数理解析研究所講究録 1323 巻 2003 年 105-123

Isolated Singularities for Some Types of Curvature Equations

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Abstract. We consider the removability of isolated singularities for the curvature equations of the form $H_k[u] = 0$, which is determined by the k-th elementary symmetric function, in an n-dimensional domain. We prove that, for $1 \leq k \leq n-1$, isolated singularities of any viscosity solutions to the curvature equations are always removable, provided the solution can be extended continuously at the singularities. We also consider the class of "generalized solutions" and prove the removability of isolated singularities.

1 Introduction

We study the removability of the isolated singularity of solutions to the curvature equations of the form

$$H_k[u] = S_k(\kappa_1, \ldots, \kappa_n) = 0 \tag{1.1}$$

in $\Omega \setminus \{0\}$, where Ω is a bounded domain in $\mathbb{R}^n$ and $0 \in \Omega$. For a function $u \in C^2(\Omega)$, $\kappa = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of the graph of the function u, namely, the eigenvalues of the matrix

$$C = D\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}\left(I - \frac{Du \otimes Du}{1+|Du|^2}\right) D^2u, \tag{1.2}$$

and $S_k, k = 1, \ldots, n$, denotes the k-th elementary symmetric function, that is,

$$S_k(\kappa) = \sum \kappa_{i_1} \cdots \kappa_{i_k}, \tag{1.3}$$

where the sum is taken over increasing k-tuples, $i_1, \ldots, i_k \subset \{1, \ldots, n\}$. The mean, scalar and Gauss curvatures correspond respectively to the special cases $k = 1, 2, n$ in (1.3).

Our aim here is to discuss the following problem.

Problem: Is it always possible to extend a "solution" of (1.1) as a "solution" of $H_k[u] = 0$ in the whole domain Ω?

In this paper, we consider two classes of solutions as a "solution" in our problem. First, except for the last two sections, we consider the class of *viscosity* solutions to (1.1), which are solutions in a certain weak sense. In many nonlinear partial differential equations, the viscosity framework allows us to obtain existence and uniqueness results under rather mild hypotheses.

We establish results concerning the removability of isolated singularities of a viscosity solution to (1.1). Here is our main theorem.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ containing the origin. Let $1 \leq k \leq n-1$ and u be a viscosity solution of (1.1). We assume that u can be extended to the continuous function $\tilde{u} \in C^0(\Omega)$. Then $\tilde{u}$ is a viscosity solution of $H_k[\tilde{u}] = 0$ in Ω. Consequently, $\tilde{u} \in C^{0,1}(\Omega)$.*

The last part of Theorem 1.1 is a consequence of [22]. Note that one cannot expect much better regularity for a viscosity solution in general. In fact, it is known that there exist an $\varepsilon > 0$ and a viscosity solution of $H_k[\tilde{u}] = 0$ in $B_\varepsilon = \{|x| < \varepsilon\}$ which does not belong to $C^{1,\alpha}(B_\varepsilon)$ for any $\alpha > 1 - \frac{2}{k}$.

For the case of $k = 1$, which corresponds to the minimal surface equation in (1.1), such removability result was proved by Bers [2], Nitsche [18], and De Giorgi and Stampacchia [12]. Serrin [19], [20] studied the same problem for a more general class of quasilinear equations of mean curvature type. He proved that any weak solution u of the mean curvature type equation in $\Omega \backslash K$ can be extended to weak solution in Ω if the singular set K is a compact set of vanishing $(n-1)$-dimensional Hausdorff measure. For various semilinear and quasilinear equations, such problems were extensively studied. See [3], [4], [26] and references therein.

Here we remark that (1.1) is a quasilinear equation for $k = 1$ while it is a *fully nonlinear* equation for $k \geq 2$. It is much harder to study the fully nonlinear equations' case. To the best of our knowledge, there are no results about the properties of isolated singularities for fully nonlinear elliptic PDEs except for the recent work of Labutin [14], [15] (for the case of uniformly elliptic equations), [16] (for the case of Hessian equations). So our main result, Theorem 1.1, is new for $2 \leq k \leq n-1$.

In the results of Bers, Serrin and others, no restrictions are imposed on the behaviour of solutions near the singularity. Therefore our result is weaker than theirs for the case of $k = 1$, but that is because their arguments rely on the quasilinear nature of the equation.

There is a standard notion of weak solutions to (1.1) for the case of $k = 1$, but it does not make sense for $k \geq 2$. So when we study the removability of isolated singularities, we consider the problem in the framework of the theory of viscosity solutions. In this framework, comparison principles play important roles. Our idea of the proof of Theorem 1.1 is adapted from that of Labutin [14], except that we have to deal with the extra difficulty coming from the non-uniform ellipticity of the equations.

We note that the case $k = n$, which corresponds to the Gauss curvature case, is excluded from Theorem 1.1. There exist solutions of (1.1) with non-removable singularities at 0. It is easily checked that a function

$$u(x) = a(|x| - 1), \quad x \in \Omega = B_1 = \{|x| < 1\} \tag{1.4}$$

where $a > 0$, satisfies the equation (1.1) with $k = n$. However, u does not satisfy $H_n[u] = 0$ in B_1 in the viscosity sense. In fact, it follows that

$$H_n[u] = \left(\frac{a}{\sqrt{1+a^2}}\right)^n \omega_n \delta_0 \tag{1.5}$$

in the *generalized* sense, where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, and δ_0 is the Dirac measure at 0. So there is a considerable difference between the cases $1 \leq k \leq n-1$ and $k = n$.

Next, we also consider the removability of isolated singularities of the *generalized solutions* to (1.1), the notion of which was introduced by the author [21]. Note that this is a weaker notion of solutions than viscosity solutions. We prove the removability result in the class of generalized solutions.

Theorem 1.2. *Let Ω be a convex domain in $\mathbb{R}^n$ containing the origin. Let $1 \leq k \leq n-1$ and u be a continuous function in $\Omega \setminus \{0\}$. We assume that for any convex subdomain $\Omega' \subset \Omega \setminus \{0\}$, u is a convex function in Ω' and a generalized solution of $H_k[u] = 0$ in Ω'. Then u can be defined at the origin as a generalized solution of $H_k[u] = 0$ in Ω.*

The technique to prove this assertion is completely different from that in the proof of Theorem 1.1. In section 4, we define the generalized solutions of the curvature equations and discuss the removability of isolated singularities of generalized solutions.

2 The notion of viscosity solutions

In this section, we define the notion of viscosity solutions of the equation

$$H_k[u] = \psi(x) \quad \text{in } \Omega, \tag{2.1}$$

where Ω is an arbitrary domain in $\mathbb{R}^n$ and $\psi \in C^0(\Omega)$ is a non-negative function. The theory of viscosity solutions to the first order equations and the second order ones was developed in the 1980's by Crandall, Evans, Ishii, Lions and others. See, for example, [9], [10], [11], [17]. For the curvature equations of the form (2.1), Trudinger [22] established existence theorems for Lipschitz solutions in the viscosity sense.

Let Ω be a domain in $\mathbb{R}^n$. First, we define the admissible set of elementary symmetric function S_k by

$$\begin{aligned}\Gamma_k &= \{\kappa \in \mathbb{R}^n \mid S_k(\kappa+\eta) \geq S_k(\kappa) \quad \text{for all } \eta_i \geq 0\} \\ &= \{\kappa \in \mathbb{R}^n \mid S_j(\kappa) \geq 0, \quad j = 1, \ldots, k\}.\end{aligned} \tag{2.2}$$

We say that a function $u \in C^2(\Omega)$ is *k-admissible* for the operator H_k if $\kappa = (\kappa_1, \ldots, \kappa_n)$ belongs to Γ_k for every point $x \in \Omega$. Except for the case $k = 1$, equation (2.1) is not elliptic on all functions $u \in C^2(\Omega)$, but Caffarelli, Nirenberg and Spruck [5], [6] have shown that (2.1) is degenerate elliptic for k-admissible functions. Obviously,

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n = \Gamma_+ = \{\kappa \in \mathbb{R}^n \mid \kappa_i \geq 0, \quad i = 1, \ldots, n\}, \tag{2.3}$$

and alternative characterizations of Γ_k are also known (see [13]).

We define a viscosity solution of (2.1). A function $u \in C^0(\Omega)$ is said to be a *viscosity subsolution* (resp. *viscosity supersolution*) of (2.1) if for any k-admissible function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$ which is a maximum (resp. minimum) point of $u - \varphi$, we have

$$H_k[\varphi](x_0) \geq \psi(x_0) \quad (\text{resp. } \leq \psi(x_0)). \tag{2.4}$$

A function u is said to be a *viscosity solution* of (2.1) if it is both a viscosity subsolution and supersolution. We note that the notion of viscosity subsolution does not change if all $C^2(\Omega)$ functions are allowed as comparison functions φ. One can prove that a function $u \in C^2(\Omega)$ is a viscosity solution of (2.1) if and only if it is a k-admissible classical solution.

The following theorem is a comparison principle for viscosity solutions of (2.1).

Theorem 2.1. *Let Ω be a bounded domain. Let ψ be a non-negative continuous function in $\overline{\Omega}$ and u, v be $C^0(\overline{\Omega})$ functions satisfying $H_k[u] \geq \psi + \delta$, $H_k[v] \leq \psi$ in Ω in the viscosity sense, for some positive constant δ. Then*

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v)^+. \tag{2.5}$$

The proof of this theorem is given in [22]. In this paper we use another type of comparison principle as follows.

Proposition 2.2. *Let Ω be a bounded domain. Let ψ be a non-negative continuous function in $\overline{\Omega}$, $u \in C^0(\overline{\Omega})$ be a viscosity subsolution of $H_k[u] = \psi$, and $v \in C^2(\overline{\Omega})$ satisfying*

$$\kappa[v(x)] \notin \{\lambda \in \Gamma_k \mid S_k(\lambda) \geq \psi(x)\} \tag{2.6}$$

for all $x \in \Omega$, where $\kappa[v(x)]$ denotes the principal curvatures of v at x. Then (2.5) holds.

Proof. We assume (2.5) does not hold. Then there exists a point $x \in \Omega$ such that

$$\sup_{\Omega}(u - v) = u(x) - v(x). \tag{2.7}$$

Since u is a viscosity subsolution of $H_k[u] = \psi$, it follows that $H_k[v](x) \geq \psi(x)$. From (2.6) we have $\kappa[v(x)] \notin \Gamma_k$. For simplicity, we write $\kappa = (\kappa_1, \ldots, \kappa_n)$ instead of $\kappa[v(x)]$. Thus it follows that there exists $i \in \{1, \ldots, n\}$ such that $S_{k-1;i}(\kappa) < 0$, where $S_{k-1;i}(\kappa) = \dfrac{\partial S_k(\kappa)}{\partial \kappa_i}$ (for, if $S_k(\kappa) \geq 0$ and $S_{k-1;i}(\kappa) \geq 0$ for all $i \in \{1, \ldots, n\}$, we get that $S_k(\kappa + \eta) \geq S_k(\kappa)$ for all $\eta_i \geq 0$.). Without loss of generality, we may suppose $i = 1$.

Then, we see that for $K \in \mathbb{R}$

$$S_k(\kappa_1 + K, \kappa_2, \ldots, \kappa_n) = S_k(\kappa) + K S_{k-1;1}(\kappa). \tag{2.8}$$

Thus if we assume

$$K > \frac{S_k(\kappa)}{-S_{k-1;1}(\kappa)} \ (> 0), \tag{2.9}$$

it holds that $S_k(\kappa_1 + K, \kappa_2, \ldots, \kappa_n) < 0$, which implies $(\kappa_1 + K, \kappa_2, \ldots, \kappa_n) \notin \Gamma_k$. We fix K satisfying (2.9).

We denote

$$X = \left(I - \frac{Dv(x) \otimes Dv(x)}{1 + |Dv(x)|^2}\right)^{1/2}. \tag{2.10}$$

Rotating the coordinate in $\mathbb{R}^n$, we may suppose

$$\frac{1}{\sqrt{1 + |Dv(x)|^2}} X \left(D^2 v(x)\right) X = \operatorname{diag}(\kappa_1, \ldots, \kappa_n). \tag{2.11}$$

We find the quadratic polynomial V which satisfies $V(x) = 0$, $DV(x) = 0$ and

$$D^2V = \sqrt{1 + |Dv(x)|^2} X^{-1} \operatorname{diag}(K, 0, \ldots, 0) X^{-1}. \tag{2.12}$$

Since $V \geq 0$ in Ω and $V(x) = 0$, $u - (v + V)$ attains a maximum value at x. Moreover, from a simple calculation, we get that the principal curvatures of $v + V$ at x are $\kappa_1 + K, \kappa_2, \ldots, \kappa_n$. Hence

$$H_k[v + V](x) = S_k(\kappa_1 + K, \kappa_2, \ldots, \kappa_n) < 0 \leq \psi(x). \tag{2.13}$$

This cannot hold since u satisfies $H_k[u] \geq \psi$ in the viscosity sense. Therefore we obtained the required inequality (2.5). □

3 Isolated singularities of viscosity solutions – Proof of Theorem 1.1

Now we prove Theorem 1.1. Without loss of generality, we may assume that $\Omega = B_1$, the unit ball in $\mathbb{R}^n$.

We show that $\tilde{u}$ is a viscosity solution of (1.1) in B_1. For the sake of simplicity, we denote u as an extended function in B_1.

Lemma 3.1. *Let $l(x) = u(0) + \sum_{i=1}^{n} \beta_i x_i$, where $\beta_1, \ldots, \beta_n \in \mathbb{R}$. Then there exist sequences $\{z_j\}, \{\tilde{z}_j\} \subset B_1 \setminus \{0\}$ such that $z_j, \tilde{z}_j \to 0$ as $j \to \infty$ and*

$$\liminf_{j \to \infty} \frac{u(z_j) - l(z_j)}{|z_j|} \leq 0, \tag{3.1}$$

$$\limsup_{j \to \infty} \frac{u(\tilde{z}_j) - l(\tilde{z}_j)}{|\tilde{z}_j|} \geq 0. \tag{3.2}$$

Proof. First we prove (3.1). To the contrary, we suppose that there exists an affine function $l(x) = u(0) + \sum_{i=1}^{n} \beta_i x_i$ such that

$$u(x) > l(x) + m|x| \quad \text{for } x \in B_\rho \setminus \{0\}, \tag{3.3}$$

for some $m, \rho > 0$. Rotating the coordinate system in $\mathbb{R}^{n+1}$ if necessary, we may assume that $Dl(x) = 0$, that is, $l(x) \equiv u(0)$.

Case 1. $k \le \frac{n}{2}$.

We fix a constant $\varepsilon > 0$ and consider the auxiliary function w_ε in $\mathbb{R}^n \setminus B_\varepsilon$ as follows:

$$w_\varepsilon(x) = u(0) + C_1 + C_2|x|^2 + C_3(\varepsilon) f_\varepsilon(x), \tag{3.4}$$

where $C_1, C_2, C_3(\varepsilon)$ are positive constants to be determined later, and

$$f_\varepsilon(x) = \int_{r_0}^{|x|} \frac{ds}{\sqrt{\left(\frac{s}{\varepsilon}\right)^{\frac{2(n-k)}{k}} - 1}} =: \int_{r_0}^{|x|} \frac{ds}{g(s)} \tag{3.5}$$

is a radially symmetric solution of (1.1) where $r_0 > 0$ will be also determined later. We write $w_\varepsilon(x) = \tilde{w}_\varepsilon(|x|)$. The principal curvatures of w_ε are

$$\kappa_1 = \frac{\tilde{w}_\varepsilon''(r)}{\left(1 + (\tilde{w}_\varepsilon'(r))^2\right)^{3/2}} = \left(2C_2 - \frac{C_3 \frac{n-k}{k} \left(\frac{r}{\varepsilon}\right)^{\frac{2(n-k)}{k}}}{r\sqrt{\left(\frac{r}{\varepsilon}\right)^{\frac{2(n-k)}{k}} - 1}^{\,3}} \right) A^{-3/2}, \tag{3.6}$$

$$\begin{aligned} \kappa_2 = \cdots = \kappa_n &= \frac{\tilde{w}_\varepsilon'(r)}{r\left(1 + (\tilde{w}_\varepsilon'(r))^2\right)^{1/2}} \\ &= \left(2C_2 + \frac{C_3}{r\sqrt{\left(\frac{r}{\varepsilon}\right)^{\frac{2(n-k)}{k}} - 1}} \right) A^{-1/2}, \end{aligned} \tag{3.7}$$

where $r = |x|$ and A is defined by

$$A = 1 + (\tilde{w}_\varepsilon'(r))^2 = 1 + \left(2C_2 r + \frac{C_3}{\sqrt{\left(\frac{r}{\varepsilon}\right)^{\frac{2(n-k)}{k}} - 1}} \right)^2. \tag{3.8}$$

Thus we obtain that

$$\begin{aligned} H_k[w_\varepsilon] &= \kappa_2^{k-1} \left(\binom{n-1}{k-1} \kappa_1 + \binom{n-1}{k} \kappa_2 \right) \\ &\ge \kappa_2^{k-1} A^{-3/2} \left(- \frac{\binom{n-1}{k-1} C_3 \frac{n-k}{k} \left(\frac{r}{\varepsilon}\right)^{\frac{2(n-k)}{k}}}{r g(r)^3} + \left(\frac{\binom{n-1}{k} C_3}{r g(r)} \right) A \right) \\ &\quad + \kappa_2^{k-1} A^{-3/2} \binom{n}{k} 2C_2 =: M_1 + M_2. \end{aligned} \tag{3.9}$$

We claim that M_1 is positive if $C_3 > 1$. In fact,

$$M_1 = \frac{\kappa_2^{k-1} A^{-3/2} \binom{n-1}{k} C_3}{r g(r)} \left(- \frac{\left(\frac{r}{\varepsilon}\right)^{\frac{2(n-k)}{k}}}{g(r)^2} + A \right) \tag{3.10}$$
$$\geq \frac{\kappa_2^{k-1} A^{-3/2} \binom{n-1}{k} C_3}{r g(r)} \left(- \frac{\left(\frac{r}{\varepsilon}\right)^{\frac{2(n-k)}{k}}}{g(r)^2} + \left(1 + \left(\frac{C_3}{g(r)} \right)^2 \right) \right)$$
$$= \frac{\kappa_2^{k-1} A^{-3/2} \binom{n-1}{k} C_3}{r g(r)} \cdot \frac{C_3^2 - 1}{g(r)^2} > 0.$$

This implies that if $C_2 > 0, C_3 > 1$,

$$H_k[w_\varepsilon] \geq \delta > 0 \quad \text{in } 2\varepsilon < |x| < \rho, \tag{3.11}$$

where δ is a positive constant depending only on $\varepsilon, C_2, C_3, \rho, k, n$. One can easily check that $\kappa = (\kappa_1, \ldots, \kappa_n) \in \Gamma_k$, i.e., w_ε is k-admissible.

Next we choose constants r_0, C_1, C_2, C_3 which have not determined yet. First, we fix $C_2 > 0$. Second, we choose $r_0 \in (0, \rho)$ so small that

$$C_2|x|^2 \leq \frac{m}{4}|x| \quad \text{in } B_{r_0}, \tag{3.12}$$

and we set $C_1 = \frac{m}{4} r_0$. From now on, we may assume that $\varepsilon < \frac{r_0}{2}$. Finally, we take the constant C so that

$$C f_\varepsilon(y) = -m r_0 \quad \text{for } |y| = 2\varepsilon, \tag{3.13}$$

and we set $C_3 = \max\{C, 1\}$. We find that a direct calculation implies

$$C_3 = \begin{cases} O(\varepsilon^{-1}) & \text{if } k < \frac{n}{2}, \\ O\left((\varepsilon \log 1/\varepsilon)^{-1}\right) & \text{if } k = \frac{n}{2}, \end{cases} \tag{3.14}$$

for sufficiently small ε.

Then, we obtain that

$$w_\varepsilon \leq u(0) + \frac{m}{4} r_0 + \frac{m}{4} r_0 < u(0) + m r_0 < u \quad \text{on } \partial B_{r_0}, \tag{3.15}$$

and that

$$w_\varepsilon \leq u(0) + \frac{m}{4} r_0 + \frac{m}{4} r_0 - m r_0 < u(0) < u \quad \text{on } \partial B_{2\varepsilon}. \tag{3.16}$$

From (3.11), (3.15), (3.16) and the comparison principle Theorem 2.1, we obtain

$$w_\varepsilon \le u \quad \text{in } \overline{B_{r_0}} \setminus B_{2\varepsilon}. \tag{3.17}$$

Now we fix $x \in B_{r_0} \setminus \{0\}$, it follows that

$$u(x) \ge w_\varepsilon(x) \ge u(0) + \frac{m}{4} r_0 + C_3 f_\varepsilon(x). \tag{3.18}$$

One can compute that

$$|f_\varepsilon(x)| = \begin{cases} O\left(\varepsilon^{\frac{n}{k}-1}\right)\left(r_0^{2-\frac{n}{k}} - |x|^{2-\frac{n}{k}}\right) & \text{if } k > \frac{n}{2}, \\ O(\varepsilon)\log r_0/|x| & \text{if } k = \frac{n}{2}, \end{cases} \tag{3.19}$$

for sufficiently small ε. Thus we obtain from (3.14) and (3.19),

$$\liminf_{\varepsilon \to 0} C_3 f_\varepsilon(x) = 0. \tag{3.20}$$

As ε tends to 0 in (3.18), we conclude from (3.20) that

$$u(x) \ge u(0) + \frac{m}{4} r_0, \tag{3.21}$$

which contradicts the continuity of u at 0.

Case 2. $k > \dfrac{n}{2}$.

For that case, we claim that

$$u(x) \ge u(0) + \tilde{C}|x|^{2-\frac{n}{k}} \quad \text{for } x \in B_\rho \setminus \{0\}, \tag{3.22}$$

for some positive constant $\tilde{C}$. To prove this claim, we introduce the auxiliary function g_ε of the form

$$g_\varepsilon(x) = u(0) + m\rho + C'(\varepsilon)\int_\rho^{|x|} \frac{ds}{\sqrt{\left(\frac{s}{\varepsilon}\right)^{\frac{2(n-k)}{k}} - 1}}, \tag{3.23}$$

where $C'(\varepsilon)$ is some positive constant. By the same manner with the above discussion, one can see that g_ε is k-admissible and that $H_k[g_\varepsilon] \ge \delta$ holds for some positive constant δ depending only on $\varepsilon, C', \rho, k, n$, provided $C' > 1$. Now we determine the constant C' by

$$C' \int_{2\varepsilon}^{\rho} \frac{ds}{\sqrt{\left(\frac{s}{\varepsilon}\right)^{\frac{2(n-k)}{k}} - 1}} = m\rho. \tag{3.24}$$

We remark that $C' > 1$ for sufficiently small ε since $C'(\varepsilon) = O\left(\varepsilon^{1-\frac{n}{k}}\right)$. So we obtain that $g_\varepsilon < u$ on $\partial B_\rho \cup \partial B_{2\varepsilon}$ from a similar argument to (3.15) and (3.16). From the comparison principle it follows that $g_\varepsilon \leq u$ in $\overline{B_\rho} \setminus B_{2\varepsilon}$. For fixed $x \in B_\rho \setminus \{0\}$ we obtain that

$$u(x) \geq g_\varepsilon(x) = u(0) + C' \int_{2\varepsilon}^{|x|} \frac{ds}{\sqrt{\left(\frac{s}{\varepsilon}\right)^{\frac{2(n-k)}{k}} - 1}}. \tag{3.25}$$

From now on the symbol C denotes a positive constant depending only on n and k. Since it holds that

$$C' = m\rho \left(\int_{2\varepsilon}^{\rho} \frac{ds}{\sqrt{\left(\frac{s}{\varepsilon}\right)^{\frac{2(n-k)}{k}} - 1}} \right)^{-1} \geq Cm \left(\frac{\rho}{\varepsilon}\right)^{\frac{n-k}{k}}, \tag{3.26}$$

and that

$$\int_{2\varepsilon}^{|x|} \frac{ds}{\sqrt{\left(\frac{s}{\varepsilon}\right)^{\frac{2(n-k)}{k}} - 1}} \geq C\varepsilon^{\frac{n-k}{k}} |x|^{2-\frac{n}{k}}, \tag{3.27}$$

for sufficiently small ε (say, $\varepsilon < |x|/2$), it follows that

$$u(x) \geq u(0) + Cm\rho^{-\frac{n-k}{k}} |x|^{2-\frac{n}{k}}, \tag{3.28}$$

for sufficiently small ε. So our claim has proved.

Now we introduce another auxiliary function w_ε as follows:

$$w_\varepsilon(x) = u(0) + C_1 + C_2|x|^\gamma + C_3(\varepsilon) \int_{r_0}^{|x|} \frac{ds}{\sqrt{\left(\frac{s}{\varepsilon}\right)^2 - 1}}, \tag{3.29}$$

where $C_1, C_2, C_3(\varepsilon), r_0$ are positive constants to be determined later, and we fix a constant γ such that

$$2 - \frac{n}{k} < \gamma < 1. \tag{3.30}$$

We get that the principal curvatures of w_ε are

$$\kappa_1 = \left(C_2\gamma(\gamma-1)r^{\gamma-2} - \frac{C_3\left(\frac{r}{\varepsilon}\right)^2}{r\sqrt{\left(\frac{r}{\varepsilon}\right)^2 - 1}^{\,3}} \right) A^{-3/2}, \tag{3.31}$$

$$\kappa_2 = \cdots = \kappa_n = \left(C_2\gamma r^{\gamma-2} + \frac{C_3}{r\sqrt{\left(\frac{r}{\varepsilon}\right)^2 - 1}} \right) A^{-1/2}. \tag{3.32}$$

where $r = |x|$ and

$$A = 1 + \left(C_2 \gamma r^{\gamma-1} + \frac{C_3}{\sqrt{\left(\frac{r}{\varepsilon}\right)^2 - 1}} \right)^2. \tag{3.33}$$

Therefore we deduce that

$$\begin{aligned} H_k[w_\varepsilon] &= \kappa_2^{k-1} A^{-3/2} \gamma r^{\gamma-2} \left(\frac{n-k}{k} A + (\gamma - 1) \right) \\ &+ \kappa_2^{k-1} A^{-3/2} \frac{C_3}{r\sqrt{\left(\frac{r}{\varepsilon}\right)^2 - 1}} \left(\frac{n-k}{k} A - \frac{\left(\frac{r}{\varepsilon}\right)^2}{\left(\frac{r}{\varepsilon}\right)^2 - 1} \right). \end{aligned} \tag{3.34}$$

We define $M_1 = \dfrac{n-k}{k} A + (\gamma - 1)$ and $M_2 = \dfrac{n-k}{k} A - \dfrac{\left(\frac{r}{\varepsilon}\right)^2}{\left(\frac{r}{\varepsilon}\right)^2 - 1}$. Then we see that

$$M_1 \geq \gamma - \left(2 - \frac{n}{k}\right) > 0, \quad \text{(from (3.30))} \tag{3.35}$$

$$\begin{aligned} M_2 &\geq \frac{n-k}{k} \left(1 + C_2^2 r^{2(\gamma-1)} + \frac{C_3}{\left(\frac{r}{\varepsilon}\right)^2 - 1} \right) - \frac{\left(\frac{r}{\varepsilon}\right)^2}{\left(\frac{r}{\varepsilon}\right)^2 - 1} \\ &\geq \frac{\left(\frac{r}{\varepsilon}\right)^2 \left[C_2^2 \gamma^2 r^{2(\gamma-1)} - \left(2 - \frac{n}{k}\right) \right] + \frac{n-k}{k} \left(C_3^2 - 1 - C_2^2 r^{2(\gamma-1)} \right)}{\left(\frac{r}{\varepsilon}\right)^2 - 1} \\ &> 0, \end{aligned} \tag{3.36}$$

assuming that $r < R_0$ for sufficiently small $R_0 \in (0, \rho)$ depending only on C_2, γ, k, n, and that $C_3 > 1 + C_2 r^{\gamma-1}$. Under these assumptions, it follows that w_ε is a k-admissible function satisfying

$$H_k[w_\varepsilon] \geq \delta > 0 \quad \text{in } 2\varepsilon < |x| < R_0 \tag{3.37}$$

for some positive constant δ.

We take constants r_0, C_1, C_2, C_3. We fix $C_2 > 0$. From (3.30) we can take $r_0 \in (0, R_0)$ such that

$$C_2 |x|^\gamma \leq \frac{\tilde{C}}{4} |x|^{2-\frac{n}{k}} \quad \text{in } B_{r_0}, \tag{3.38}$$

where $\tilde{C}$ is a constant in the previous claim, and we set $C_1 = \dfrac{\tilde{C}}{4} r_0^{2-\frac{n}{k}}$. Then we take C_3 so that

$$C_3 \int_{2\varepsilon}^{r_0} \frac{ds}{\sqrt{\left(\frac{s}{\varepsilon}\right)^2 - 1}} = \tilde{C} r_0^{2-\frac{n}{k}}. \tag{3.39}$$

From (3.14), $C_3 = O\left((\varepsilon \log 1/\varepsilon)^{-1}\right)$, so that if $r \in (2\varepsilon, r_0)$, $C_3 > 1 + C_2 r^{\gamma-1}$ holds for small ε. Since it holds that $w_\varepsilon < u$ on $\partial B_{r_0} \cup \partial B_{2\varepsilon}$, which we can prove as (3.15) and (3.16), we find that $w_\varepsilon \leq u$ in $\overline{B_{r_0}} \setminus B_{2\varepsilon}$ from the comparison principle.

We repeat a similar argument to (3.19), (3.20), (3.21). Fixing $x \in B_{r_0} \setminus \{0\}$ and taking $\varepsilon \to 0$, we obtain that

$$u(x) \geq u(0) + C_1 = u(0) + \frac{\tilde{C}}{4} r_0^{2-\frac{n}{k}}. \tag{3.40}$$

This is contradictory to the continuity of u. The proof that there exists a sequence $\{z_j\}$ satisfying (3.1) is complete.

It remains to show that there exists a sequence $\{\tilde{z}_j\}$ such that (3.2) holds. But we can prove it similarly. For example, in the case of $k \leq \frac{n}{2}$, we use the auxiliary function of the form

$$w_\varepsilon(x) = u(0) - C_1 - C_2|x|^2 - C_3 \int_{r_0}^{|x|} \frac{ds}{\sqrt{\left(\frac{s}{\varepsilon}\right)^{\frac{2(n-k)}{k}} - 1}}, \tag{3.41}$$

and Proposition 2.2 as the comparison principle instead of Theorem 2.1. Then we can see that $\kappa[w_\varepsilon] \notin \Gamma_k$ and $w_\varepsilon \geq u$ on $\partial B_{r_0} \cup \partial B_{2\varepsilon}$, which implies that $w_\varepsilon \geq u$ in $\overline{B_{r_0}} \setminus B_{2\varepsilon}$ from Proposition 2.2. We omit its proof. □

We proceed to prove Theorem 1.1. To show that u is a viscosity subsolution of (1.1) in B_1, we need to prove that

$$H_k[P] \geq 0 \tag{3.42}$$

for any k-admissible quadratic polynomial P satisfying $u(0) = P(0)$ and $u \leq P$ in B_{r_0} for some $r_0 > 0$ (We say that P touches u at 0 from above). First we fix $\delta > 0$ and set $P_\delta(x) = P(x) + \frac{\delta}{2}|x|^2$. Then $P_\delta(x)$ satisfies the following properties:

$$P_\delta(0) = u(0), \quad P_\delta > u \quad \text{in } B_{r_0} \setminus \{0\}. \tag{3.43}$$

Next there exists $\varepsilon = \varepsilon(\delta) > 0$ such that $P_{\delta,\varepsilon}(x) = P_\delta(x) - \varepsilon(x_1 + \cdots + x_n)$ satisfies

$$P_{\delta,\varepsilon}(0) = u(0), \quad u < P_{\delta,\varepsilon} \quad \text{on } \partial B_{r_0}. \tag{3.44}$$

We notice that $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Now we apply the Lemma 3.1 for $l(x) = \langle DP_\delta(0), x\rangle + P_\delta(0)$. Passing to a subsequence if necessary, there exists a sequence $\{z_j\}$, $z_j \to 0$ as $j \to \infty$ such that all coordinates of every z_j are non-negative, and

$$u(z_j) - P_{\delta,\varepsilon}(z_j) > 0 \tag{3.45}$$

for any sufficiently large j. Thus from (3.44) there exists a point $x^\varepsilon \in B_{r_0}\backslash\{0\}$ such that

$$u(x^\varepsilon) - P_{\delta,\varepsilon}(x^\varepsilon) = \max_{B_{r_0}}(u - P_{\delta,\varepsilon}) > 0. \tag{3.46}$$

We introduce the polynomial

$$Q_{\delta,\varepsilon}(x) = P_{\delta,\varepsilon}(x) + u(x^\varepsilon) - P_{\delta,\varepsilon}(x^\varepsilon). \tag{3.47}$$

From (3.44), (3.46), we see that $Q_{\delta,\varepsilon}$ touches u at $x^\varepsilon \neq 0$ from above. Since u is a subsolution of (1.1) in $B_1 \setminus \{0\}$, we deduce that

$$0 \leq H_k[Q] = H_k\left[P + \frac{\delta}{2}|x|^2 - \varepsilon(x_1 + \cdots + x_n)\right]. \tag{3.48}$$

Finally, as $\delta \to 0$, we conclude that (3.42) holds.

It can be proved by analogous arguments that u is a supersolution of (1.1) in B_1. This completes the proof of Theorem 1.1.

4 Generalized solutions of curvature equations

For a large class of elliptic PDEs, there are various notions of solutions in a generalized sense, such as *weak solutions* for quasilinear equations, *distributional solutions* for semilinear equations, and *viscosity solutions* for fully nonlinear equations. Weak solutions and distributional solutions have an integral nature, and this makes it difficult to define such concepts of solutions for fully nonlinear PDEs. However, for some special types of fully nonlinear PDEs, one can introduce an appropriate notion of "solutions" that have an integral nature. For example, for Monge-Ampère type equations, the notion of generalized solutions was introduced and their properties have been studied intensively by Aleksandrov, Pogorelov, Bakel'man, Cheng and Yau, and others. For details, see [1], [7]. Recently Colesanti and Salani [8] considered generalized solutions in the case of Hessian equations (see also [23], [24], [25]). For the curvature equations, the author [21] introduced the notion of *generalized solutions* which form a wider class than viscosity solutions under the convexity assumptions. So it is natural to ask if the removability of singularities also holds in the framework of generalized solutions to (1.1).

First we define generalized solutions of (1.1). We assume that Ω is an open, convex and bounded subset of $\mathbb{R}^n$ and we look for solutions in the class of convex and (uniformly) Lipschitz functions defined on Ω. For a point $x \in \Omega$, let $\mathrm{Nor}(u;x)$ be the set of downward normal unit vectors to u at $(x,u(x))$. For a non-negative number ρ and a Borel subset η of Ω, we set

$$Q_\rho(u;\eta) = \{z \in \mathbb{R}^n \mid z = x + \rho v,\ x \in \eta,\ v \in \gamma_u(x)\}, \tag{4.1}$$

where $\gamma_u(x)$ is a subset of $\mathbb{R}^n$ defined by

$$\gamma_u(x) = \{(a_1,\dots,a_n) \mid (a_1,\dots,a_n,a_{n+1}) \in \mathrm{Nor}(u;x)\}. \tag{4.2}$$

The following theorem, which the author has proved in [21], plays a key role in the definition of generalized solutions.

Theorem 4.1. *Let Ω be an open convex bounded set in $\mathbb{R}^n$, and let u be a convex and Lipschitz function defined on Ω. Then the following hold.*

(i) For every Borel subset η of Ω and for every $\rho \geq 0$, the set $Q_\rho(u;\eta)$ is Lebesgue measurable.

(ii) There exist $n+1$ non-negative, finite Borel measures $\sigma_0(u;\cdot),\dots,$ $\sigma_n(u;\cdot)$ such that

$$\mathcal{L}^n(Q_\rho(u;\eta)) = \sum_{k=0}^{n} \binom{n}{k} \sigma_k(u;\eta)\rho^m \tag{4.3}$$

for every $\rho \geq 0$ and for every Borel subset η of Ω, where $\mathcal{L}^n$ denotes the Lebesgue n-dimensional measure.

The measures $\sigma_k(u;\cdot)$ determined by u are characterized by the following two properties.

(i) If $u \in C^2(\Omega)$, then for every Borel subset η of Ω,

$$\sigma_k(u;\eta) = \int_\eta H_k[u](x)\,dx, \tag{4.4}$$

(see Propositon [21], Proposition 2.1);

(ii) If u_i converges uniformly to u on every compact subset of Ω, then

$$\sigma_k(u_i;\cdot) \rightharpoonup \sigma_k(u;\cdot) \quad \text{(weakly)}. \tag{4.5}$$

Therefore we can say that for $k = 0,\dots,n$, the measure $\binom{n}{k}\sigma_k(u;\cdot)$ generalizes the integral of the function $H_k[u]$.

We state the definition of a generalized solution of curvature equations.

Definition 4.2. Let Ω be an open convex bounded set in $\mathbb{R}^n$ and let ν be a non-negative, finite Borel measure in Ω. A convex and Lipschitz function $u \in C^{0,1}(\Omega)$ is said to be a *generalized solution* of

$$H_k[u] = \nu \quad \text{in } \Omega, \tag{4.6}$$

if it holds that

$$\binom{n}{k}\sigma_k(u;\eta) = \nu(\eta) \tag{4.7}$$

for every Borel subset η of Ω.

There is a notion of generalized solutions to the Gauss curvature equations which correspond to the case of $k = n$ in (4.6), since they are in a class of Monge-Ampère type. As far as the Gauss curvature equation, namely,

$$\frac{\det(D^2u)}{(1+|Du|^2)^{\frac{n+2}{2}}} = \nu, \tag{4.8}$$

is concerned, the definition of generalized solutions introduced by Aleksandrov and others coincides with the one stated above.

One can show that if $\nu = \psi(x)dx$ for $\psi \in C^0(\Omega)$, a convex viscosity solution of $H_k[u] = \psi$ is a generalized solution of $H_k[u] = \nu$. Thus the notion of generalized solutions is weaker (hence wider) than that of viscosity solutions under the convexity assumptions.

Now we prove Theorem 1.2 which means that the removability of isolated singularities also holds for generalized solutions of (1.1) for $1 \le k \le n-1$. The technique to prove this result is different from what we have used in the proof of Theorem 1.1. It relies heavily on the integral nature of generalized solutions.

Proof of Theorem 1.2. Since u is locally convex in $\Omega \setminus \{0\}$, one can easily see that u can be defined at 0 continuously and the extended function (we denote it by the same symbol u) is convex and Lipschitz in Ω. We may assume that $\Omega = B_1$. Hence Theorem 4.1 implies that there exists a constant $C \ge 0$ such that in the generalized sense $H_k[u] = C\delta_0$ in $B_{1/2}$, where δ_0 is Dirac delta measure at 0. That is,

$$\binom{n}{k}\sigma_k(u;B_r) = C \tag{4.9}$$

for arbitrary $r \in (0, 1/2)$.

We deduce from (4.3) and (4.9) that

$$\begin{aligned}\omega_n(r+\rho)^n &\geq \mathcal{L}^n\left(Q_\rho(u;B_r)\right) \\ &= \sum_{m=0}^{n}\binom{n}{m}\sigma_m(u;B_r)\rho^m \\ &\geq \binom{n}{k}\sigma_k(u;B_r)\rho^k = C\rho^k.\end{aligned} \tag{4.10}$$

The first inequality in (4.10) is due to the fact that $Q_\rho(u;B_r) \subset B_{r+\rho}$, since taking any $z \in Q_\rho(u;B_r)$, we obtain

$$|z| = |x+\rho v| \leq |x| + \rho|v| \leq r+\rho, \tag{4.11}$$

for some $x \in B_r$, $v \in \gamma_u(x)$. Taking $r \to 0$ in (4.10), we obtain that

$$\omega_n\rho^n \geq C\rho^k. \tag{4.12}$$

Since (4.12) holds for arbitrary $\rho \geq 0$, C must be 0. Therefore we have proved that $H_k[u] = 0$ in the entire ball $B_{1/2}$, so that the origin is removable. □

Remark 4.1. (1) Examining the above proof carefully, we find that the inhomogeneous term 0 in Theorem 1.2 can be replaced by a measurable function f which is non-negative and belongs to $L^1(\Omega)$.
(2) We can extend the function space to which u belongs in the theorems and definition of this section to the space of semiconvex functions (see [21]).

As we have seen in section 1, Theorem 1.2 does not hold for $k = n$. So we have generalized solutions of (4.6), where the inhomogeneous term ν is a Dirac delta measure. One may consider the existence and uniqueness of generalized solutions to the Dirichlet problem for (4.6) in a bounded convex domain when ν is a Borel measure. Many mathematicians have discussed this problem. For details, see [1]. However, there are few results about the solvability of the Dirichlet problem in the generalized sense for the case of $1 \leq k \leq n-1$.

5 Conjectures and open problems

In this section, we make some conjectures and state some open problems we would like to study in a future.

(1) To remove the continuity assumption on u in Theorem 1.1.

As we have mentioned in the introduction, for the case of $k = 1$, Theorem 1.1 holds even if no restrictions are imposed on the behaviour of solutions near the singularities. We conjecture that isolated singularities of (1.1) are always removable without any assumptions on the behaviour of the solution near the singularities.

(2) To study the removability of a set, instead of a single point.

It is also interesting to study the removability of a singular set whose α-dimensional Hausdorff measure is zero for some $\alpha > 0$.

(3) To study properties of generalized solutions to (4.6).

We would like to know if the notion of generalized solutions is truly weaker than that of viscosity solution for the case of $1 \leq k \leq n-1$ in (4.6), that is, if there exists a generalized solution u of (4.6) such that ν cannot be expressed as $\psi(x)dx$ for any $\psi \in C^0$. We think that this question is closely related to the above problem of removability of singular sets.

(4) Problems of isolated singularities for other fully nonlinear equations.

For example, we would like to consider the case of the curvature quotient equations, $\dfrac{H_k[u]}{H_l[u]} = \psi(x)$ where $0 < l < k \leq n$, or that of $F_k(D^2u) + f(u) = 0$ where $F_k(D^2u)$ is the k-th elementary symmetric function of the eigenvalues of D^2u (F_k is called k-Hessian operator).

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