^q L_x^p$ denote the Banach spaces $L_x^p(\mathbf{R}; L^q(0,T))$ and $L^q(0,T; L_x^p(\mathbf{R}))$, respectively. L_t^q denotes L_T^q for $T = \infty$. We set $\langle x \rangle = (1+|x|^2)^{1/2}$. The operators D_x^s and $\langle D_x \rangle^s$ stand for the fractional order derivative of the Riesz and Bessel potential in the space variable, respectively; i.e, $D_x^s = \mathcal{F}_\xi^{-1} |\xi|^s \mathcal{F}_x$ and $\langle D_x \rangle^s = \mathcal{F}_\xi^{-1} \langle \xi \rangle^s \mathcal{F}_x$. For $b, s \in \mathbf{R}$, we define the following Hilbert space $H_t^b(\mathbf{R}; H_x^s(\mathbf{R})) = \{f \in \mathcal{S}'(\mathbf{R}^2); \langle D_t \rangle^b f \in H_x^s(\mathbf{R})\}$. For $\nu \in \mathbf{R}$, we denote by $W_\nu(t)$ the unitary group generated by linear part of (1.1) i.e.

$$W_\nu(t) v(x) = C \int_{\mathbf{R}} e^{ix\xi - it(\xi^2 - \nu\xi^4)} \mathcal{F}_x v(\xi) d\xi.$$

Let $\psi(t)$ be a smooth cut-off function to the interval $[-1,1]$ i.e.$\psi \in C_0^\infty(\mathbf{R})$ and $\psi(t) \equiv 1$ for $|t| \leq 1$, $\equiv 0$ for $|t| \geq 2$. For $\delta > 0$, we donote $\psi_\delta(t) = \psi(t/\delta)$.

Equation (1.1) can be rewritten in the following integral equation:

$$u(t) = W_\nu(t) u_0 - i \int_0^t W_\nu(t-t') F(t') dt'. \tag{1.9}$$

for $t \in [-T, T]$. Hereafter, we define the solution to initial value problem (1.1) as the solution to the integral equation (1.9).

Our well-posedness result is as follows:

Theorem 1.1. Let $\nu < 0,\ \lambda_5 = -2\mu + \nu = 0,\ s \geq 1/2$ and $b \in (1/2, 5/8)$. Then for $u_0 \in H^s(\mathbf{R})$, there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution $u(t)$ of the initial value problem (1.1) in $t \in [-T, T]$ satisfying

$$\begin{aligned} u &\in C([-T,T]; H^s(\mathbf{R})), \\ \psi_T W_\nu(-t)u &\in H_t^b(\mathbf{R}; H_x^s(\mathbf{R})), \\ \psi_T W_\nu(-t)F &\in H_t^b(\mathbf{R}; H_x^s(\mathbf{R})). \end{aligned}$$

Moreover, for given $T' \in (0, T)$, two maps $u_0 \longmapsto u$ from $H^s(\mathbf{R})$ to $C([-T', T']; H^s(\mathbf{R}))$ and $u_0 \longmapsto \psi_{T'} W_\nu(-t)u$ from $H^s(\mathbf{R})$ to $H_t^b(\mathbf{R}; H_x^s(\mathbf{R}))$ are Lipschitz continuous, respectively.

Remarks. Here, we summarize several remarks.

(a) Here the restriction $\nu < 0$ with $\mu - \nu/2 = 0$ is only required because of mathematical point of view. So far, it is not clear whether this restriction is meaningful in physical phenomena. However as is mentioned in Remark (d) below, the case $\mu - \nu/2 = 0$ corresponds the non-completely integrable case of the equation, which is covered our theorem.

(b) Because of technical reason the case $\nu \geq 0$ or $\lambda_5 = -2\mu + \nu \neq 0$ is not covered by our theorem. If we allow the regilarity of initial data more smooth, then the general theory of semilinear equation (see, e.g., [11]) show the time local well-posedness.

(c) By employ analogous method in Tzvetkov [28], we are able to show the optimality of our method to show the well-posedness, that is, we are not able to guarantee the time local well-posedness via our method below. For this detail, see section 6 in [21].

(d) When $\mu + \nu/2 = 0$ in (1.8), it is known that (1.8) is the completely integrable equation (see Fukumoto [6]) and have infinity many conserved quantities (see Langer-Perline [17]); namely,

$$\begin{aligned} \Phi_1(u) &= \frac{1}{2}\int |u|^2 dx, \qquad \Phi_2(u) = -\frac{i}{2}\int (\partial_x u)\bar{u}dx, \\ \Phi_3(u) &= -\frac{1}{2}\int (\partial_x^2 u)\bar{u}dx - \frac{1}{8}\int |u|^4 dx, \cdots \end{aligned}$$

Nevertheless, in Theorem 1.1, we assume $\lambda_5 = 0$ in (1.1), i.e. $\mu - \nu/2 = 0$ and $\nu \neq 0$ in (1.8). Therefore, we are not able to extend the local results to global one via the above conservation laws.

To prove the well-posedness, we will use the Fourier Restriction method. To explain the method of Fourier restriction norm, let us consider the following initial value ploblem for semilinear dispersive equation:

$$(1.10) \qquad \begin{cases} i\partial_t u + P(i^{-1}\partial_x)u = F(u, \partial_x u), & x, t \in \mathbf{R}, \\ u(x, 0) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

where $P(i^{-1}\partial_x) = \mathcal{F}_\xi^{-1} p(\xi) \mathcal{F}_x$ with real polynomial p and F is a nonlinear term. The many methods are known to prove the time local well-posedness for the initial value problem (1.10) (see [1-4,9,11,13-16,19,22,24-27]). In [2,3], Bourgain introduced new method to guarantee the well-posedness for (1.10) in lower order

Sobolev space. He introduced the following norm called "Fourier restriction norm": for $s, b \in \mathbf{R}$

$$\begin{aligned}\|u\|_{X_{s,b}} =& \|\langle\tau - p(\xi)\rangle^b \langle\xi\rangle^s \widehat{\widehat{u}}(\xi,\tau)\|_{L^2_\tau L^2_\xi}\\ =& \|e^{itP(i^{-1}\partial_x)}u\|_{H^b_t H^s_x},\end{aligned}$$

where $e^{itP(i^{-1}\partial_x)}$ is the unitary group generated by the linearized equation of (1.10). Here we note that $\tau - p(\xi)$ represents above norm is a symbol of the linear part of (1.10). By using this norm, he guaranteed the time local well-posedness for the nonlinear Schrödinger equation with power type nonlinearity i.e. $P(i^{-1}\partial_x) = \partial_x^2$ and $F(u, \partial_x u) = |u|^{p-1}u$ and the Korteweg-de Vries eqaution i.e. $P(i^{-1}\partial_x) = i\partial_x^3$ and $F(u, \partial_x u) = -iu\partial_x u$. Later on, Kenig-Ponce-Vega [14,15,16] refined this method in one spacial dimension case by making use of the algebraic structure of the symbol associated with the linearlized equation and Kato type smoothing effect. They applied this method to the quadratic Schrödinger equaiton and the Korteweg-de Vries eqaution. Therefore, this norm is heavily depend on the algebraic structure of the symbol and nonlinearity. Indeed, in section 3 below, we have some crucial tri-linear estimate concerning to the Fourier restriction norm for the nonlinear term $u^2\partial_x\overline{u}$. However, we don't know the same estimate for the term $|u|^2\partial_x^2 u$. Fourier restriction method has recently been applied to various nonlinear equations by many authors (see, e,g., [1,22,24,25]).

Concerning to the equation (1.1), let us define the corresponding Fourier restriction norm: For $s, b, \nu \in \mathbf{R}$ define the Hilbert space $X^\nu_{s,b}$ as follows

$$X^\nu_{s,b} = \{u \in \mathcal{S}'(\mathbf{R}^2); \|u\|_{X^\nu_{s,b}} < \infty\},$$

$$\begin{aligned}\|u\|_{X^\nu_{s,b}} &= \|\langle\tau + \xi^2 - \nu\xi^4\rangle^b \langle\xi\rangle^s \widehat{\widehat{u}}(\xi,\tau)\|_{L^2_\xi L^2_\tau}\\ &= \|W_\nu(-t)u(t)\|_{H^b_t H^s_x}.\end{aligned}$$

Our plan in this paper is the following. In section 2 and 3, we prove the preliminary estimates to Theorem 1.1. Especially in section 2 and 3, we prove the linear estimates and the nonlinear estimates, respectively. In section 4 we reduce Theorem1.1.

2. Linear Estimates

In this section we prove the linear estimates needed for the proof of Theorem 1.1. Through sections 2-4, we prove only the case when $\nu = -1$. We denote $W_{-1}(t) = W(t)$ and $X^{-1}_{s,b} = X_{s,b}$.

First, we state some known linear estimates that is used for proof.

Proposition 2.1. (Kenig-Ponce-Vega [14,15]) For $s \in \mathbf{R}$, $1/2 < b < b' < 1$, $\delta \in$

$(0,1)$. Then we have

$$\|\psi_\delta W(t)u_0\|_{X_{s,b}} \leq C\delta^{(1-2b)/2}\|u_0\|_{H^s}, \tag{2.1}$$

$$\|\psi_\delta F\|_{X_{s,b-1}} \leq C\delta^{b'-b}\|F\|_{X_{s,b'-1}}, \tag{2.2}$$

$$\left\|\psi_\delta \int_0^t W(t-t')F(t')dt'\right\|_{X_{s,b}} \leq C\delta^{(1-2b)/2}\|F\|_{X_{s,b-1}}, \tag{2.3}$$

$$\left\|\psi_\delta \int_0^t W(t-t')F(t')dt'\right\|_{L_t^\infty((0,T);H_x^s)} \leq C\delta^{(1-2b)/2}\|F\|_{X_{s,b-1}}. \tag{2.4}$$

Next, we give the estimates for the unitary group generated by linearized equation of (1.1).

Proposition 2.2. (Kenig-Ponce-Vega [12]) Let $u_0 \in L_x^2(\mathbf{R})$. Then

$$\|D_x^{\zeta\eta\theta/2}W(t)u_0\|_{L_t^q L_x^p} \leq C\|u_0\|_{L_x^2}, \tag{2.5}$$

$$\|W(t)u_0\|_{L_x^4 L_t^\infty} \leq C\|D_x^{1/4}u_0\|_{L_x^2}, \tag{2.6}$$

$$\|D_x^{3/2}W(t)u_0\|_{L_x^\infty L_t^2} \leq C\|u_0\|_{L_x^2}. \tag{2.7}$$

where $(\zeta,\eta,\theta) \in [0,1]\times[0,1]\times[0,1]$ and $(q,p) = (8/\zeta(\eta+1), 2/(1-\zeta))$.

Remark 2.3. The time decay estimate for the linear fourth order Schödinger type equation is as follows (see Kenig-Ponce-Vega [12]):

$$\|D_x^\theta W(t)u_0\|_{L_x^{2/(1-\theta)}} \leq C|t|^{-\theta/2}\|u_0\|_{L_x^{2/(1+\theta)}},$$

for any $\theta \in [0,1]$. In particular, by taking $\theta = 1$ in above inequality, we have $L^\infty - L^1$ estimate and $\|D_xW(t)u_0\|_{L_x^\infty}$ decays in time as $|t|^{-1/2}$.

The inequalities (2.5), (2.6) and (2.7) are called Strichartz [23] type estimate, Kenig-Ruiz [17] type estmate and Kato [11] type smoothing effect, respectively. For the proof of those estimates (2.5)-(2.7), see Thoerem 2.1, Theorem 2.5 and Thorem 4.1 in [15], respectevely. In Theorem 1.1, we stated the results in the case $\nu < 0$ for the initial value problem (1.1). The one of reasons on this restriction is that those estimates (2.5)-(2.7) hold for the case $\nu < 0$ and, however, we don't know same one for the case $\nu > 0$.

Define F_b, F_b^* and G_b via the Fourier transform

$$\begin{aligned} &\widehat{F_b}(\xi,\tau) = \frac{f(\xi,\tau)}{\langle\tau+\xi^2+\xi^4\rangle^b}, \qquad \widehat{F_b^*}(\xi,\tau) = \frac{\overline{f}(\xi,\tau)}{\langle\tau-\xi^2-\xi^4\rangle^b}, \\ &\widehat{G_b}(\xi,\tau) = \frac{g(\xi,\tau)}{\langle\tau+\xi^2+\xi^4\rangle^b}. \end{aligned} \tag{2.8}$$

By Proposition 2.2, we have immediately following Lemma.

Lemma 2.4. Let $b > 1/2,\ f \in L^2_\xi L^2_\tau$. Then

$$\|F_b\|_{L^6_x L^6_t} \leq C\|f\|_{L^2_\xi L^2_\tau}, \tag{2.9}$$

$$\|F_b\|_{L^{10}_x L^{10}_t} \leq C\|f\|_{L^2_\xi L^2_\tau}, \tag{2.10}$$

$$\|D_x^{-1/4} F_b\|_{L^4_x L^\infty_t} \leq C\|f\|_{L^2_\xi L^2_\tau}, \tag{2.11}$$

$$\|D_x^{3/2} F_b\|_{L^\infty_x L^2_t} \leq C\|f\|_{L^2_\xi L^2_\tau}. \tag{2.12}$$

Similar estimates hold for F_b^* and G_b replacing F_b and $\overline{f}$ and g replacing f respectively.

For the proof of Lemma 2.4, See e.q., [1,21].

3. Nonlinear Estimates

By using the linear estimates in Lemma 2.4, we derive the crusial nonlinear estimates.

Proposition 3.1. Let $s \geq -1/2,\ a \in (-1/2, -1/4]$ and $b > 1/2$. Then for any $u \in X_{s,b}$, we have

$$\||u|^2 u\|_{X_{s,b}} \leq C\|u\|^3_{X_{s,b}}. \tag{3.1}$$

Let $s \geq 0,\ a \leq 0,\ b > 1/2$. Then for any $u \in X_{s,b}$,

$$\||u|^4 u\|_{X_{s,a}} \leq C\|u\|^5_{X_{s,b}}. \tag{3.2}$$

Let $s \geq 1/2,\ a \in (-1/2, -1/4],\ b > 1/2$. Then for any $u \in X_{s,b}$, we have

$$\|(\partial_x u)^2 \overline{u}\|_{X_{s,a}} \leq C\|u\|^3_{X_{s,b}}, \tag{3.3}$$

$$\||\partial_x u|^2 u\|_{X_{s,a}} \leq C\|u\|^3_{X_{s,b}}. \tag{3.4}$$

Let $s \geq 1/2,\ a \in (-1/2, -3/8],\ b > 1/2$. Then for any $u \in X_{s,b}$, we have

$$\|u^2 \partial_x^2 \overline{u}\|_{X_{s,a}} \leq C\|u\|^3_{X_{s,b}}. \tag{3.5}$$

Remark 3.2. The case $\lambda_5 \neq 0$, we do not have the results. Because we do not know the "Tri-linear Estimates":

$$\||u|^2 \partial_x^2 u\|_{X_{s,a}} \leq C\|u\|^3_{X_{s,b}}, \tag{3.6}$$

where s, a and b satisfies same condition as in (3.5).

Proof of Proposition 3.1. By density, we have only prove (3.1)-(3.5) for $u \in \mathcal{S}(\mathbf{R}^2)$. Let

$$f(\xi,\tau) = \langle\sigma\rangle^b\langle\xi\rangle^s|\widehat{u}|, \qquad \overline{f}(\xi,\tau) = \langle\overline{\sigma}\rangle^b\langle\xi\rangle^s|\widehat{\overline{u}}|.$$

where

$$\sigma = \tau + \xi^2 + \xi^4, \qquad \overline{\sigma} = \tau - \xi^2 - \xi^4.$$

By duality argument, (3.1) is reduced to showing that for any $0 \le g \in L^2_\xi L^2_\tau$,

$$\begin{aligned}&\int_{\mathbf{R}^6} \frac{g(\xi,\tau)}{\langle\sigma\rangle^{|a|}\langle\xi\rangle^{|s|}} \frac{\langle\xi-\xi_1\rangle^{|s|}f(\xi-\xi_1,\tau-\tau_1)}{\langle\sigma_1\rangle^b}\\ &\qquad\times\frac{\langle\xi_1-\xi_2\rangle^{|s|}f(\xi_1-\xi_2,\tau_1-\tau_2)}{\langle\sigma_2\rangle^b}\frac{\langle\xi_2\rangle^{|s|}f^*(\xi_2,\tau_2)}{\langle\overline{\sigma}_3\rangle^b}\\ &\le C\|g\|_{L^2_\xi L^2_\tau}\|f\|^2_{L^2_\xi L^2_\tau}\|\overline{f}\|_{L^2_\xi L^2_\tau},\end{aligned} \tag{3.7}$$

where σ and σ_i are given by (3.8) below. Similary, (3.2)-(3.5) are reduced to showing that for any $0 \le g \in L^2_\xi L^2_\tau$,

$$\begin{aligned}&\int_{\mathbf{R}^{10}} \frac{g(\xi,\tau)}{\langle\rho\rangle^{|a|}}\frac{f(\xi-\xi_1,\tau-\tau_1)}{\langle\rho_1\rangle^b}\frac{\overline{f}(\xi_1-\xi_2,\tau_1-\tau_2)}{\langle\overline{\rho}_2\rangle^b}\\ &\qquad\times\frac{f(\xi_2-\xi_3,\tau_2-\tau_3)}{\langle\rho_3\rangle^b}\frac{\overline{f}(\xi_3-\xi_4,\tau_3-\tau_4)}{\langle\overline{\rho}_4\rangle^b}\frac{f(\xi_4,\tau_4)}{\langle\rho_5\rangle^b},\end{aligned}$$

$$\int_{\mathbf{R}^6} \frac{\langle\xi\rangle^s g(\xi,\tau)}{\langle\sigma\rangle^{|a|}}\frac{|\xi-\xi_1|f(\xi-\xi_1,\tau-\tau_1)}{\langle\sigma_1\rangle^b\langle\xi-\xi_1\rangle^s}\frac{|\xi_1-\xi_2|f(\xi_1-\xi_2,\tau_1-\tau_2)}{\langle\sigma_2\rangle^b\langle\xi_1-\xi_2\rangle^s}\frac{\overline{f}(\xi_2,\tau_2)}{\langle\overline{\sigma}_3\rangle^b\langle\xi_2\rangle^s},$$

$$\int_{\mathbf{R}^6} \frac{\langle\xi\rangle^s g(\xi,\tau)}{\langle\sigma\rangle^{|a|}}\frac{f(\xi-\xi_1,\tau-\tau_1)}{\langle\sigma_1\rangle^b\langle\xi-\xi_1\rangle^s}\frac{|\xi_1-\xi_2|f(\xi_1-\xi_2,\tau_1-\tau_2)}{\langle\sigma_2\rangle^b\langle\xi_1-\xi_2\rangle^s}\frac{|\xi_2|\overline{f}(\xi_2,\tau_2)}{\langle\overline{\sigma}_3\rangle^b\langle\xi_2\rangle^s},$$

$$\int_{\mathbf{R}^6} \frac{\langle\xi\rangle^s g(\xi,\tau)}{\langle\sigma\rangle^{|a|}}\frac{f(\xi-\xi_1,\tau-\tau_1)}{\langle\sigma_1\rangle^b\langle\xi-\xi_1\rangle^s}\frac{f(\xi_1-\xi_2,\tau_1-\tau_2)}{\langle\sigma_2\rangle^b\langle\xi_1-\xi_2\rangle^s}\frac{|\xi_2|^2\overline{f}(\xi_2,\tau_2)}{\langle\overline{\sigma}_3\rangle^b\langle\xi_2\rangle^s},$$

are bounded by

$$C\|g\|_{L^2_\xi L^2_\tau}(\|f\|^2_{L^2_\xi L^2_\tau} + \|f\|^4_{L^2_\xi L^2_\tau})\|\overline{f}\|_{L^2_\xi L^2_\tau},$$

respectively. Here,

$$\int_{\mathbf{R}^6} = \int_{\mathbf{R}^6} d\xi d\tau d\xi_1 d\tau_1 d\xi_2 d\tau_2, \text{ and } \int_{\mathbf{R}^{10}} = \int_{\mathbf{R}^{10}} d\xi d\tau d\xi_1 d\tau_1 \cdots d\xi_4 d\tau_4,$$

$$\begin{cases}\sigma = \tau + \xi^2 + \xi^4, & \sigma_1 = \tau - \tau_1 + (\xi-\xi_1)^2 + (\xi-\xi_1)^4,\\ \sigma_2 = \tau_1 - \tau_2 + (\xi_1-\xi_2)^2 + (\xi_1-\xi_2)^4, & \overline{\sigma}_3 = \tau_2 - \xi_2^2 - \xi_2^4,\end{cases} \tag{3.8}$$

and

$$\begin{cases}\rho = \tau + \xi^2 + \xi^4, & \rho_1 = \tau - \tau_1 + (\xi-\xi_1)^2 + (\xi-\xi_1)^4,\\ \overline{\rho}_2 = \tau_1 - \tau_2 - (\xi_1-\xi_2)^2 - (\xi_1-\xi_2)^4, & \\ \rho_3 = \tau_2 - \tau_3 + (\xi_2-\xi_3)^2 + (\xi_2-\xi_3)^4, & \\ \overline{\rho}_4 = \tau_3 - \tau_4 - (\xi_3-\xi_4)^2 - (\xi_3-\xi_4)^4, & \rho_5 = \tau_4 + \xi_4^2 + \xi_4^4.\end{cases} \tag{3.9}$$

Above estimates are obtained by section 4 in [21]. As we stated in introduction in the present paper, Fourier restriction method is very sensitive to the connection between the symbol and a shape of nonlinear term. Indeed, in the proof of Proposition 3.1, we use the following inequality:

$$
\begin{aligned}
&|\tau+\xi^2+\xi^4|+|\tau-\tau_1+(\xi-\xi_1)^2+(\xi-\xi_1)^4| \\
&\qquad +|\tau_1-\tau_2+(\xi_1-\xi_2)^2+(\xi_1-\xi_2)^4|+|\tau_2-\xi_2^2-\xi_2^4| \\
&\qquad \geq |\xi^2+\xi^4-(\xi-\xi_1)^2-(\xi-\xi_1)^4-(\xi_1-\xi_2)^2-(\xi_1-\xi_2)^4+\xi_2^2+\xi_2^4| \\
&\qquad =2|\xi_1||\xi-\xi_1+\xi_2||2\xi^2+\xi_1^2+2\xi_2^2-\xi\xi_1-\xi_1\xi_2-2\xi_2\xi+1| \\
&\qquad =2|\xi_1||\xi-\xi_1+\xi_2| \\
&\qquad\qquad \times\left\{\frac{1}{2}\xi^2+\frac{1}{2}(\xi-\xi_1)^2+\frac{1}{2}(\xi_1-\xi_2)^2+\frac{1}{2}\xi_2^2+(\xi_2-\xi)^2+1\right\}.
\end{aligned}
$$

We reduce the Proposition 3.1 by combining the above inequality with Strichartz, Kenig-Ruiz and Kato type smoothing effect (2.9)-(2.12).in Lemma 2.4.

4. Proof of Well-posedness

The integral equation associated with (1.1) is expressed by (1.9). First, we proved the existence of a solution u to integral equation (1.9).
Let $r=\|u_0\|_{H^s}$. For $\delta\in(0,1)$, define

$$\mathcal{B}(r)=\{u\in X_{s,b}:\ \|u\|_{X_{s,b}}\leq 2C_0 r\},$$

$$\Phi(u)=\psi(t)W(t)u_0-i\psi(t)\int_0^t W(t-t')\psi_\delta(t')F(t')dt'.$$

According to Proposition 2.1.(2.1)-(2.3) and Proposition 3.1. (3.1)-(3.5). ,we have for b,b' with $1/2<b<b'<5/8$ and for $u\in\mathcal{B}(r)$,

$$\|\Phi(u)\|_{X_{s,b}}\leq C_0 r+C_1\delta^{b'-b}(1+r^2)r^3.$$

Similarly, we have for $u,\tilde{u}\in\mathcal{B}(r)$,

$$\|\Phi(u)-\Phi(\tilde{u})\|_{X_{s,b}}\leq C_2\delta^{b'-b}(1+r^2)r^2\|u-\tilde{u}\|_{X_{s,b}}.$$

By choosing $\delta>0$ such that

$$\delta^{b'-b}\leq\min\left\{\frac{C_0}{(1+r^2)r^2C_1},\ \frac{1}{2(1+r^2)r^2C_2}\right\},$$

Φ is a contraction map on $\mathcal{B}(r)$. Therefore there exists a unique solution $u\in\mathcal{B}(r)$ satisfying

$$u(t)=\psi(t)W(t)u_0-i\psi(t)\int_0^t W(t-t')\psi_\delta(t')F(t')dt'.$$

We choose $T < \delta$, then $u(t)$ satisfies (1.9) in $t \in [-T, T]$.

Concerning the proof of the uniqueness of the solution, we define the following auxiliary norms introduced by Bekiranov-Ogawa-Ponce [1]. For $T > 0$, we let

$$\|u\|_{X_T} = \inf_{w} \{\|w\|_{X_{s,b}} : \ w \in X_{s,b} \text{ such that } u(t) = w(t), \ t \in [-T, T] \text{ in } H^s(\mathbf{R})\}.$$

If $\|u - \tilde{u}\|_{X_T} = 0$, we have $u(t) = \tilde{u}(t)$ in $H^s(\mathbf{R})$ for $t \in [-T, T]$. By similar argument as in [1], we reduce the uniqueness.

The persistence property follows the Sobolev's embedding $H_t^b(\mathbf{R}; H_x^s(\mathbf{R})) \subset C(\mathbf{R}; H^s(\mathbf{R}))$. The continuous dependence upon data also follows from a similar argument. $\square$

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