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Uniqueness of Renormalized Solutions for Nonlinear Degenerate Problems

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$, and let $T > 0$. When $N \geq 2$ we assume that Ω has a Lipschitz boundary $\partial\Omega$. We consider the initial-boundary value problem

$$
\text{(E)} \quad \begin{cases} \dfrac{\partial g(u)}{\partial t} - \triangle b(u) + \operatorname{div} \phi(u) = f & \text{in} \quad Q = (0,T) \times \Omega, \\ b(u) = 0 & \text{on} \quad \Sigma = (0,T) \times \partial\Omega, \\ g(u)(0,\cdot) = g(u_0) & \text{in} \quad \Omega, \end{cases}
$$

where

(H1) $g, b : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing functions satisfying the normalization conditions $g(0) = b(0) = 0$, and $\phi : \mathbb{R} \to \mathbb{R}^N$ is a continuous N-dimensional vector-valued function satisfying $\phi(0) = 0$.

(H2) $f \in L^1(Q)$ and $u_0 : \Omega \to \overline{\mathbb{R}}$ is measurable with $g(u_0) \in L^1(\Omega)$, where $\overline{\mathbb{R}} = [-\infty, \infty]$.

(H3) For any measurable functions u, $v : Q \to \mathbb{R}$

$$
\begin{aligned} &((\nabla b(u) - \phi(u)) - (\nabla b(v) - \phi(v)))\cdot(\nabla u - \nabla v) \\ &\quad + C(u,v)(1 + |\nabla b(u) - \phi(u)|^2 + |\nabla b(v) - \phi(v)|^2)|u - v| \;\geq\; 0, \end{aligned}
$$

where $C : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is continuous.

Many authors have considered the problems like (E) as well as the stationary problems under various assumptions on the vector field and have introduced several different notions of solutions for these problems in order to prove existence and uniqueness of such solutions, see [1]-[3], [6], [10] and [14], for example.

Due to the possible degeneracy of b and g, in general, we are not able to expect that solution in the sense of distribution for (E) is unique. We thus consider the problem (E) adopting the notion of renormalized solutions. The notion of renormalized solutions was introduced by DiPerna and Lions in their papers [8] and [9] dealing with existence of a solution for the Boltzmann equation. We can also treat the problem in the case of large data in a sense by utilizing this theory. In this report we shall prove uniqueness and a comparison result of renormalized solutions for the problem (E) with no growth condition applying the method of doubling variables both in space and time introduced by Kruzhkov [12]. As to some studies of renormalized solutions, see [4], [5], [7], [11], [13], [15] and [16], for example.

We shall mention the notations and definitions. For $k > 0$ we define a truncate function T_k by

$$T_k(u) = \begin{cases} k & \text{if } u > k \\ u & \text{if } |u| \le k \\ -k & \text{if } u < -k \end{cases}$$

as usual. We introduce the following functions

$$S(r) = \begin{cases} 1 & \text{if } r > 0 \\ [0,1] & \text{if } r = 0 \\ 0 & \text{if } r < 0 \end{cases}$$

and

$$S_0(r) = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{if } r \le 0 \end{cases},$$

and also define nonnegative functions r^+ and r^- by $r^+ = \max(r,0)$ and $r^- = -\min(r,0)$, respectively.

We now define a renormalized solution as in [7].

Definition 1.1. *A renormalized solution of* (E) *is a measurable function* $u : Q \to \mathbb{R}$ *satisfying*

(R1) $g(u) \in L^1(Q)$,

(R2) $T_k(u) \in L^2(0,T;H_0^1(\Omega))$ *for any* $k > 0$,

(R3) $b(T_k(u)) \in L^2(0,T;H_0^1(\Omega))$ *for any* $k > 0$,

(R4) $\phi(T_k(u)) \in L^2(Q)^N$ *for any* $k > 0$, (R5) *for all* $h \in C_0^1(\mathbb{R})$ *and* $\xi \in C_0^\infty([0,T) \times \Omega)$,

$$\int_Q \xi_t \int_{u_0}^{u} h(r)dg(r)\,dxdt + \int_Q \xi f h(u)\,dxdt = \int_Q (\nabla b(u) - \phi(u))\cdot\nabla(h(u)\xi)\,dxdt, \tag{1.1}$$

moreover,

$$\int_{Q\cap\{n\le|u|\le n+1\}} \nabla b(u)\cdot\nabla u\,dxdt \to 0 \quad \text{as } n \to \infty. \tag{1.2}$$

Remark 1.2. *Note that each integral in* (1.1) *and* (1.2) *is well-defined. In fact, the right-hand side of* (1.1) *is identified with*

$$\int_{Q\cap\{|u|<k\}} (\nabla b(T_k(u)) - \phi(T_k(u)))\cdot\nabla(h(T_k(u))\xi)\,dxdt$$

for $k > 0$ *such that* $\operatorname{supp} h \subset (-k,k)$. *Similarly, the integral in* (1.2) *has to be understood as*

$$\int_{Q\cap\{n\le|u|\le n+1\}} \nabla b(T_{n+1}(u))\cdot\nabla T_{n+1}(u)\,dxdt.$$

2 Main theorem

We obtain the following comparison result.

Theorem 2.1. *Suppose that* (H1) *and* (H3) *hold. Let* $u_{0i} : \Omega \to \overline{\mathbb{R}}$ *be measurable with* $g(u_{0i}) \in L^1(\Omega)$, $f_i \in L^1(Q)$ *and let* u_i *be a renormalized solution of* (E_i) *for* $i = 1, 2$, *where*

$$(\mathrm{E}_i)\quad \begin{cases} \dfrac{\partial g(u_i)}{\partial t} - \Delta b(u_i) + \operatorname{div}\phi(u_i) = f_i & \text{in } Q = (0,T)\times\Omega, \\ b(u_i) = 0 & \text{on } \Sigma = (0,T)\times\partial\Omega, \\ g(u_i)(0,\cdot) = g(u_{0i}) & \text{in } \Omega. \end{cases}$$

Then there exists $\kappa \in S(u_1 - u_2)$ *such that for a.e.* $\tau \in (0,T)$,

$$\int_\Omega (g(u_1)(\tau,x) - g(u_2)(\tau,x))^+\,dx \le \int_\Omega (g(u_{01})(x) - g(u_{02})(x))^+\,dx + \int_0^\tau\int_\Omega \kappa(f_1(t,x) - f_2(t,x))\,dxdt. \tag{2.1}$$

Moreover, for any u_0 *satisfying* (H2) *there exists a unique solution for* (E).

In order to prove this theorem, we start with the following lemma.

Lemma 2.2. *Let u be a renormalized solution of* (E). *Then*

$$\int_Q S_0(u-k)\Big((h(u)(\nabla b(u)-\phi(u))+h(k)\phi(k))\cdot\nabla\xi$$
$$-\xi f h(u)-\xi_t\int_k^u h(r)dg(r)+\xi h'(u)(\nabla b(u)-\phi(u))\cdot\nabla u\Big)\,dxdt$$
$$\leq \int_\Omega \xi(0,x)S_0(u_0-k)\int_k^{u_0} h(r)dg(r)\,dx \tag{2.2}$$

and

$$\int_Q S_0(-k-u)\Big((h(u)(\nabla b(u)-\phi(u))+h(-k)\phi(-k))\cdot\nabla\xi$$
$$-\xi f h(u)-\xi_t\int_{-k}^u h(r)dg(r)+\xi h'(u)(\nabla b(u)-\phi(u))\cdot\nabla u\Big)\,dxdt$$
$$\geq \int_\Omega \xi(0,x)S_0(-k-u_0)\int_{-k}^{u_0} h(r)dg(r)\,dx \tag{2.3}$$

for any $h\in C_0^1(\mathbb{R})^+$ and for any pair (k,ξ) satisfying

$$(k,\xi)\in\mathbb{R}\times C_0^\infty([0,T)\times\Omega)^+ \quad\text{or}\quad (k,\xi)\in\mathbb{R}^+\times C_0^\infty([0,T)\times\overline{\Omega})^+, \tag{2.4}$$

where $\mathbb{R}^+=[0,\infty)$ and X^+ denotes all nonnegative functions which belong to X with $X=C_0^1(\mathbb{R})$, $C_0^\infty([0,T)\times\Omega)$ or $C_0^\infty([0,T)\times\overline{\Omega})$.

Remark 2.3. *Note that if u is a renormalized solution of* (E), *then $-u$ is a renormalized solution of the problem associated with the equation $\widetilde{g}(v)_t-\Delta\widetilde{b}(v)+\operatorname{div}\widetilde{\phi}(v)=\widetilde{f}$, where $\widetilde{g}(r)=-g(-r)$, $\widetilde{b}(r)=-b(-r)$, $\widetilde{\phi}(r)=-\phi(-r)$, $\widetilde{f}=-f$ and initial data $\widetilde{u_0}=-u_0$.*

Sketch of the proof of Lemma 2.2. Due to Remark 2.3 it is sufficient to show (2.2). Let $h\in C_0^1(\mathbb{R})^+$. For $\varepsilon>0$ let $N_\varepsilon\in W^{1,\infty}(\mathbb{R})$ be defined by $N_\varepsilon(r)=\inf(r^+/\varepsilon,1)$. For $\varepsilon>0$ we see that $N_\varepsilon(u-k)\xi\in L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)$ for any pair (k,ξ) satisfying (2.4). Since u is a renormalized solution we find

$$\begin{aligned}
G_h(u) &:= \int_0^u h(r)dg(r) \in L^1(Q),\\
\frac{\partial G_h(u)}{\partial t} &\in L^2(0,T;H^{-1}(\Omega))+L^1(Q)\\
\text{and}\quad G_h(u)(0,\cdot) &= \int_0^{u_0} h(r)dg(r) \in H^{-1}(\Omega)+L^1(\Omega).
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
-\int_0^T \langle G_h(u)_t, N_\varepsilon(u-k)\xi\rangle \, dt
&= \int_Q \xi_t \int_{u_0}^{u} N_\varepsilon(r-k)dG_h(r)\, dxdt \\
&= \int_Q \xi_t \int_{u_0}^{u} N_\varepsilon(r-k)h(r)dg(r)\, dxdt.
\end{aligned}$$

Letting $\varepsilon \to 0$ on the right we obtain

$$\begin{aligned}
\int_Q \xi_t \int_{u_0}^{u} S_0(r-k)h(r)dg(r)\, dxdt
&= \int_\Omega \xi(0,x)S_0(u_0-k)\int_k^{u_0} h(r)dg(r)\, dx \\
&\quad + \int_Q \xi_t S_0(u-k)\int_k^{u} h(r)dg(r)\, dxdt.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
-\int_0^T \langle G_h(u)_t, N_\varepsilon(u-k)\xi\rangle \, dt
&= \int_Q (N_\varepsilon(u-k)\xi)_t \int_{u_0}^{u} h(r)dg(r)\, dxdt \\
&= -\int_Q fh(u)N_\varepsilon(u-k)\xi\, dxdt \\
&\quad + \int_Q (\nabla b(u)-\phi(u))\cdot\nabla(h(u)N_\varepsilon(u-k)\xi)\, dxdt
\end{aligned}$$

and since $fh(u)N_\varepsilon(u-k)\xi \in L^1(Q)$ from the Lebesgue convergence theorem it follows that

$$\lim_{\varepsilon\to 0}\left(-\int_Q fh(u)N_\varepsilon(u-k)\xi\, dxdt\right) = -\int_Q fh(u)S_0(u-k)\xi\, dxdt.$$

As to the second integral we find

$$\begin{aligned}
&\int_Q (\nabla b(u)-\phi(u))\cdot\nabla(h(u)N_\varepsilon(u-k)\xi)\, dxdt \\
&= \int_Q N_\varepsilon(u-k)(\xi h'(u)(\nabla b(u)-\phi(u))\cdot\nabla u + h(u)(\nabla b(u)-\phi(u))\cdot\nabla\xi)\, dxdt \\
&\quad + \frac{1}{\varepsilon}\int_{Q\cap\{0<u-k<\varepsilon\}} \xi h(u)(\nabla b(u)-\phi(u))\cdot\nabla u\, dxdt \\
&\to \int_Q S_0(u-k)(\xi h'(u)(\nabla b(u)-\phi(u))\cdot\nabla u + h(u)(\nabla b(u)-\phi(u))\cdot\nabla\xi)\, dxdt \\
&\quad + \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{Q\cap\{0<u-k<\varepsilon\}} \xi h(u)(\nabla b(u)-\phi(u))\cdot\nabla u\, dxdt \qquad \text{as } \varepsilon\to 0.
\end{aligned}$$

Due to the divergence theorem we obtain

$$\begin{aligned}
0 &= \int_Q \operatorname{div}\left(\xi \int_0^{N_\varepsilon(u-k)} h(\varepsilon r + k)(\nabla b(\varepsilon r + k) - \phi(\varepsilon r + k))\, dr \right) dxdt \\
&= \int_Q \int_0^{N_\varepsilon(u-k)} h(\varepsilon r + k)(\nabla b(\varepsilon r + k) - \phi(\varepsilon r + k))\cdot \nabla \xi \, drdxdt \\
&\quad + \frac{1}{\varepsilon} \int_{Q\cap\{0<u-k<\varepsilon\}} \xi h(u)(\nabla b(u) - \phi(u))\cdot \nabla u \, dxdt
\end{aligned}$$

whenever the pair (k, ξ) satisfies (2.4), hence

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{Q\cap\{0<u-k<\varepsilon\}} \xi h(u)(\nabla b(u) - \phi(u))\cdot \nabla u \, dxdt \\
&\qquad \geq \int_Q S_0(u-k) h(k)\phi(k)\cdot \nabla \xi.
\end{aligned}$$

Combining these estimates above we finally obtain (2.2). □

We next prove the following renormalized Kato inequality.

Lemma 2.4. *Let $u_{0i} : \Omega \to \overline{\mathbb{R}}$ be measurable with $g(u_{0i}) \in L^1(\Omega)$, $f_i \in L^1(Q)$ and let u_i be a renormalized solution of* (E_i) *for $i = 1, 2$. Then there exists $\kappa \in S(u_1 - u_2)$ such that for a.e. $t \in (0, T)$,*

$$\begin{aligned}
&- \int_{Q\cap\{u_1>u_2\}} \xi_t \int_{u_2}^{u_1} h(r) dg(r)\, dxdt \\
&\quad - \int_{\Omega\cap\{u_{01}>u_{02}\}} \xi(0,x) \int_{u_{02}}^{u_{01}} h(r) dg(r)\, dx \\
&\quad + \int_{Q\cap\{u_1>u_2\}} (h(u_1)(\nabla b(u_1) - \phi(u_1)) \\
&\qquad\qquad - h(u_2)(\nabla b(u_2) - \phi(u_2)))\cdot \nabla \xi \, dxdt \\
&\quad + \int_{Q\cap\{u_1>u_2\}} \xi (h'(u_1)(\nabla b(u_1) - \phi(u_1))\cdot \nabla u_1 \\
&\qquad\qquad - h'(u_2)(\nabla b(u_2) - \phi(u_2))\cdot \nabla u_2)\, dxdt \\
&\leq \int_Q \xi \kappa (f_1 h(u_1) - f_2 h(u_2))\, dxdt \qquad (2.5)
\end{aligned}$$

for all $h \in C_0^1(\mathbb{R})^+$ and all $\xi \in C_0^\infty([0,T) \times \overline{\Omega})^+$.

Sketch of the proof of Lemma 2.4. We adopt the method of doubling variables introduced by Kruzhkov. Thus we choose two different pairs of variables (s, y) and (t, x) in Q and consider u_1, f_1 as functions in (s, y), u_2, f_2 in (t, x). Let $\xi \in C_0^\infty([0,T) \times \mathbb{R}^N)^+$ be such that

$$\operatorname{supp} \xi \cap ([0,T) \times \mathbb{R}^N) \subset ([0,T) \times B)$$

where B is a ball for which

$$\text{either}\quad B\cap\partial\Omega=\emptyset\quad\text{or}\quad B\subset\subset B'\quad\text{and}$$
$$B'\cap\partial\Omega\quad\text{is a part of the graph of a Lipschitz continuous function.}\tag{2.6}$$

Then there exists a sequence of mollifiers σ_l defined in $\mathbb{R}$ with $\operatorname{supp}\sigma_l\subset(-2/l,0)$ and there exists a sequence of mollifiers ρ_n in $\mathbb{R}^N$ such that $x\mapsto\rho_n(x-y)\in C_0^\infty(\Omega)$ for any $y\in B\cap\Omega$,

$$\mu_n(x)=\int_\Omega\rho_n(x-y)dy$$

is an increasing sequence for any $x\in B$ and $\mu_n(x)=1$ for any $x\in B$ with $d(x,\mathbb{R}^N\setminus\Omega)>c/n$, where c is a positive constant depending on B. Further, for sufficiently large l and n, the function $\xi^{(l,n)}$ defined by

$$\xi^{(l,n)}(t,x,s,y)=\xi(t,x)\rho_n(x-y)\sigma_l(t-s)$$

satisfies

$$\begin{aligned}(s,y)&\mapsto\xi^{(l,n)}(t,x,s,y)\in C_0^\infty([0,T)\times\overline{\Omega})\qquad\text{for any }(t,x)\in Q,\\(t,x)&\mapsto\xi^{(l,n)}(t,x,s,y)\in C_0^\infty([0,T)\times\Omega)\qquad\text{for any }(s,y)\in Q,\end{aligned}$$

and the function $\xi^{(n)}$ defined by

$$\xi^{(n)}\;=\;\int_Q\xi^{(l,n)}(t,x,s,y)\,dyds\;=\;\xi\mu_n$$

satisfies

$$\xi^{(n)}\in C_0^\infty([0,T)\times\Omega),\qquad 0\le\xi^{(m)}\le\xi^{(n)}\le\xi\qquad\text{for any }m\le n.$$

We thus apply Lemma 2.2 with $u=u_1$, $k=0$, $f=f_1$, $\xi=\xi^{(l,n)}(t,x,\cdot)$ and $h(\cdot)N_\varepsilon(\cdot-u_2^+)$ in the place of h, and we have

$$\begin{aligned}&\int_Q(\xi^{(l,n)})_s\int_{u_2^+}^{u_1^+}h(r)N_\varepsilon(r-u_2^+)dg(r)\,dyds\\&\qquad+\int_\Omega\xi^{(l,n)}(t,x,0,y)\int_{u_2^+}^{u_{01}^+}h(r)N_\varepsilon(r-u_2^+)dg(r)\,dy\\&\qquad+\int_Q f_1h(u_1)N_\varepsilon(u_1^+-u_2^+)\xi^{(l,n)}\,dyds\\&\ge\int_Q(\nabla b(u_1)-\phi(u_1))\cdot\nabla_y(h(u_1)N_\varepsilon(u_1^+-u_2^+)\xi^{(l,n)})\,dyds\end{aligned}\tag{2.7}$$

and since u_2 is a renormalized solution of (E_2) we obtain from (1.1) that

$$\begin{aligned}
&\int_Q (\xi^{(l,n)})_t \int_{u_1^+}^{u_2} h(r)N_\varepsilon(u_1^+ - r^+)dg(r)\,dxdt \\
&\quad + \int_\Omega \xi^{(l,n)}(0,x,s,y) \int_{u_1^+}^{u_{02}} h(r)N_\varepsilon(u_1^+ - r^+)dg(r)\,dx \\
&\quad + \int_Q f_2 h(u_2)N_\varepsilon(u_1^+ - u_2^+)\xi^{(l,n)}\,dxdt \\
&= \int_Q (\nabla b(u_2) - \phi(u_2))\cdot\nabla_x(h(u_2)N_\varepsilon(u_1^+ - u_2^+)\xi^{(l,n)})\,dxdt. \qquad (2.8)
\end{aligned}$$

Integrating (2.7) in (t,x) and (2.8) in (s,y), respectively, over Q and taking their difference we obtain

$$\begin{aligned}
&\int_{Q\times Q}\Big((\xi^{(l,n)})_s \int_{u_2^+}^{u_1^+} h(r)N_\varepsilon(r - u_2^+)dg(r) \\
&\qquad\qquad - (\xi^{(l,n)})_t \int_{u_1^+}^{u_2} h(r)N_\varepsilon(u_1^+ - r^+)dg(r)\Big)\,dydsdxdt \\
&+\Big(\int_{Q\times\Omega} \xi^{(l,n)}(t,x,0,y) \int_{u_2^+}^{u_{01}^+} h(r)N_\varepsilon(r - u_2^+)dg(r)\,dydxdt \\
&\qquad - \int_{\Omega\times Q} \xi^{(l,n)}(0,x,s,y) \int_{u_1^+}^{u_{02}} h(r)N_\varepsilon(u_1^+ - r^+)dg(r)\,dydsdx\Big) \\
&+ \int_{Q\times Q} (f_1 h(u_1) - f_2 h(u_2))N_\varepsilon(u_1^+ - u_2^+)\xi^{(l,n)}\,dydsdxdt \\
\geq\ & \int_{Q\times Q}\Big((\nabla b(u_1) - \phi(u_1))\cdot\nabla_y(h(u_1)N_\varepsilon(u_1^+ - u_2^+)\xi^{(l,n)}) \\
&\qquad - (\nabla b(u_2) - \phi(u_2))\cdot\nabla_x(h(u_2)N_\varepsilon(u_1^+ - u_2^+)\xi^{(l,n)})\Big)\,dydsdxdt.
\end{aligned}$$

We shall denote the three integrals on the left by J_1, J_2 and J_3, the integral on the right by J_4, respectively. We begin with the first term J_1.

$$\begin{aligned}
\lim_{\varepsilon\to 0} J_1 &= \int_{Q\times Q}\Big((\xi^{(l,n)})_s S_0(u_1^+ - u_2^+) \int_{u_2^+}^{u_1^+} h(r)dg(r) \\
&\qquad\qquad - (\xi^{(l,n)})_t S_0(u_1^+ - u_2^+) \int_{u_1^+}^{u_2} h(r)dg(r)\Big)\,dydsdxdt \\
&= \int_{Q\times Q} \xi_t \rho_n \sigma_l S_0(u_1^+ - u_2^+) \int_{u_2^+}^{u_1^+} h(r)dg(r)\,dydsdxdt \\
&\quad - \int_{Q\times Q\cap\{u_1>0\}\cap\{u_2<0\}} (\xi^{(l,n)})_t \int_0^{u_2} h(r)dg(r)\,dydsdxdt. \qquad (2.9)
\end{aligned}$$

As to J_2 we see from $\operatorname{supp}\sigma_l \subset (-2/l, 0)$ that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} J_2 \\
&= \int_{\Omega\times(0,2/l)\times\Omega} \xi^{(l,n)}(0,x,s,y) S_0(u_1^+ - u_{02}^+) \int_{u_{02}}^{u_1^+} h(r)dg(r)\, dydsdx \\
&= \int_{\Omega\times(0,2/l)\times\Omega} \xi^{(l,n)}(0,x,s,y) S_0(u_1^+ - u_{02}^+) \int_{u_{02}^+}^{u_1^+} h(r)dg(r)\, dydsdx \\
&\quad + \int_{\Omega\times(0,2/l)\times\Omega\cap\{u_1>0\}\cap\{u_{02}<0\}} \xi^{(l,n)}(0,x,s,y) \int_{u_{02}}^{0} h(r)dg(r)\, dydsdx. \qquad (2.10)
\end{aligned}$$

In the third term we deduce that

$$\lim_{\varepsilon\to 0} J_3 = \int_{Q\times Q\cap\{u_1^+>u_2^+\}} (f_1 h(u_1) - f_2 h(u_2))\xi^{(l,n)}\, dydsdxdt. \qquad (2.11)$$

It remainds to consider J_4. In terms of the divergence theorem we have

$$\begin{aligned}
J_4 &= \int_{Q\times Q} \xi^{(l,n)} N_\varepsilon(u_1 - u_2^+) h'(u_1)(\nabla b(u_1) - \phi(u_1))\cdot\nabla u_1\, dydsdxdt \\
&\quad + \int_{Q\times Q} h(u_1)(\nabla b(u_1) - \phi(u_1))\cdot\nabla_y (N_\varepsilon(u_1 - u_2^+)\xi^{(l,n)})\, dydsdxdt \\
&\quad - \int_{Q\times Q} (\nabla b(u_2^+) - \phi(u_2^+))\cdot\nabla_x (h(u_2^+) N_\varepsilon(u_1^+ - u_2^+)\xi^{(l,n)})\, dydsdxdt \\
&\quad + \int_{Q\times Q\cap\{u_2<0\}} (\nabla b(u_2^+) - \phi(u_2^+))\cdot\nabla_x (h(u_2^+) N_\varepsilon(u_1^+ - u_2^+)\xi^{(l,n)})\, dydsdxdt \\
&\quad - \int_{Q\times Q\cap\{u_2<0\}} (\nabla b(u_2) - \phi(u_2))\cdot\nabla_x (h(u_2) N_\varepsilon(u_1^+ - u_2)\xi^{(l,n)})\, dydsdxdt \\
&= \int_{Q\times Q} \xi^{(l,n)} N_\varepsilon(u_1 - u_2^+)(h'(u_1)(\nabla b(u_1) - \phi(u_1))\cdot\nabla u_1 \\
&\qquad\qquad - h'(u_2^+)(\nabla b(u_2^+) - \phi(u_2^+))\cdot\nabla u_2^+)\, dydsdxdt \\
&\quad + \int_{Q\times Q} (h(u_1)(\nabla b(u_1) - \phi(u_1)) - h(u_2^+)(\nabla b(u_2^+) - \phi(u_2^+))) \\
&\qquad\qquad \cdot(\nabla_x + \nabla_y)(N_\varepsilon(u_1 - u_2^+)\xi^{(l,n)})\, dydsdxdt \\
&\quad - \int_{Q\times Q\cap\{u_2<0\}} N_\varepsilon(u_1^+)(\nabla b(u_2) - \phi(u_2))\cdot\nabla_x (h(u_2)\xi^{(l,n)})\, dydsdxdt
\end{aligned}$$

Then we find

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_{Q\times Q} (h(u_1)(\nabla b(u_1) - \phi(u_1)) - h(u_2^+)(\nabla b(u_2^+) - \phi(u_2^+))) \\
&\qquad\qquad \cdot(\nabla_x + \nabla_y)(N_\varepsilon(u_1 - u_2^+)\xi^{(l,n)})\, dydsdxdt \\
&\geq \int_{Q\times Q\cap\{u_1>u_2^+\}} \rho_n\sigma_l (h(u_1)(\nabla b(u_1) - \phi(u_1)) \\
&\qquad\qquad - h(u_2^+)(\nabla b(u_2^+) - \phi(u_2^+)))\cdot\nabla\xi dydsdxdt \qquad (2.12)
\end{aligned}$$

$$\liminf_{\varepsilon\to 0}\int_{Q\times Q} N_\varepsilon(u_1-u_2^+)\xi^{(l,n)}(h'(u_1)(\nabla b(u_1)-\phi(u_1))\cdot\nabla u_1$$
$$-h'(u_2)(\nabla b(u_2)-\phi(u_2))\cdot\nabla u_2)\,dydsdxdt$$
$$\geq \int_{Q\times Q\cap\{u_1>u_2^+\}} \xi^{(l,n)}(h'(u_1)(\nabla b(u_1)-\phi(u_1))\cdot\nabla u_1$$
$$-h'(u_2^+)(\nabla b(u_2^+)-\phi(u_2^+))\cdot\nabla u_2)dydsdxdt. \qquad (2.13)$$

As to the remaining term we obtain from Lemma 2.2 that

$$\int_{Q\cap\{u_2<0\}}\Big((\xi^{(l,n)})_t\int_{u_{02}}^{u_2} h(r)dg(r)+\xi^{(l,n)}f_2h(u_2)$$
$$-(\nabla b(u_2)-\phi(u_2))\cdot\nabla(h(u_2)\xi^{(l,n)})\Big)\,dxdt \;\leq\; 0.$$

Since $1-N_\varepsilon(u_1)\geq 0$, multiplying $(1-N_\varepsilon(u_1))$ to the previous inequality and integrating in (s,y) over Q we have

$$-\int_{Q\times Q\cap\{u_2<0\}} N_\varepsilon(u_1)(\nabla b(u_2)-\phi(u_2))\cdot\nabla_x(h(u_2)\xi^{(l,n)})\,dydsdxdt$$
$$\geq \int_{Q\times Q\cap\{u_2<0\}}\Big((\xi^{(l,n)})_t\int_{u_{02}}^{u_2} h(r)dg(r)+\xi^{(l,n)}f_2h(u_2)$$
$$-(\nabla b(u_2)-\phi(u_2))\cdot\nabla_x(h(u_2)\xi^{(l,n)})\Big)\,dydsdxdt$$
$$-\int_{Q\times Q\cap\{u_2<0\}} N_\varepsilon(u_1)(\xi^{(l,n)})_t\int_{u_{02}}^{u_2} h(r)dg(r)\,dydsdxdt$$
$$-\int_{Q\times Q\cap\{u_2<0\}} N_\varepsilon(u_1)\xi^{(l,n)}f_2h(u_2)\,dydsdxdt$$

$$\to \int_{Q\times Q\cap\{u_2<0\}}\Big((\xi^{(l,n)})_t\int_{u_{02}}^{u_2} h(r)dg(r)+\xi^{(l,n)}f_2h(u_2)$$
$$-(\nabla b(u_2)-\phi(u_2))\cdot\nabla_x(h(u_2)\xi^{(l,n)})\Big)\,dydsdxdt$$
$$-\int_{Q\times Q\cap\{u_1>0\}\cap\{u_2<0\}}(\xi^{(l,n)})_t\int_{u_{02}}^{u_2} h(r)dg(r)\,dydsdxdt$$
$$-\int_{Q\times Q\cap\{u_1>0\}\cap\{u_2<0\}}\xi^{(l,n)}f_2h(u_2)\,dydsdxdt \qquad \text{as } \varepsilon\to 0. \qquad (2.14)$$

Combining these estimates (2.9) - (2.14) we deduce that

$$\begin{aligned}
&\int_Q \xi_t S_0(u_1^+ - u_2^+) \int_{u_2^+}^{u_1^+} h(r)dg(r)\,dxdt \\
&\quad + \int_\Omega \xi(0,x) S_0(u_{01}^+ - u_{02}^+) \int_{u_{02}^+}^{u_{01}^+} h(r)dg(r)\,dx \\
&\quad + \int_Q \xi\kappa_+ S_0(u_1)(f_1 h(u_1) - (1 - S_0(-u_2)) f_2 h(u_2))\,dxdt \\
&\geq \int_{Q\cap\{u_1>u_2^+\}} (h(u_1)(\nabla b(u_1) - \phi(u_1)) - h(u_2^+)(\nabla b(u_2^+) - \phi(u_2^+)))\cdot\nabla\xi\,dxdt \\
&\quad + \int_{Q\cap\{u_1>u_2^+\}} \xi(h'(u_1)(\nabla b(u_1) - \phi(u_1))\cdot\nabla u_1 \\
&\qquad\qquad -h'(u_2^+)(\nabla b(u_2^+) - \phi(u_2^+))\cdot\nabla u_2)\,dxdt \\
&\quad + \lim_{n\to\infty} \int_{Q\cap\{u_2<0\}} \Big((\xi^{(n)})_t \int_{u_{02}}^{u_2} h(r)dg(r) + f_2 h(u_2)\xi^{(n)} \\
&\qquad\qquad -(\nabla b(u_2) - \phi(u_2))\cdot\nabla(h(u_2)\xi^{(n)}) \Big)\,dxdt
\end{aligned}$$

for any $\xi \in C_0^\infty([0,T) \times B)^+$, where $\kappa_+ \in S(u_1^+ - u_2^+)$. We also obtain from Remark 2.3 that there exists $\kappa_- \in S(u_2^- - u_1^-)$ such that

$$\begin{aligned}
&\int_Q \xi_t S_0(u_2^- - u_1^-) \int_{-u_2^-}^{-u_1^-} h(r)dg(r)\,dxdt \\
&\quad + \int_\Omega \xi(0,x) S_0(u_{02}^- - u_{01}^-) \int_{-u_{02}^-}^{-u_{01}^-} h(r)dg(r)\,dx \\
&\quad + \int_Q \xi\kappa_- S_0(u_2^-)((1 - S_0(u_1^+)) f_1 h(u_1) - f_2 h(u_2))\,dxdt \\
&\geq \int_{Q\cap\{u_2^->u_1^-\}} (h(u_1)(\nabla b(-u_1^-) - \phi(-u_1^-)) \\
&\qquad\qquad -h(u_2)(\nabla b(u_2) - \phi(u_2)))\cdot\nabla\xi\,dxdt \\
&\quad + \int_{Q\cap\{u_2^->u_1^-\}} \xi(h'(u_1)(\nabla b(-u_1^-) - \phi(-u_1^-))\cdot\nabla u_1 \\
&\qquad\qquad -h'(u_2)(\nabla b(u_2) - \phi(u_2))\cdot\nabla u_2)\,dxdt \\
&\quad - \lim_{n\to\infty} \int_{Q\cap\{u_1>0\}} \Big((\xi^{(n)})_t \int_{u_{01}}^{u_1} h(r)dg(r) + f_1 h(u_1)\xi^{(n)} \\
&\qquad\qquad -(\nabla b(u_1) - \phi(u_1))\cdot\nabla(h(u_1)\xi^{(n)}) \Big)\,dxdt
\end{aligned}$$

for any $\xi \in C_0^\infty([0,T) \times B)^+$. Since $\widetilde{\kappa} = (1 - S_0(u_1^+))S_0(-u_2^+)\kappa_- + S_0(u_1^+)\kappa_+ = (1 - S_0(-u_2))S_0(u_1)\kappa_+ + S_0(-u_2)\kappa_- \in S(u_1 - u_2)$, summing up the previous two inequalities we have

$$
\begin{aligned}
&\int_Q \xi_t S_0(u_1-u_2)\int_{u_2}^{u_1} h(r)dg(r)\,dxdt \\
&\quad + \int_\Omega \xi(0,x)S_0(u_{01}-u_{02})\int_{u_{02}}^{u_{01}} h(r)dg(r)\,dx \\
&\quad + \int_Q \xi\widetilde{\kappa}(f_1h(u_1)-f_2h(u_2))\,dxdt \\
&\quad - \int_{Q\cap\{u_1>u_2\}} (h(u_1)(\nabla b(u_1)-\phi(u_1)) - h(u_2)(\nabla b(u_2)-\phi(u_2)))\cdot\nabla\xi\,dxdt \\
&\quad - \int_{Q\cap\{u_1>u_2\}} \xi(h'(u_1)(\nabla b(u_1)-\phi(u_1))\cdot\nabla u_1 \\
&\qquad\qquad -h'(u_2)(\nabla b(u_2)-\phi(u_2))\cdot\nabla u_2)\,dxdt \\
\geq\ & \lim_{n\to\infty}\int_{Q\cap\{u_2<0\}}\Bigg((\xi^{(n)})_t\int_{u_{02}}^{u_2} h(r)dg(r) + \xi^{(n)} f_2 h(u_2) \\
&\qquad\qquad -(\nabla b(u_2)-\phi(u_2))\cdot\nabla(h(u_2)\xi^{(n)})\Bigg)\,dxdt \\
&\quad - \lim_{n\to\infty}\int_{Q\cap\{u_1>0\}}\Bigg((\xi^{(n)})_t\int_{u_{01}}^{u_1} h(r)dg(r) + \xi^{(n)} f_1 h(u_1) \\
&\qquad\qquad -(\nabla b(u_1)-\phi(u_1))\cdot\nabla(h(u_1)\xi^{(n)})\Bigg)\,dxdt \qquad (2.15)
\end{aligned}
$$

for any $\xi \in C_0^\infty([0,T)\times B)^+$.

Now let $\xi \in C_0^\infty([0,T)\times B)^+$. Then $\xi^{(m)} = \xi\mu_m \in C_0^\infty([0,T)\times\Omega)$ and we see that

$$
\begin{aligned}
&-\int_{Q\cap\{u_1>u_2\}} {\xi^{(m)}}_t \int_{u_2}^{u_1} h(r)dg(r)\,dxdt \\
&\quad -\int_{\Omega\cap\{u_{01}>u_{02}\}} \xi^{(m)}(0,x)\int_{u_{02}}^{u_{01}} h(r)dg(r)\,dx \\
&\quad +\int_{Q\cap\{u_1>u_2\}} (h(u_1)(\nabla b(u_1)-\phi(u_1)) \\
&\qquad\qquad - h(u_2)(\nabla b(u_2)-\phi(u_2)))\cdot\nabla\xi^{(m)}\,dxdt \\
&\quad +\int_{Q\cap\{u_1>u_2\}} \xi^{(m)}(h'(u_1)(\nabla b(u_1)-\phi(u_1))\cdot\nabla u_1 \\
&\qquad\qquad - h'(u_2)(\nabla b(u_2)-\phi(u_2))\cdot\nabla u_2)\,dxdt \\
\leq\ & \int_Q \xi^{(m)}\kappa(f_1h(u_1)-f_2h(u_2))\,dxdt.
\end{aligned}
$$

nce $\xi^{(m)} = \xi - \xi(1-\mu_m)$ we obtain from (2.15) that

$$
\begin{aligned}
&\int_{Q\cap\{u_1>u_2\}} \xi_t \int_{u_2}^{u_1} h(r)dg(r)\,dxdt \\
&\qquad + \int_{\Omega\cap\{u_{01}>u_{02}\}} \xi(0,x) \int_{u_{02}}^{u_{01}} h(r)dg(r)\,dx \\
&\qquad + \int_Q \xi\widetilde{\kappa}(f_1h(u_1) - f_2h(u_2))\,dxdt \\
&\qquad + \int_Q \xi(\kappa-\widetilde{\kappa})(f_1h(u_1) - f_2h(u_2))\,dxdt \\
&\qquad - \int_{Q\cap\{u_1>u_2\}} (h(u_1)(\nabla b(u_1) - \phi(u_1)) \\
&\qquad\qquad\qquad - h(u_2)(\nabla b(u_2) - \phi(u_2)))\cdot\nabla\xi\,dxdt \\
&\qquad - \int_{Q\cap\{u_1>u_2\}} \xi(h'(u_1)(\nabla b(u_1) - \phi(u_1))\cdot\nabla u_1 \\
&\qquad\qquad\qquad - h'(u_2)(\nabla b(u_2) - \phi(u_2))\cdot\nabla u_2)\,dxdt \\
&\geq \int_{Q\cap\{u_1>u_2\}} (\xi(1-\mu_m))_t \int_{u_2}^{u_1} h(r)dg(r)\,dxdt \\
&\qquad + \int_{\Omega\cap\{u_{01}>u_{02}\}} (\xi(1-\mu_m))(0,x) \int_{u_{02}}^{u_{01}} h(r)dg(r)\,dx \\
&\qquad + \int_Q (\xi(1-\mu_m))\widetilde{\kappa}(f_1h(u_1) - f_2h(u_2))\,dxdt \\
&\qquad + \int_Q (\xi(1-\mu_m))(\kappa-\widetilde{\kappa})(f_1h(u_1) - f_2h(u_2))\,dxdt \\
&\qquad - \int_{Q\cap\{u_1>u_2\}} (h(u_1)(\nabla b(u_1) - \phi(u_1)) \\
&\qquad\qquad\qquad - h(u_2)(\nabla b(u_2) - \phi(u_2)))\cdot\nabla(\xi(1-\mu_m))\,dxdt \\
&\quad - \int_{Q\cap\{u_1>u_2\}} (\xi(1-\mu_m))(h'(u_1)(\nabla b(u_1) - \phi(u_1))\cdot\nabla u_1 \\
&\qquad\qquad\qquad - h'(u_2)(\nabla b(u_2) - \phi(u_2))\cdot\nabla u_2)\,dxdt \\
&\geq \lim_{n\to\infty} \int_{Q\cap\{u_2<0\}} \Big((\xi(1-\mu_m)\mu_n)_t \int_{u_{02}}^{u_2} h(r)dg(r) + (\xi(1-\mu_m)\mu_n)f_2h(u_2) \\
&\qquad\qquad\qquad -(\nabla b(u_2) - \phi(u_2))\cdot\nabla(h(u_2)\xi(1-\mu_m)\mu_n)\Big)\,dxdt \\
&\quad - \lim_{n\to\infty} \int_{Q\cap\{u_1>0\}} \Big((\xi(1-\mu_m)\mu_n)_t \int_{u_{01}}^{u_1} h(r)dg(r) + (\xi(1-\mu_m)\mu_n)f_1h(u_1) \\
&\qquad\qquad\qquad -(\nabla b(u_1) - \phi(u_1))\cdot\nabla(h(u_1)\xi(1-\mu_m)\mu_n)\Big)\,dxdt \\
&\quad + \int_Q (\xi(1-\mu_m))(\kappa-\widetilde{\kappa})(f_1h(u_1) - f_2h(u_2))\,dxdt.
\end{aligned}
$$

In the last term on the right it is clear that the integral converges to 0 as $m \to \infty$. Since if n is large enough then $\mu_n = 1$ on $\operatorname{supp}\mu_m$ we find that $(1-\mu_m)\mu_n = \mu_n - \mu_m$. Therefore the remaining terms tend to 0 as $m \to \infty$. It implies that

$$\begin{aligned}
&\int_Q \xi_t S_0(u_1-u_2)\int_{u_2}^{u_1} h(r)dg(r)\,dxdt \\
&\quad+\int_\Omega \xi(0,x)S_0(u_{01}-u_{02})\int_{u_{02}}^{u_{01}} h(r)dg(r)\,dx \\
&\quad+\int_Q \xi\kappa(f_1h(u_1)-f_2h(u_2))\,dxdt \\
&\quad-\int_{Q\cap\{u_1>u_2\}} (h(u_1)(\nabla b(u_1)-\phi(u_1))-h(u_2)(\nabla b(u_2)-\phi(u_2)))\cdot\nabla\xi\,dxdt \\
&\quad-\int_{Q\cap\{u_1>u_2\}} \xi(h'(u_1)(\nabla b(u_1)-\phi(u_1))\cdot\nabla u_1 \\
&\qquad\qquad -h'(u_2)(\nabla b(u_2)-\phi(u_2))\cdot\nabla u_2)\,dxdt \;\geq\; 0,
\end{aligned}$$

with $\kappa \in S(u_1-u_2)$.

To this end, let $B_0 \subset\subset \Omega$ be such that $\cup_{i=0}^n B_i$ is a covering of Ω, where B_i is a ball satisfing (2.6) for $i = 1,\cdots,n$. Let $\{\nu_i\}_{i=0}^n$ be such that $\nu_i \in C_0^\infty(B_i)$ for $i = 0,1,\cdots,n$ and let $\xi \in C_0^\infty([0,T)\times\overline{\Omega})^+$. Then for $i = 0,1,\cdots,n$ we have

$$\begin{aligned}
&\int_Q (\xi\nu_i)_t S_0(u_1-u_2)\int_{u_2}^{u_1} h(r)dg(r)\,dxdt \\
&\quad+\int_\Omega (\xi\nu_i)(0,x)S_0(u_{01}-u_{02})\int_{u_{02}}^{u_{01}} h(r)dg(r)\,dx \\
&\quad+\int_Q (\xi\nu_i)\kappa(f_1h(u_1)-f_2h(u_2))\,dxdt \\
&\quad-\int_{Q\cap\{u_1>u_2\}} (h(u_1)(\nabla b(u_1)-\phi(u_1))-h(u_2)(\nabla b(u_2)-\phi(u_2)))\cdot\nabla(\xi\nu_i)\,dxdt \\
&\quad-\int_{Q\cap\{u_1>u_2\}} (\xi\nu_i)(h'(u_1)(\nabla b(u_1)-\phi(u_1))\cdot\nabla u_1 \\
&\qquad\qquad -h'(u_2)(\nabla b(u_2)-\phi(u_2))\cdot\nabla u_2)\,dxdt \;\geq\; 0.
\end{aligned}$$

Since $\xi = \sum_{i=0}^n(\xi\nu_i)$ we obtain (2.5) for any $\xi \in C_0^\infty([0,T)\times\overline{\Omega})^+$. □

3 Proof of the main theorem

We finally give the proof of our main result.

Proof of Theorem 2.1. Let u_i be a renormalized solution of (E_i) for $i = 1, 2$. Choosing $\xi = \alpha \otimes 1$ with $\alpha \in C_0^\infty([0,T))$ in (2.5) there exists $\kappa \in S(u_1 - u_2)$

$$\begin{aligned} &-\int_Q \alpha_t \Big(S_0(u_1-u_2) \int_{u_2}^{u_1} h(r)dg(r) - S_0(u_{01}-u_{02}) \int_{u_{02}}^{u_{01}} h(r)dg(r) \Big)\, dxdt \\ &\qquad + \int_{Q\cap\{u_1>u_2\}} \alpha(h'(u_1)(\nabla b(u_1) - \phi(u_1))\cdot\nabla u_1 \\ &\qquad\qquad\qquad -h'(u_2)(\nabla b(u_2) - \phi(u_2))\cdot \nabla u_2)\, dxdt \\ &\leq \int_Q \alpha\kappa(f_1 h(u_1) - f_2 h(u_2))\, dxdt \end{aligned} \tag{3.1}$$

for any $h \in W^{1,\infty}(\mathbb{R})$ with compact support. We now define the function $h_n \in W^{1,\infty}(\mathbb{R})$ by $h_n(r) = \inf((n+1-|r|)^+, 1)$ and replace h by h_n in (3.1). As to the second integral on the left we divide as

$$\begin{aligned} &\int_{Q\cap\{u_1>u_2\}} \alpha h_n{}'(u_1)\nabla b(u_1)\cdot\nabla u_1\, dxdt - \int_{Q\cap\{u_1>u_2\}} \alpha h_n{}'(u_2)\nabla b(u_2)\cdot\nabla u_2\, dxdt \\ &\quad - \int_{Q\cap\{u_1>u_2\}} \alpha(h_n{}'(u_1)\phi(u_1)\cdot\nabla u_1 - h_n{}'(u_2)\phi(u_2)\cdot\nabla u_2)\, dxdt. \end{aligned}$$

Since u_1, u_2 are renormalized solutions we see from (1.2) that the first two integrals on the right tend to 0 as $n \to \infty$. Moreover, thanks to the divergence theorem we have

$$\begin{aligned} &-\int_{Q\cap\{u_1>u_2\}} \alpha(h_n{}'(u_1)\phi(u_1)\cdot\nabla u_1 - h_n{}'(u_2)\phi(u_2)\cdot\nabla u_2)\, dxdt \\ &\qquad = \int_Q \alpha \operatorname{div}\Big(-\int_{\inf(u_1,u_2)}^{u_1} h_n{}'(r)\phi(r)dr \Big)\, dxdt \;=\; 0. \end{aligned}$$

Therefore the second integral on the left in (3.1) converges to 0 as $h = h_n \to 1$ and it implies that

$$\begin{aligned} &-\int_Q \alpha_t((g(u_1)(t,x) - g(u_2)(t,x))^+ - (g(u_{01})(x) - g(u_{02})(x))^+)\, dxdt \\ &\qquad \leq \int_Q \alpha\kappa(f_1(t,x) - f_2(t,x))\, dxdt \end{aligned}$$

for all $\alpha \in C_0^\infty([0,T))$. We thus conclude the proof of our main theorem. $\square$

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