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On time-periodic solutions for a weakly anisotropic curvature flow equation with driving force term

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§1. Introduction and main result

Frank [3] claimed that the presence of steps associated with screw dislocations plays a key role for the growth of crystal surfaces; see also [1]. In geometric model the location of the steps on a crystal surface is represented as a curve $\Gamma(t)$ depending on time t. In [2] it was proposed that the evolution of $\Gamma(t)$ is governed by a curvature flow equation with a driving force term:

$$V = V_0(1 + \ell_0\kappa). \tag{1}$$

Here V and κ denotes the normal velocity and the curvature of $\Gamma(t)$ respectively in the direction of the unit normal vector field $\boldsymbol{n}$ of $\Gamma(t)$; V_0 is a positive constant, which is the normal velocity of the straight step; and ℓ_0 is also a positive constant, which is the critical radius of the curvature. If 0 is the only dislocation, we consider (1) in $\mathbb{R}^2 \setminus \{0\}$ such that one of the end point of $\Gamma(t)$ is zero. In [2] it is suggested that there is an essentially unique rotating solution for (1). Such a solution is called a spiral(-shaped) solution. In modern analysis this problem can be solved by a shooting method as suggested in [16, Appendix AVI, p.190-203]; see also [14].

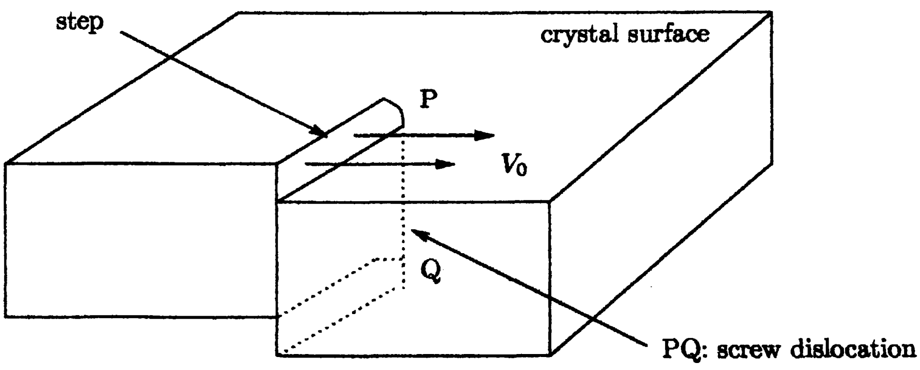

Figure 1: Screw dislocation

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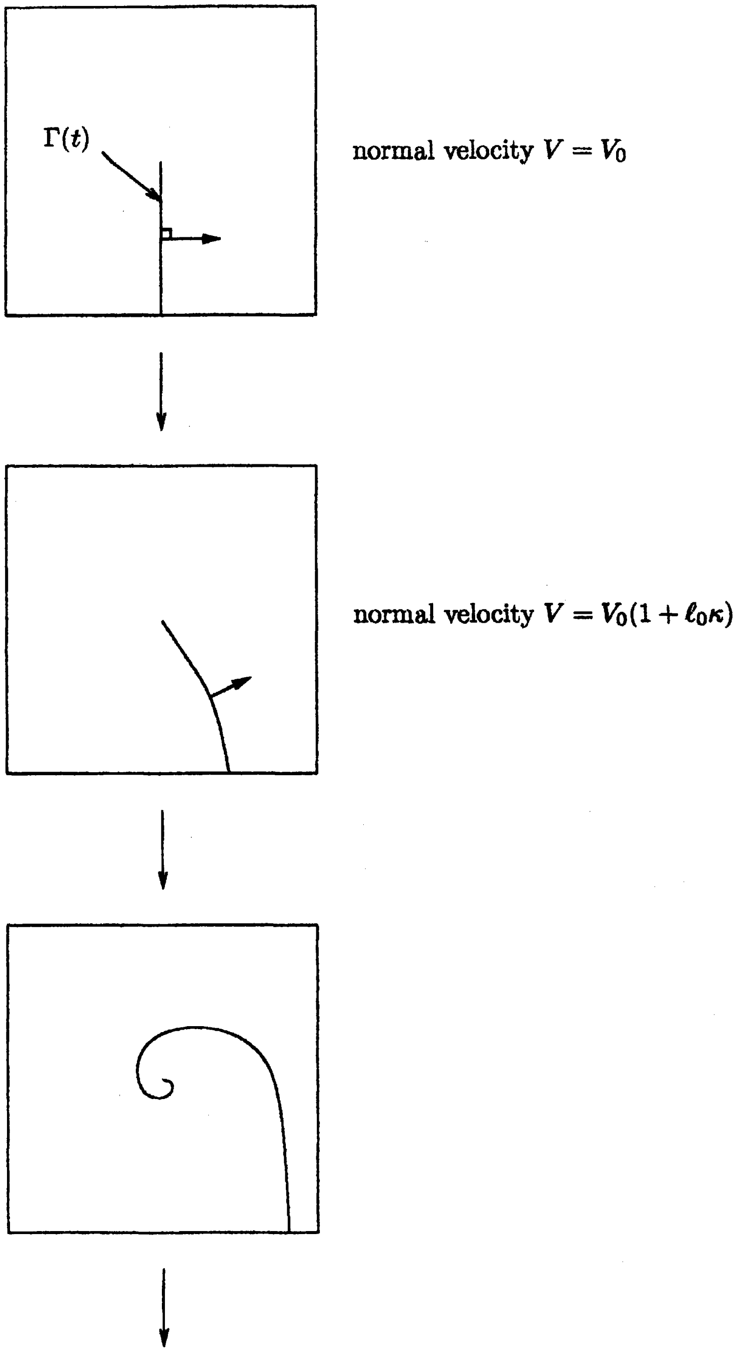

The step due to the dislocation will rapidly wind itself up into a spiral centered on the dislocation, until the curvature at the center reaches the critical value $-1/\ell_0$, at which the normal velocity falls to zero; the whole spiral will then rotate steadily with stationary shape.

Figure 2: The location of the step

In this proceeding we study the existence of spiral(-shaped) solution when the growth equation (1) takes the anisotropy into account. Such an extension is very natural in the theory of crystal growth. For technical reasons we postulate that the dislocation is not a point but a closed disk B_ρ, the center of which is 0, and that crystal surface B_R is a large disk having common center with dislocation disk. Moreover, we postulate that $\Gamma(t)$ is orthogonal to the boundary of the crystal surface B_R and B_ρ. Under these assumptions evolution of $\Gamma(t)$ in an annulus $\Omega = \{x \in \mathbb{R}^2 \mid \rho < |x| < R\}(= B_R \setminus B_\rho)$ is governed by

$$\begin{cases} V = M(\boldsymbol{n})(D_0\kappa_\gamma + V_0) \quad \text{on} \quad \Gamma(t), \\ \Gamma(t) \perp \partial\Omega, \end{cases} \tag{2}$$

Here κ_γ denotes the anisotropic curvature of $\Gamma(t)$ in the direction of $\boldsymbol{n}$. It is defined by

$$\kappa_\gamma := -\mathrm{div}_s \nabla\gamma(\boldsymbol{n}), \tag{3}$$

with the interfacial energy density $\gamma : \mathbb{R}^2 \to \mathbb{R}^+ = \{\sigma \in \mathbb{R} \mid \sigma \geq 0\}$ which is positively homogeneous of degree one, i.e., $\gamma(\lambda p) = \lambda\gamma(p)$ for all $p \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}^+$; ∇ denotes the gradient and div_s denotes the surface divergence. For a vector field f on a curve in $\mathbb{R}^2$ the surface divergence is defined by

$$\mathrm{div}_s f := \langle \partial_s f, \boldsymbol{\tau} \rangle,$$

where ∂_s is the derivative with respect to arclength and $\boldsymbol{\tau}$ is the unit tangent vector to the curve; $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{R}^2$. The function $M(\boldsymbol{n})$ is called the mobility, which is positively homogeneous of degree zero and depends on $\boldsymbol{n}$; D_0 is a positive constant. If $M(\boldsymbol{n}) \equiv 1$ and $\gamma(p) = |p|$, then the curvature flow equation in (2) is nothing but (1) with $D_0 = \ell_0 V_0$. For more applications of these equations the reader is referred to a nice monograph of M. E. Gurtin [4] and a review article by J. Taylor, J. Cahn and A. Handwerker [19].

Our goal in this proceeding is to seek a spiral(-shaped) solution. Contrary to isotropic case there might be no rotating solution. For anisotropic case it is natural to say that $\Gamma(t)$ is spiral(-shaped) solution if $\Gamma(t)$ is a periodic-in-time solution of (2). In this proceeding we consider more special spiral solution of the form

$$\Gamma(t) = \{(r\cos\theta(r,t), r\sin\theta(r,t)) \mid \rho \leq r \leq R\} \tag{4}$$

where $r = |x|$ and θ represents the argument or the angle of $x \in \mathbb{R}^2$.

Definition. *We call $\Gamma(t)$ a spiral solution of (2) of $\theta(r,t)$ in (4) if $\theta(r,t)$ is monotone with respect to r, and periodic in time t, that is, there exists $T > 0$ such that $\theta(r, t+T) = \theta(r,t) + 2\pi$ for all $t > 0$.*

We remark that other kind of spirals, such as those are not included in above category, do exist in reality; certain crystals usually involve facets, where the angle θ is not a monotone function of r. It is not expected that a spiral solution of form (4) always exists unless anisotropic effect is small. We here confine ourselves to investigating the existence of spirals within the above somewhat restricted family when anisotropic effect is small. Our main results read as follows.

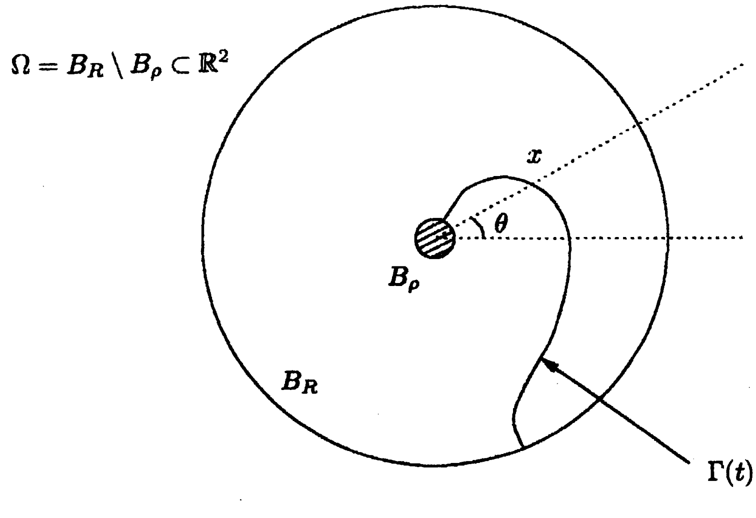

Figure 3: The representation of the form (4)

Main Theorem. *(Existence of a spiral solution). Assume that M and γ are smooth on a unit circle. Assume that the equation (2) is close to isotropy in the sense defined in Section 3 below. Then there exists a spiral solution $\hat{\Gamma}(t)$ of (2), which is unique up to translation of time.*

The assumption that the equation (2) is close to isotropy is necessary to obtain a global-in-time solution of form (4).

We briefly describe our strategy of the proof. First we derive the equation for $\theta(r,t)$ appeared in the formula (4), and establish its gradient estimates under the condition that the equation (2) is close to isotropy in some sense. The precise assumption is presented in Section 3. The gradient estimate is a consequence of the use of the weak maximum principle, and plays a key role to obtain a global-in-time solution to (2). Next, we show a time-monotonicity of the infimum of $\theta(r,t)$ on $\bar{I} := \{r \in \mathbb{R} \mid \rho \leq r \leq R\}$ and an order-preserving property of θ. The strong maximum principle is involved in the proof. Based on the gradient estimate as well as these properties on θ we apply the theory developed in [14] to obtain a time-periodic solution, which is unique up to translation of time.

In [14] they studied the Neumann problem for the Allen–Cahn type equation

$$u_t = \Delta u + g(u - \theta), \quad x \in \Omega$$

when g is 2π-periodic function. Here θ denotes the angle of x. The function g is the derivative of a multi-well potential and $\int_0^{2\pi} g(v)dv > 0$. They proved the unique existence of a spiral traveling wave solution u in the sense that u is of the form

$$u(x,t) = \varphi(r, \theta - \omega t) + \omega t$$

with some $\omega > 0$ (independent of x and t) and a function φ with $r = |x|$. In [15] a similar result is derived for more general semilinear parabolic equations and it is proved that such a solution is asymptotically stable. To construct a spiral solution, they use strong maximum principle in a smart way. We shall use their argument in the proof of Theorem 1 as mentioned in Section 4. In [14] they also prove the existence of rotating solution for (2) when it is isotropic (i.e. $\gamma(p) = |p|$, $M(\boldsymbol{n}) \equiv 1$) by a shooting argument for an ordinary differential equation. We remark that such an ODE argument does not work when the equation is anisotropic (see Remark 4).

We take this opportunity to mention several related works on spirals. There are numerical calculations based on the Allen–Cahn equation by A. Karma and M. Plapp [9], R. Kobayashi [10]; the latter also treats the case when there are several dislocation. In this case two steps may collide. To treat such a phenomena two level set methods are proposed numerically by P. Smereka [18] and analytically by T. Ohtsuka [17]; their methods are different each other. Other aspects of spiral shaped solutions for various interface equations, we refer for instance to [5, 6, 7, 8, 20] and references therein.

§2. Derivation of the equation

Let us derive the equations for $\theta(r,t)$ in the expression (4) of $\Gamma(t)$. Since γ is homogeneous of degree 1, we first get

$$\langle \nabla\gamma(p), p\rangle = \gamma(p). \tag{5}$$

for $p \in \mathbb{R}^2$. Let $p^\perp$ be rotation of p by $-\pi/2$. We observe that

$$\nabla\gamma(p) = \langle \nabla\gamma(p), \frac{p}{|p|}\rangle \frac{p}{|p|} + \langle \nabla\gamma(p), \frac{p^\perp}{|p^\perp|}\rangle \frac{p^\perp}{|p^\perp|}.$$

Since $|p| = |p^\perp|$, we obtain

$$|p|^2\nabla\gamma(p) = \langle \nabla\gamma(p), p\rangle p + \langle \nabla\gamma(p), p^\perp\rangle p^\perp. \tag{6}$$

Combining (5) and (6), we are led to

$$|p|^2\nabla\gamma(p) = \gamma(p)p + \langle \nabla\gamma(p), p^\perp\rangle p^\perp. \tag{7}$$

Here, the unit normal vector $\boldsymbol{n}$ and the unit tangent vector $\boldsymbol{\tau}$ of $\Gamma(t)$ is represented as

$$\boldsymbol{n} = \boldsymbol{n}(r,\theta,\theta_r) = \frac{1}{(1+r^2\theta_r^2)^{1/2}}\begin{pmatrix} -\sin\theta - r\theta_r\cos\theta \\ \cos\theta - r\theta_r\sin\theta \end{pmatrix},$$
$$\boldsymbol{\tau} = \boldsymbol{\tau}(r,\theta,\theta_r) = (\boldsymbol{n}(r,\theta,\theta_r))^\perp = \frac{1}{(1+r^2\theta_r^2)^{1/2}}\begin{pmatrix} \cos\theta - r\theta_r\sin\theta \\ \sin\theta + r\theta_r\cos\theta \end{pmatrix}.$$

This implies

$$\frac{\partial\boldsymbol{n}}{\partial\theta} = -\boldsymbol{\tau}, \quad \frac{\partial\boldsymbol{\tau}}{\partial\theta} = \boldsymbol{n}.$$

Setting $\hat\gamma(\theta) := \gamma(\boldsymbol{n}(\cdot,\theta,\cdot))$, we find

$$\hat\gamma'(\theta) = \langle \nabla\gamma(\boldsymbol{n}), \frac{\partial\boldsymbol{n}}{\partial\theta}\rangle = -\langle \nabla\gamma(\boldsymbol{n}), \boldsymbol{\tau}\rangle \tag{8}$$

where $'$ denotes the derivative with respect to θ. Thus, by virtue of (7) and (8), we are led to

$$\nabla\gamma(\boldsymbol{n}) = \gamma(\boldsymbol{n})\boldsymbol{n} - \hat\gamma'(\theta)\boldsymbol{\tau} = \hat\gamma(\theta)\boldsymbol{n} - \hat\gamma'(\theta)\boldsymbol{\tau}. \tag{9}$$

We also note

$$\partial_s \boldsymbol{n} = \frac{1}{(1+r^2\theta_r^2)^{1/2}}\frac{\partial \boldsymbol{n}}{\partial r} = -\frac{r\theta_{rr} + r^2\theta_r^3 + 2\theta_r}{(1+r^2\theta_r^2)^{3/2}}\boldsymbol{\tau},$$
$$\partial_s \boldsymbol{\tau} = \frac{1}{(1+r^2\theta_r^2)^{1/2}}\frac{\partial \boldsymbol{\tau}}{\partial r} = \frac{r\theta_{rr} + r^2\theta_r^3 + 2\theta_r}{(1+r^2\theta_r^2)^{3/2}}\boldsymbol{n}.$$

Then, by (3) and (9), we derive

$$\begin{aligned} \kappa_\gamma &= -\mathrm{div}_s(\hat{\gamma}(\theta)\boldsymbol{n} - \hat{\gamma}'(\theta)\boldsymbol{\tau}) \\ &= -\langle \partial_s(\hat{\gamma}(\theta)\boldsymbol{n} - \hat{\gamma}'(\theta)\boldsymbol{\tau}), \boldsymbol{\tau}\rangle \\ &= -\left\{-\hat{\gamma}(\theta)\frac{r\theta_{rr} + r^2\theta_r^3 + 2\theta_r}{(1+r^2\theta_r^2)^{3/2}} - \partial_s\hat{\gamma}'(\theta)\right\}|\boldsymbol{\tau}|^2. \end{aligned}$$

On the other hand, differentiating the both sides of (8) with respect to the arclength parameter s, we deduce that

$$\partial_s\hat{\gamma}'(\theta) = \frac{r\theta_{rr} + r^2\theta_r^3 + 2\theta_r}{(1+r^2\theta_r^2)^{3/2}}\{\langle(\nabla^2\gamma(\boldsymbol{n}))\boldsymbol{\tau}, \boldsymbol{\tau}\rangle - \langle\nabla\gamma(\boldsymbol{n}), \boldsymbol{n}\rangle\}. \tag{10}$$

Moreover, differentiating the both sides of (8) with respect to θ, we obtain

$$\hat{\gamma}''(\theta) = \langle(\nabla^2\gamma(\boldsymbol{n}))\boldsymbol{\tau}, \boldsymbol{\tau}\rangle - \langle\nabla\gamma(\boldsymbol{n}), \boldsymbol{n}\rangle. \tag{11}$$

Thanks to (10) and (11), we are led to

$$\partial_s\hat{\gamma}'(\theta) = \frac{r\theta_{rr} + r^2\theta_r^3 + 2\theta_r}{(1+r^2\theta_r^2)^{3/2}}\hat{\gamma}''(\theta),$$

from where we conclude that

$$\kappa_\gamma = (\hat{\gamma}(\theta) + \hat{\gamma}''(\theta))\frac{r\theta_{rr} + r^2\theta_r^3 + 2\theta_r}{(1+r^2\theta_r^2)^{3/2}}. \tag{12}$$

Consequently, since the normal velocity of $\Gamma(t)$ is

$$V = \langle\frac{\partial\Gamma}{\partial t}, \boldsymbol{n}\rangle = \frac{r\theta_t}{(1+r^2\theta_r^2)^{1/2}},$$

so that the interface equation (2) become

$$\begin{cases} \theta_t = M(\boldsymbol{n})\left(\dfrac{a(\boldsymbol{n})(r\theta_{rr} + r^2\theta_r^3 + 2\theta_r)}{r(1+r^2\theta_r^2)} + \dfrac{V_0(1+r^2\theta_r^2)^{1/2}}{r}\right), \\ \theta_r(\rho, t) = \theta_r(R, t) = 0. \end{cases} \tag{13}$$

where $a(\boldsymbol{n}) := D_0(\gamma(\theta) + \gamma''(\theta)) = D_0\langle[\nabla^2\gamma(\boldsymbol{n})]\boldsymbol{\tau}, \boldsymbol{\tau}\rangle$.

Remark 1. *(Local existence of the solution of (13)).* We describe the local existence of the solution of (13). By using the optimal regularity theory of analytic semigroups as in [13], we get a unique and smooth local-in-time solution of (13) with existence time T, which depends on $1/\|\theta_0\|_{C^{1+\alpha}(I)}$. This implies that if we obtain $C^{1+\alpha}$-a priori estimate of the solution $\theta(\cdot, t)$ for $t > 0$, there exists a unique global-in-time solution of (13). □

§3. Gradient estimate

The goal of this section is to obtain the gradient estimate. For this purpose, we first provide the precise assumption that the equation of (13) is close to isotropy.

For λ, μ, $\varepsilon > 0$ and $\Lambda < \infty$, we assume

$$\text{(A-1)} \quad \lambda \le M(p) \le \Lambda \quad \text{and} \quad \mu \le a(p) \le \Lambda \quad \text{for all } p \in \mathbb{S}^1,$$

$$\text{(A-2)} \quad \left| \frac{\langle \nabla M(p), p^{\perp} \rangle}{M(p)} \right| + \left| \frac{\langle \nabla a(p), p^{\perp} \rangle}{a(p)} \right| \le \varepsilon \quad \text{for all } p \in \mathbb{S}^1.$$

If (A-2) holds for small $\varepsilon > 0$, this implies that M and a are close to isotropy. The next proposition is the main ingredient of our current exposition.

Proposition 1. *(Gradient estimate). Let*

$$\varepsilon_* = \frac{\mu\rho}{2R^2V_0(d + \sqrt{d^2-1})} (> 0)$$

where

$$d = 1 + \frac{2R^2}{\rho^2}\left(2 + \frac{RV_0}{\mu}\right) (> 1).$$

Assume that $M(\boldsymbol{n})$ and $a(\boldsymbol{n})$ satisfy (A-1) and also fulfill (A-2) for some $\varepsilon \in (0, \varepsilon_*]$. Then if $\theta(r,t)$ is a solution of (13) with the initial data $\theta(\cdot, 0) = \theta_0$ such that $-L \le \theta_{0r} \le 0$ for $L \in [L_1, L_2]$, the gradient estimate $-L \le \theta_r(\cdot, t) \le 0$ holds for $t > 0$. Here, constants L_1 and L_2 $(0 < L_1 \le L_2)$ are solutions of quadratic equation

$$\frac{2\varepsilon}{\rho}\left(2 + \frac{RV_0}{\mu}\right)L^2 - \left(\frac{1}{R^2} - \frac{2\varepsilon V_0}{\mu\rho}\right)L + \frac{V_0}{\mu\rho^2} = 0. \tag{14}$$

Remark 2. If $0 < \varepsilon \le \varepsilon_*$, the quadratic equation (14) has two positive real-valued solutions. Indeed, (14) has two positive real-valued solutions if and only if

$$\frac{1}{R^2} - \frac{2\varepsilon V_0}{\mu\rho} > 0, \quad \text{and} \quad \left(\frac{1}{R^2} - \frac{2\varepsilon V_0}{\mu\rho}\right)^2 - 4 \cdot \frac{2\varepsilon}{\rho}\left(2 + \frac{RV_0}{\mu}\right) \cdot \frac{V_0}{\mu\rho^2} \ge 0.$$

These conditions holds for $0 < \varepsilon \le \varepsilon_*$. We also stress that ε in (A-2) is determined a posteriori from ε_* in Proposition 1. □

Proof of Proposition 1. Differentiating both sides of (13) and setting $v := \theta_r$, we have

$$\alpha(r,t)v_{rr} + \beta(r,t)v_r + \gamma(r,t)v - v_t = \frac{M(\boldsymbol{n})V_0}{r^2(1 + r^2v^2)^{1/2}} \ge 0,$$

$$
\begin{aligned}
\alpha(r,t) &= \frac{M(\boldsymbol{n})a(\boldsymbol{n})}{1+r^2v^2},\\
\beta(r,t) &= M(\boldsymbol{n})a(\boldsymbol{n})\frac{2+r^4v^4-2r^3vv_r-r^2v^2}{r(1+r^2v^2)^2}\\
&\quad -M(\boldsymbol{n})a(\boldsymbol{n})\left(\frac{\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{M(\boldsymbol{n})}+\frac{\langle\nabla a(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{a(\boldsymbol{n})}\right)\frac{r^2v_r+2rv(2+r^2v^2)}{r(1+r^2v^2)^2}\\
&\quad +\frac{M(\boldsymbol{n})V_0}{(1+r^2v^2)^{1/2}}\left(v-\frac{\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{M(\boldsymbol{n})}\right),\\
\gamma(r,t) &= -\frac{M(\boldsymbol{n})a(\boldsymbol{n})}{r^2}\left(1+\frac{1+3r^2v^2}{(1+r^2v^2)^2}\right)\\
&\quad -M(\boldsymbol{n})a(\boldsymbol{n})\left(\frac{\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{M(\boldsymbol{n})}+\frac{\langle\nabla a(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{a(\boldsymbol{n})}\right)\frac{(2+r^2v^2)^2v}{(1+r^2v^2)^2r}\\
&\quad -\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle\frac{V_0(1+r^2v^2)^{1/2}(2+r^2v^2)}{r(1+r^2v^2)}
\end{aligned}
$$

with $v=v(r,t)$, $\boldsymbol{n}=\boldsymbol{n}(r,t)$, and $\boldsymbol{\tau}=\boldsymbol{\tau}(r,t)$. Since $\alpha(r,t)>0$ and $|\alpha(r,t)|$, $|\beta(r,t)|$, $|\gamma(r,t)|<\infty$, we can appeal to the weak maximum principle; we deduce that $v(\cdot,t)\le 0$ for $t>0$ if the initial data of v satisfies $v(\cdot,0)\le 0$. That is, we are led to $\theta_r(\cdot,t)\le 0$ for $t>0$ if the initial data θ_0 satisfies $\theta_{0r}\le 0$.

Next, we prove that if the initial data θ_0 satisfies $\theta_{0r}\ge -L$ for $L\in[L_1,L_2]$, the minimum of $\theta_r(\cdot,t)$ is estimated by $-L$ for $t>0$. To prove this, we set $w:=-v-L$. Then w satisfies

$$
\begin{aligned}
&\hat\alpha(r,t)w_{rr}+\hat\beta(r,t)w_r+\hat\gamma(r,t)w-w_t\\
&= \frac{M(\boldsymbol{n})a(\boldsymbol{n})}{r^2}\left(1+\frac{1+3r^2(w+L)^2}{(1+r^2(w+L)^2)^2}\right)\cdot L\\
&\quad -\frac{M(\boldsymbol{n})a(\boldsymbol{n})}{r}\left(\frac{\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{M(\boldsymbol{n})}+\frac{\langle\nabla a(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{a(\boldsymbol{n})}\right)\frac{(2+r^2(w+L)^2)^2}{(1+r^2(w+L)^2)^2}\cdot L^2\\
&\quad +\frac{M(\boldsymbol{n})V_0(1+r^2(w+L)^2)^{1/2}}{r}\cdot\frac{\langle\nabla M(\boldsymbol{n}),\boldsymbol{\tau}\rangle}{M(\boldsymbol{n})}\cdot\frac{2+r^2(w+L)^2}{1+r^2(w+L)^2}\cdot L\\
&\quad -\frac{M(\boldsymbol{n})V_0}{r^2(1+r^2(w+L)^2)^{1/2}}\\
&\ge M(\boldsymbol{n})a(\boldsymbol{n})\left\{\frac{1}{R^2}L-\frac{4\varepsilon}{\rho}L^2-\frac{2\varepsilon V_0(1+R(w+L))}{\mu\rho}L-\frac{V_0}{\mu\rho^2}\right\}
\end{aligned}
$$

where $\hat\alpha$, $\hat\beta$ and $\hat\gamma$ are represented by putting $-w-L$ instead of v in α, β, and γ, respectively. Transposing the term of w in the right hand side to the left hand side, we

$$\begin{aligned}&\hat{\alpha}(r,t)w_{rr}+\hat{\beta}(r,t)w_r+\left(\hat{\gamma}(r,t)+\frac{2\varepsilon RV_0L}{\mu\rho}\right)w-w_t\\&\geq -M(\boldsymbol{n})a(\boldsymbol{n})\left\{\frac{2\varepsilon}{\rho}\left(2+\frac{RV_0}{\mu}\right)L^2-\left(\frac{1}{R^2}-\frac{2\varepsilon V_0}{\mu\rho}\right)L+\frac{V_0}{\mu\rho^2}\right\}\\&=: -M(\boldsymbol{n})a(\boldsymbol{n})Q(L).\end{aligned}\tag{15}$$

According to the condition of L, we find that $Q(L) \leq 0$. Since $M(\boldsymbol{n})$ and $a(\boldsymbol{n})$ are positive, it follows that

$$\hat{\alpha}(r,t)w_{rr}+\hat{\beta}(r,t)w_r+\left(\hat{\gamma}(r,t)+\frac{2\varepsilon RV_0L}{\mu\rho}\right)w-w_t \geq -M(\boldsymbol{n})a(\boldsymbol{n})Q(L) \geq 0.$$

Using the weak maximum principle for this equation, we obtain that $w(\cdot,t) \leq 0$ for $t > 0$ if the initial data of w satisfies $w(\cdot,0) \leq 0$. The proof is now complete. □

§4. Existence of spiral solutions

In this section, our goal is to obtain a periodic-in-time solution of (13). For this purpose, we shall apply the idea of [14]. Let us first derive the useful properties on θ.

Lemma 1. *(i) (Monotonicity for time). Let $\theta(\cdot,t)$ be solutions of (13) with the initial data θ_0 and $m(t) := \inf\{\theta(r,t) \mid \rho \leq r \leq R\}$. Assume that $\theta_{0r} \leq 0$. Then there exists a constant $\nu > 0$ such that*

$$\frac{d}{dt}m(t) \geq \nu \quad \text{for} \quad t > 0.$$

(ii) (Order-preserving). Let $\theta^{(1)}(\cdot,t)$ and $\theta^{(2)}(\cdot,t)$ be solutions of (13) with the initial data $\theta_0^{(1)}$ and $\theta_0^{(2)}$, respectively. If $\theta_0^{(1)} \leq \theta_0^{(2)}$ with $\theta_0^{(1)} \not\equiv \theta_0^{(2)}$, then the order is preserved for $t > 0$, i.e.,

$$\theta^{(1)}(\cdot,t) < \theta^{(2)}(\cdot,t) \quad \text{for} \quad t > 0.$$

Proof. We first prove (i). By virtue of Proposition 1, we have $\theta_r(\cdot,t) \leq 0$ for $t > 0$. This implies that

$$\inf_{\rho\leq r\leq R}\theta(r,t) = \theta(R,t)(=: m(t)).$$

Letting $r \uparrow R$ in (13), we have

$$\theta_t(R,t) = M(\boldsymbol{n})\left(a(\boldsymbol{n})\theta_{rr}(R,t)+\frac{V_0}{R}\right). \tag{16}$$

Now we claim that

$$\theta_{rr}(R,t) \geq 0 \quad \text{for} \quad t > 0. \tag{17}$$

In fact, suppose that $\theta_{rr}(R,t) < 0$. Then there exists a constant $\delta > 0$ such that $\theta_{rr}(r,t) < 0$ for $R-\delta < r \leq R$. Since $\theta_r(R,t) = 0$, we see that $\theta_r(r,t) > 0$ for

$R-\delta < r \leq R$. This contradicts $\theta_r(r,t) \leq 0$ for $\rho \leq r \leq R$, which verifies (17). In view of (16) and (17), we thus obtain

$$\theta_t(R,t) \geq M(\boldsymbol{n})\frac{V_0}{R} \geq \frac{\lambda V_0}{R} > 0.$$

Since θ is a C^1-function with respect to t for $t>0$, the desired inequality is established.

(ii) is proved by using the strong maximum principle. We may safely omit the details. □

Moreover we obtain the estimate of θ as follows.

Lemma 2. *Let $\theta(\cdot,t)$ be solutions of (13) with the initial data θ_0. Assume that $\theta_{0r} \leq 0$. Then we have*

$$\theta_0(R) + \nu_1 t \leq \theta(\cdot,t) \leq \theta_0(\rho) + \nu_2 t \quad \textit{for} \quad t>0.$$

where $\nu_1 = \lambda V_0/R$ and $\nu_2 = \Lambda V_0/\rho$.

Proof. By Proposition 1 we see $\theta_r(\cdot,t) \leq 0$ for $t>0$. Then we have

$$\theta(R,t) \leq \theta(r,t) \leq \theta(\rho,t) \quad \text{for} \quad \rho \leq r \leq R.$$

It follows from Lemma 1(i) that $\theta_0(R) + \nu_1 t \leq \theta(R,t)$. On the other hand, applying the similar argument to the proof of Lemma 1(i), $\theta_{rr}(\rho,t) \leq 0$ is verified. Letting $r \downarrow \rho$ in (13), we have

$$\theta_t(\rho,t) = M(\boldsymbol{n})\left(a(\boldsymbol{n})\theta_{rr}(\rho,t) + \frac{V_0}{\rho}\right) \leq M(\boldsymbol{n})\frac{V_0}{\rho} \leq \frac{\Lambda V_0}{\rho}.$$

This implies that $\theta(\rho,t) \leq \theta_0(\rho) + \nu_2 t$ and completes the proof. □

Remark 3. *(Global existence of the solution of (13)).* According to the theory of parabolic equations (see [11, 12]), the Hölder norm of the gradient is estimated by a constant depending on the maximum norm of the gradient and some given constants in the assumptions for $M(\boldsymbol{n})$ and $a(\boldsymbol{n})$. Thus the existence of the global-in-time solution of (13) is assured by Remark 1, Proposition 1, and Lemma 2. The uniqueness of the solution of (13) is also assured by the order preserving property. □

We set

$$\mathcal{D} = \{\psi \in C^{1+\alpha}(\bar{I}) \mid -L \leq \psi_r \leq 0,\ \psi_r(\rho) = \psi_r(R) = 0\}$$

where $I = \{r \in \mathbb{R} \mid \rho < r < R\}$ and define the map Φ_t on $\mathcal{D}$ as

$$\Phi_t(\theta_0) = \theta(\cdot,t) \quad \text{for each} \quad t>0 \tag{18}$$

where $\theta(\cdot,t)$ is the solution of (13) with the initial data $\theta(\cdot,0) = \theta_0$. Note that if the initial data θ_0 is in $\mathcal{D}$, then we have the estimates of $\theta_r(\cdot,t)$ and $\theta(\cdot,t)$ for $t>0$ by Proposition 1 and Lemma 2; that is, there exists a unique global-in-time solution of (13) (see Remark 3). Moreover, if the initial data θ_0 is in $\mathcal{D}$, the solution $\theta(\cdot,t)$ stays also in

$\mathcal{D}$. The definition of the mapping Φ_t and the uniqueness of the solution of (13) imply that a family of the mappings Φ_t from $\mathcal{D}$ to itself satisfies the semigroup property:

$$\Phi_0(\theta) = \theta \quad \text{for all} \quad \theta \in \mathcal{D}, \qquad \Phi_t \circ \Phi_s = \Phi_{t+s} \quad \text{for any} \quad t, s \in [0, \infty). \tag{19}$$

The invariance of (13) for 2π-periodicity and the uniqueness of the solution of (13) justify

$$\Phi_t(\theta \pm 2n\pi) = \Phi_t(\theta) \pm 2n\pi \quad \text{for any} \quad t > 0,\ \theta \in \mathcal{D} \text{ and } n \in \mathbb{N}. \tag{20}$$

In addition, the standard parabolic estimate implies that Φ_t is a compact map on $\mathcal{D}$ for each $t > 0$. Recalling Lemma 1(ii), Φ_t is also order-preserving for each $t > 0$, which means that $\theta_1 < \theta_2$ implies $\Phi_t(\theta_1) < \Phi_t(\theta_2)$ for each $t > 0$. To obtain a periodic solution of (13), we need the following proposition.

Proposition 2. *Let $\{\Phi_t\}_{t \in [0,\infty)}$ be a family of mappings Φ_t defined by (18). Then there exists a unique $T_0 > 0$ such that $\varphi + 2\pi = \Phi_{T_0}(\varphi)$ for some function $\varphi \in \mathcal{D}$.*

In order to prove this proposition, we apply the idea of [14].

Proof. Let θ be a solution of (13) with the initial data $\theta(\cdot, 0) = \theta_0 \in \mathcal{D}$. According to Proposition 1, we have

$$\max\{\theta(r,t) \mid r \in \bar{I}\} - \min\{\theta(r,t) \mid r \in \bar{I}\} \le 2LR \quad \text{for} \quad t > 0.$$

Set $\theta_k(r) := \theta(r,k) - 2\pi n_k$ and choose $n_k \in \mathbb{Z}$ satisfying

$$\theta_k(r) \in [0, 2LR + 2\pi].$$

Note that $\theta_k \in \mathcal{D}$. Let $s \in (0,1)$ be fixed. Since $\{\Phi_s(\theta_k)\}_{k=1}^{\infty}$ is relatively compact in $C^{1+\alpha}(\bar{I})$ for each $s \in [\varepsilon, 1)$ where $\varepsilon > 0$ is arbitrary, there exists a subsequence $\{\Phi_s(\theta_{k_j})\}_{j=1}^{\infty} \subset \{\Phi_s(\theta_k)\}_{k=1}^{\infty}$ and a function $\varphi \in C^{1+\alpha}(\bar{I})$ such that

$$\|\Phi_s(\theta_{k_j}) - \varphi\|_{C^{1+\alpha}(\bar{I})} \to 0 \quad \text{as} \quad j \to \infty.$$

Note that also $\varphi \in \mathcal{D}$ because a constant $L > 0$ is independent of time t.

We first show that $\varphi + 2\pi \le \Phi_T(\varphi)$ for a $T > 0$. Lemma 1(i) implies that $\theta_0 + 2\pi < \theta(\cdot, T)$ for a $T > 0$. Using Lemma 1(ii) and (19), $\Phi_{s+k_j}(\theta_0) + 2\pi < \Phi_{s+k_j+T}(\theta_0)$. Adding $-2\pi n_{k_j}$ to the both side of this inequality and recalling $\Phi_{s+k_j}(\theta_0) - 2\pi n_{k_j} = \Phi_s(\theta_{k_j})$, we have

$$\Phi_s(\theta_{k_j}) + 2\pi < \Phi_T(\Phi_s(\theta_{k_j})).$$

Letting $j \uparrow \infty$, we see $\varphi + 2\pi \le \Phi_T(\varphi)$.

Now we define

$$T_0 := \inf\{t \ge 0 \mid \varphi + 2\pi \le \Phi_t(\varphi)\}.$$

It is the completely same argument as in [14, Section 3] to prove that $\varphi + 2\pi = \Phi_{T_0}(\varphi)$ and $T_0 > 0$ is unique. Thus we omit their proof. □

Proposition 2 implies the following theorem.

Theorem 1. *(Existence of a periodic-in-time solution). There exists a periodic-in-time solution $\hat{\theta}$ of (13) satisfying $\hat{\theta}(\cdot, t) \in \mathcal{D}$ for $t \in \mathbb{R}$, which is unique up to translation of*

Remark 4. When (2) is isotropic, the equation for θ is denoted by

$$\begin{cases} \theta_t = \dfrac{D_0(r\theta_{rr} + r^2\theta_r^3 + 2\theta_r)}{r(1+r^2\theta_r^2)} + \dfrac{V_0(1+r^2\theta_r^2)^{1/2}}{r}, \\ \theta_r(\rho,t) = \theta_r(R,t) = 0. \end{cases} \tag{21}$$

According to [14], the periodic-in-time solution of (21) is represented precisely as

$$\hat{\theta}(r,t) = \xi(r) + \omega t$$

for some function $\xi(r)$ and some constant ω. Indeed, setting $h(r) := r\xi'(r)$, the problem is reduced to solve the following ordinary differential equation:

$$h' = f(r,h;\omega), \quad h(\rho) = h(R) = 0,$$

where

$$f(r,h;\omega) = \frac{1}{D_0}(1+h^2)\left\{-V_0(1+h^2)^{1/2} - D_0\frac{h}{r} + r\omega\right\}.$$

On the other hand, when the equation is anisotropic, to find the periodic-in-time solution can not be reduced to an ODE argument due to the functions $M(\boldsymbol{n})$ and $a(\boldsymbol{n})$. □

Proof of Theorem 1. Choose $\varphi \in \mathcal{D}$, which satisfies $\varphi + 2\pi = \Phi_{T_0}(\varphi)$, as the initial data. Then, we can obtain a solution $\hat{\theta}$ of (13) with $\hat{\theta}(\cdot,0) = \varphi$ and $\hat{\theta}(\cdot,t) \in \mathcal{D}$ for $t \geq 0$. This solution fulfills

$$\hat{\theta}(\cdot,t) + 2\pi = \Phi_t(\varphi) + 2\pi = \Phi_t(\varphi + 2\pi) = \Phi_t(\Phi_{T_0}(\varphi)) = \Phi_{t+T_0}(\varphi) = \hat{\theta}(\cdot, t+T_0).$$

That is, $\hat{\theta}$ is a periodic solution of (13) for $t \geq 0$. Using the uniqueness of the solution of (13) and the periodicity of the map Φ_t, we can extend this periodic solution $\hat{\theta}$ to $t < 0$. Note that Φ_t is also order-preserving for $t < 0$.

Finally we prove that this periodic solution is unique up to translation to the t-direction for $t \in \mathbb{R}$. The argument is essentially similar to that of [14, Section 3]. However, since the argument in [14] is based on an abstract theory, we reproduce their idea directly without appealing the abstract theory. Assume that $\hat{\theta}_i(\cdot,t) \in \mathcal{D}$ $(i = 1,2)$ are the periodic solutions of (13) with the period $T_0 > 0$. We can take $\hat{\theta}_1(r,-kT_0) \leq \hat{\theta}_2(r,0)$ for sufficiently large $k \in \mathbb{N}$. Rewrite $\hat{\theta}_1(r,t-kT_0)$ as $\hat{\theta}_1(r,t)$. Then $\hat{\theta}_1(r,0) \leq \hat{\theta}_2(r,0)$. The order-preserving property implies $\hat{\theta}_1(r,t) \leq \hat{\theta}_2(r,t)$ for $r \in \bar{I}$ and $t \in \mathbb{R}$. We may consider the attainable time t in the time-interval $[0,T_0]$ instead of $\mathbb{R}$ by the periodicity. Set

$$s_0 := \sup\{s \geq 0 \mid \hat{\theta}_1(r,t+s) \leq \hat{\theta}_2(r,t), \ r \in \bar{I}, \ t \in [0,T_0]\}.$$

Clearly $\hat{\theta}_1(r,t+s_0) \leq \hat{\theta}_2(r,t)$ and it follows from the compactness of $\bar{I} \times [0,T_0]$ that there exists $(r_0,t_0) \in I \times [0,T_0]$ satisfying $\hat{\theta}_1(r_0,t_0+s_0) = \hat{\theta}_2(r_0,t_0)$. Applying the strong maximum principle, we have $\hat{\theta}_1(r,t+s_0) \equiv \hat{\theta}_2(r,t)$ for all $r \in \bar{I}$ and $t \leq t_0$. Using the weak maximum principle as the initial time t_0, we see $\hat{\theta}_1(r,t+s_0) \equiv \hat{\theta}_2(r,t)$ for all $r \in \bar{I}$ and $t \geq t_0$. Thus, we obtain the desired result. □

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Consequently, by virtue of Theorem 1, we can obtain the spiral solution of (2) of the form (4), which completes the proof of Main Theorem.

Remark 5. *(Stability of the spiral solution).* We can derive the stability of the spiral solution given by Main Theorem. That is, for any $\varepsilon_0 > 0$ there exists a $\delta_0 > 0$ such that if $d(\Gamma(0), \hat{\Gamma}(0)) < \delta_0$, then $d(\Gamma(t), \hat{\Gamma}(t)) < \varepsilon_0$ for all $t > 0$. Indeed, by means of applying the similar argument as in [14, Section 3], we deduce that for any $\varepsilon_0 > 0$ there exists a $\delta_0 > 0$ such that $\|\theta(\cdot,t)-\hat{\theta}(\cdot,t)\|_{C(\bar{I})} < \varepsilon_0$ for all $t > 0$ whenever $\|\theta(\cdot,0)-\hat{\theta}(\cdot,0)\|_{C(\bar{I})} < \delta_0$. Since

$$d(\Gamma(t), \hat{\Gamma}(t)) \leq C\|\theta(\cdot,t) - \hat{\theta}(\cdot,t)\|_{C(\bar{I})} \quad \text{for all} \quad t \geq 0$$

where C is a positive constant independent of t, it follows that the spiral solution given by Main Theorem is stable. □

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