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Total variation flow with value constraints

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1 Introduction

This is a continuation of our work [KG], [GGK], where we studied a gradient (flow) system of an energy whose energy density is not C^1 so that the diffusivity in the equation is very strong and its effect is even nonlocal. In this paper we consider the case when the values of unknowns are constrained. To be specific we consider a gradient (flow) system of the total variations of mappings with constraint of their values. Let us write the equation formally. For a mapping $u : \Omega \to \mathbf{R}^N$ with a domain Ω in $\mathbf{R}^n$ let $E_1(u)$ denote its total variation, *i.e.*,

$$E_1(u) = \int_\Omega |\nabla u| dx. \tag{1.1}$$

Let $\delta E_1/\delta u$ denote its 'first variation' (with respect to L^2 inner product). Then the unconstrained gradient system is formally written in the form

$$u_t = -\delta E_1/\delta u \tag{1.2}$$

for $u = u(x,t), x \in \Omega, t > 0$, where u_t denotes the time derivative, *i.e.*, $u_t = \partial u/\partial t$. If the values of u is constrained in some fixed (Riemannian) manifold M embedded in $\mathbf{R}^N$, the first variation $\delta E_1/\delta_M u$ with

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this constaint is of the form

$$\delta E_1/\delta_M u = P_u(\delta E_1/\delta u),$$

where P_u is the orthogonal projection to the tangent space of M at the value of u. Thus our constrained gradient system is of the form

$$u_t = -P_u(\delta E_1/\delta u). \tag{1.3}$$

The explicit form of (1.2) is

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{1.4}$$

If M is a unit sphere S^{N-1}, then the explicit form of (1.3) is

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u|u \tag{1.5}$$

as explained in Example 2 in Section 2. An explicit calculation for (1.3) is for example in [MSO]. Although the notion of solution of (1.4) is not a priorily clear because of singularity at $\nabla u = 0$, a general nonlinear semigroups theory (initiated by Y. Kōmura [Ko]) applies under appropriate boundary conditions since the energy is convex. The theory yields the unique global solvability of the initial value problem for (1.2) under the Dirichlet boundary condition; see e.g. [Ba] and also [KG], [GGK], [HZ], [ACM]. However, for (1.3) such a theory does not apply since it cannot be viewed as a gradient system of a convex functional. Even for smooth energy a constrained gradient system needs individual study for well-posedness. A typical example is the harmonic map flow equation. It is formally written in the form (1.3) where E_1 in (1.1) is replaced by the Dirichlet energy

$$E_2(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx.$$

Its initial value problem is well-studied, for example, in [ES], [St], [Cg], [Ch], [C], [CDY], [F]. The solution is independent of the way how M is embedded in $\mathbf{R}^N$. For the gradient system of the total variation (1.3) even the notion of solution is unclear because of singularity at $\nabla u = 0$.

In this paper, as a first attempt, we propose to formulate a constrained gradient system when the energy φ is convex but having singularities by using subdifferentials $\partial\varphi$. It is formally written as

$$u_t \in -P_u(\partial\varphi(u)).$$

The speed u_t looks undertermined. However, under some regularity condition of u we prove that the right derivative d^+u/dt is uniquely detemined. Like unconstrained problems it equals the minus of 'minimal section' of the convex set $P_u(\partial\varphi(u))$.

Unfortunately, even unique local solvability of the initial value problem for (1.3) is not clear. We restrict ourselves to consider a piecewise constant initial data in a one dimensional domain — an open interval. We calculate subdifferentials $\partial\varphi$ when φ is the total variation at a piecewise constant function. We further calculate the minimal section of $P_u(\partial\varphi(u))$ and construct a global solution for (1.3) with the Dirichlet condition by reducing the problem to a system of ordinary differential equations (ODEs). A key observation is that the minimal section is constant on each maximal spatial interval where the solution is constant so that the solution must stay as piecewise constant and the jump discontinuities are included in those of the initial data. This yields the uniqueness of a solution at least among piecewise constant functions. We say that each connected component of the graph of a piecewise constant function as a plateau.

We also study the behaviour of solution when M is the unit circle S^1. The equation of the motion of the plateau is presented, which is written in the form of *reducing* ODE. We identify the form of stationary solutions and prove that the solution becomes a stationary solution in finite time.

Unlike the harmonic map flow, the notion of solution depends also on the ambient space $\mathbf{R}^N$ not only on M itself. Moreover, there are several ways to define the notion of total variation for mappings to M. The corresponding gradient system may differ. The definition of the total variation in this paper is not intrinsic; it depends on distance of the ambient space $\mathbf{R}^N$. For S^1-valued problem one is tempting to define the total variation of $u = (\cos\theta, \sin\theta)$ by $\int_\Omega |\nabla\theta| dx$. However, this energy is also singular when the jump of argument is π, so the dynamics starting with such jumps cannot be determined uniquely. There are several dicussion to define the notion of mapping of bounded variation with valued in S^1. In [GMS] a class of mappings approximated by smooth S^1 mapping was characterized.

Although there are huge literature on quasilinear parabolic equations with singularity at $\nabla u = 0$, the singularity is weaker than ours in the sense that the diffusion effect is still local; see e.g. [D], [G]. There are several fields where equations with nonloal singular diffusivity are proposed. The first example stems from material sciences for describing motin of antiphase grain boundaries [Gu]. In fact, a crystalline curvature flow equation was proposed [AG], [T] as an example of anisotropic curvature flow equations [G], [Gu] with singular interfacial energy. When the interface is a curve given as the graph of a function, one of simplest examples is of the form (1.4) with $n = 1$ [FG]. The second example stems from image analysis. In [ROF] it was proposed to use gradient flow system of the total variation with L^2-constraint for a grey level function u to remove noises from images. The third example stems from plastisity problem [HZ]. The fourth example is derived from the phase field model of grain structure evolution which include grain boundary migrations and grain rotation [KWC],[WKC],[LW],[GBP]. The equation of orientation with singular diffusibity is coupled with the equation of ordering parameter. This model yields a mathematical subproblem with spatially non-uniform energy. We developed a mathematical theory which handles such a non-uniform equation with sigular diffusivity in [KG] and [GGK] together with the case of the uniform energy. By now well-posedness for unconstained gradient system (1.3) is established by many authors [FG], [HZ], [ACM], [ChW]...

Although the curvature flow equations with singular diffusivity do not have the divergence structure of the form (1.2), they are well-studied for evolution of curves [GG1] based on order-preserving structure. For a surface evolution the corresponding theory is widely open; see e.g. [BN], [GPR]. There are several other applications of singular diffusivity, for example for formation of shocks of conservation laws [GG2], [TGO].

The problem with value constraint naturally arises in image processing to remove noise from direction field of color gray-level mappings $u = (u_1, u_2, u_3)$ keeping its strength $u_1^2 + u_2^2 + u_3^2 = 1$. There is a nice book for background of the problems form image processing. As mentioned in [S, §6.3] the well-posedness for the initial-boundary problem for constrained problem (1.3) has not yet been studied even for (1.5). This type of constrained problems also naturally arise in multi-grain problems [KWC] where u is an angle of averaged crystagraphical directions.

2 Gradient system with constraint

We prepare an abstract framework for studying gradient systems of a convex functional. Let $\varphi(\not\equiv \infty)$ be a convex, lower semicontinuous function on a Hilbert space H with values in $\mathbf{R} \cup \{\infty\}$. The gradient system for φ is of the form

$$\frac{du}{dt}(t) \in -\partial\varphi(u(t)) \quad \text{for} \quad t > 0, \tag{2.1}$$

where $\partial\varphi(v)$ denotes the subdifferential of φ at v, *i.e.*,

$$\partial\varphi(v) := \{\xi \in H;\ \varphi(v+h) - \varphi(v) \geq \langle h, \xi\rangle \quad \text{for all} \quad h \in H\}$$

and u is a function from $(0,\infty)$ to H. It is well known (see e.g. [Ba]) that the problem (2.1) admits a unique global solution for any given initial data in H. We next consider a gradient system with constraints on values of functions. Let (v,w) denote the standard inner product of $v,w\in\mathbf{R}^N$. Let Ω be a smoothly bounded domain in $\mathbf{R}^n$. The space of $\mathbf{R}^N$-valued square integrable functions is denoted by $L^2(\Omega;\mathbf{R}^N)$. As a Hilbert space H we take $L^2(\Omega;\mathbf{R}^N)$ equipped with the inner product

$$\langle f,g\rangle=\int_\Omega (f(x),g(x))dx \quad \text{for} \quad f,g\in H.$$

Let M be a smoothly embedded complete manifold in $\mathbf{R}^N$. For a given point $v\in M$ let π_v denote the orthogonal projection from $\mathbf{R}^N=T_v\mathbf{R}^N$ to the tangent space T_vM of M at v. Let $\mathcal{M}$ be the space of L^2-mappings from Ω to M *i.e.*,

$$\mathcal{M}=\{f\in H;\ f(x)\in M \quad \text{for} \quad \text{a.e. } x\in\Omega\}.$$

For $g\in\mathcal{M}$ we define a mapping from H to H by

$$P_g(f)(x)=\pi_{g(x)}(f(x)) \quad \text{for} \quad \text{a.e. } x\in\Omega,$$

where $f\in H$. By definition P_g is an orthogonal projection of H so that its image H_g is a closed subspace of H. (Actually, it is the tangent space of the Hilbert manifold $\mathcal{M}$ at g.)

A constained (by M) gradient systems is of the form

$$\frac{du}{dt}(t)\in -P_{u(t)}(\partial\varphi(u(t))) \quad \text{for} \quad t>0. \tag{2.2}$$

This problem is no longer dissipative so unique globally solvability is not expected even if φ is smooth so that no singular diffusivity appears. In fact, there is a counterexample for global solvability of a smooth solution and uniqueness for the harmonic map flow in Example 1.

Example 1 (Harmonic map flow). Let g be a Lipschitz map from $\partial\Omega$ to M. For $v\in H$ we set

$$\varphi(v)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega|\nabla v|^2dx, & \text{if } \partial_{x_i}v,\ v\in H\ (1\le i\le n) \text{ with } v=g \text{ on } \Omega,\\ +\infty, & \text{otherwise.}\end{cases}$$

Then (2.2) is the harmonic map flow equation with the Dirichlet condition $v=g$ on $\partial\Omega$. Here $\nabla v=(\partial_{x_1}v,\dots,\partial_{x_n}v)$ and $\partial_{x_i}=\partial/\partial x_i$ and $|\nabla v|^2$ denotes the sum of all squares of $\partial_{x_i}v^s$ for $v=(v^1,\dots,v^N)$. Unconstained problem (2.1) for this φ is the heat equation with the Dirichlet condition. Of course, φ is a lower semicontinuous, convex function in H.

The harmonic map flow equation is well-studied by many authors. Uniqueness and global solvability depends on dimension of Ω and also geometric properties of manifold M. For example if Ω is two-dimensional, *i.e.*, $n = 2$, there is a unique global weak solution which is regular except a finite number of isolated points and the energy is decreasing in time [St], [Cg], [F]. When $n \geq 3$, although there exists a global weak solution, it may not be unique [Ch], [C]. If $M = S^1$, then the global solution is smooth. However, if $M = S^2$, there exists a smooth local solution which develops singularities in finite time [CDY] when Ω is a two dimensional disk. See, for example, [S] for more complete list of references on this topics.

If

$$M = S^{N-1} = \{w \in \mathbf{R}^N;\ |w| = 1\} \tag{2.3}$$

i.e. M is the unit sphere, then for $z \in M$

$$\pi_z(y) = y - (y,z)z \quad \text{for} \quad y \in \mathbf{R}^N.$$

Since $\partial\varphi(v) = \{-\Delta v\}$ for v (belonging to the domain of $\partial\varphi$),

$$-P_v(\partial\varphi(v)) = \{\Delta v - (\Delta v, v)v\}.$$

Since $|v| = 1$ so that $(\Delta v, v) = \operatorname{div}(\nabla v, v) - |\nabla v|^2 = -|\nabla v|^2$, we observe that

$$-P_v(\partial\varphi(v)) = \{\Delta v + |\nabla v|^2 v\}.$$

So (2.2) is formally written as

$$\frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2 u.$$

Example 2 (Total variation flow with constraint). Let g be a Lipschitz map form $\partial\Omega$ to M. Let $\tilde{g}$ denote a Lipschitz extension of g to $\mathbf{R}^n$. For $v \in H$ let $\tilde{v}$ be its extension to $\mathbf{R}^n$ such that $\tilde{v}(x) = \tilde{g}(x)$ for $x \in \mathbf{R}^n \setminus \Omega$. We set

$$\varphi(v) = \begin{cases} \displaystyle\int_{\bar{\Omega}} |\nabla\tilde{v}(x)|dx, & \text{if} \quad \tilde{v} \in BV(\Omega;\mathbf{R}^N) \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.4}$$

where BV denotes the space of functions of total variations. The quantity $\varphi(v)$ is the total variation of the measure ∇v in $\mathbf{R}^n$. The reason we extend v to $\tilde{v}$ is that we would rather measure the discrepancy of v from g on the boundary. By this choice of φ (2.1) is the total variation flow equation with Dirichlet condition. Its formal form is

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{2.5}$$

It is easy to see that φ is a convex, lower semicontinuous function in H [GGK]. The equation (2.2) is the Dirichlet problem for the total variation flow equation with constaint. If M is the unit sphere (2.3), then its formal form is

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u| u$$

since $(\operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right), v) = -|\nabla v|$ for v satisfying $|v| = 1$.

Example 3 (A simple inhomogeneous example). Let a be a positive continuous function in $\bar{\Omega}$. Instead of Example 2 we set

$$\varphi(v) = \int_{\Omega} a(x)|\nabla \tilde{v}(x)| dx$$

for $v \in BV(\Omega, \mathbf{R}^N)$ and $\varphi(v) = +\infty$ otherwise. This φ is also a convex, lower semicontinuous function in H. This type of inhomogenous one is important in application to multi-grain problem [GGK], [KG] and also image processing e.g. [ChW].

3 Characterization of speed

The evolution laws (2.1) and (2.2) look ambiguous since $\partial\varphi$ is multivalued. Like (2.1) the speed du/dt of the evolution by (2.2) is actually uniquely detemined under the stronger assumptions than those for (2.1). We state such a charactarization of the speed in this section. Unfortunately, it does not yield the uniqueness of a solution of the initial value problem for (2.2).

We prepare several notations. For a closed convex set A in a Hilbert space there exists a unique point z closest to the origin. We shall write z by ${}^0A$. Since $\partial\varphi(v)$ is always a closed convex set in H, ${}^0(\partial\varphi(v))$ is well-defined and is denoted by $\partial^0\varphi(v)$. It is called the canonical restriction (or minimal section) of $\partial\varphi(v)$. The set $P_v(\partial\varphi(v))$ is also a convex set in H_v for $v \in \mathcal{M}$ since P_v is an orthogonal projection. However, it may not be closed. If there exists a point $z' \in P_v(\partial\varphi(v))$ which is closest to the origin of H_v, it must be unique since the set is convex. We shall denote z' by ${}^0P_v(\partial\varphi(v))$. We call this element the *minimal section* (of $P_v(\partial\varphi(v))$.

Theorem 3.1. *Assume that $\delta > 0$ and that M is compact. Assume that $u : [t_0, t_0 + \delta] \to \mathcal{M} \subset H$ is continuous and right differentiable. Assume that the right derivative d^+u/dt is continuous in $[t_0, t_0 + \delta]$*

$$\{\partial^0\varphi\left(u(t)+P_{u(t)}(u(t+\tau)-u(t))\right);\ t,\ t+\tau\in[t_0,t_0+\delta],\ \tau\in\mathbf{R}\}$$

is bounded in H. If u satisfies

$$\frac{d^+u}{dt}(t)\in -P_{u(t)}(\partial\varphi(u(t)) \quad \text{for} \quad t\in[t_0,t_0+\delta), \tag{3.1}$$

then

$$\frac{d^+u}{dt}(t)=-{}^0P_{u(t)}(\partial\varphi(u(t)) \quad \text{for} \quad t\in[t_0,t_0+\delta). \tag{3.2}$$

In particular, the minimal section of $-P_{u(t)}(\partial\varphi(u(t)))$ always exists for $t\in[t_0,t_0+\delta)$.

Proof. It suffices to prove (3.2) for $t=t_0$. We may assume that $t_0=0$. we set

$$h(s)=u(s)-u(0), \quad P_s=P_{u(s)} \quad \text{for} \quad s\in[0,\delta)$$

to simplify the notation. By (3.1)

$$\langle\frac{d^+u}{dt}(s),\ h(s)\rangle=\langle-\frac{d^+u}{dt}(s),\ -P_sh(s)\rangle\le\varphi(u(s)-P_sh(s))-\varphi(u(s)). \tag{3.3}$$

By definition for $\xi\in P_0(\partial\varphi(u(0))$ we have

$$\langle-\xi,h(s)\rangle=\langle-\xi,\ P_0h(s)\rangle\le\varphi(u(0))-\varphi(u(0)+P_0h(s)). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\langle\frac{d^+u}{dt}(s),\ h(s)\rangle\le\langle-\xi,\ h(s)\rangle+\Phi(s)+\Psi(s) \tag{3.5}$$

with

$$\Phi(s)=\varphi(u(s)-P_sh(s))-\varphi(u(0)),$$

$$\Psi(s)=\varphi(u(0)+P_0h(s))-\varphi(u(s)).$$

We divide both hand sides by $s>0$. Sending s to zero yields

$$\left\|\frac{d^+u}{dt}(0)\right\|^2\le\langle-\xi,\ \frac{d^+u}{dt}(0)\rangle\le\|\xi\|\ \left\|\frac{d^+u}{dt}(0)\right\| \tag{3.5}$$

if we admit

$$\varlimsup_{s\downarrow 0}\Phi(s)/s=0 \quad \text{and} \quad \varlimsup_{s\downarrow 0}\Psi(s)/s=0, \tag{3.6}$$

where $\|\cdot\|$ denotes the norm in H. By (3.5) we observe that $d^+u(0)/dt$ is the minimal section of $P_0(\partial\varphi(u(0))$.

It remains to prove (3.6). We shall present the proof for Φ since the proof for Ψ is similar. By defintion of subdifferentials

$$\varphi(u(s) - P_s h(s)) - \varphi(u(0)) \le \langle (1-P_s)h(s), \partial^0\varphi(u(s) - P_s h(s))\rangle$$

By our boundedness assumption on $\partial^0\varphi$ it suffices to prove that

$$\lim_{s\downarrow 0} \|(1-P_s)h(s)\|/s = 0. \tag{3.7}$$

By definition of the tangent space there exists a constant C that satisfies

$$|(1-\pi_v)\zeta| \le C|\pi_v\zeta|^2$$

for all $\zeta \in \mathbf{R}^N, v \in M$ satistying $\zeta + v \in M$. Thus

$$\|(1-P_s)h(s)\|^2/s^2 \le C\int_\Omega \frac{|h(s)|^2}{s^2}|h(s)|^2\,dx \tag{3.8}$$

Since $h(s)/s \to d^+u(0)/dt$ as $s \downarrow 0$ in H, $|h(s)|^2/s^2 \to |du^+(0)/dt|^2$ in L^1 sense. Since M is bounded, $|h(s)|$ is bounded in L^∞ for small s. So the right hand side of (3.8) converges to zero as $s \to 0$ since $h(x,s) \to 0$ a.e. $x \in \Omega$ by taking a subsequence. Thus we have proved (3.7) so we obtain (3.6). □

4 One dimensional piecewise constant evolution

We now consider the total variation flow with constraint (Example 2) when the domain Ω is an interval (z_0, z_1). We consider the initial value problem

$$\frac{du}{dt}(t) \in -P_{u(t)}(\partial\varphi(u(t))),\ u|_{t=0} = u_0 \tag{4.1}$$

with φ defined by (2.4) with $\Omega = (z_0, z_1)$. We consider a piecewise constant initial data

$$u_0(x) = h_i^0 \in \mathbf{R}^N \quad \text{on} \quad (x_i, x_{i+1}),\ i = 0, 1, \ldots, \ell-1, \ell \ge 2, \tag{4.2}$$

where $z_0 = x_0 < x_1 < x_2 < \cdots < x_\ell = z_1$. The boundary values $h_0^0, h_{\ell-1}^0$ are taken so that $h_0^0 = g(z_0)$ and $h_{\ell-1}^0 = g(z_1)$. We also assume that $h_i^0 \ne h_{i+1}^0$ for $i = 0, 1, \ldots, \ell-2$.

We shall seek a solution $u(t) = u(x,t)$ of (4.1)-(4.2) when $u(x,t)$ is piecewise constant and its jump discontinuities are included in $\{x_i\}_{i=1}^{\ell}$.

4.1 Subdifferentials

We first calculate the subdifferential $\partial\varphi$ of φ defined by (2.4) at a piecewise linear function u_0 defined by (4.2). We set

$$m_i^0 = (h_i^0 - h_{i-1}^0)/|h_i^0 - h_{i-1}^0|,\ i = 1, \dots, \ell - 1. \tag{4.3}$$

Lemma 4.1 *Let $f \in L^2(\Omega; \mathbf{R}^N)$ be of the form*

$$f(x) = -(\xi(x))_x,\ |\xi(x)| \le 1,\ x \in \Omega = (z_0, z_1) \tag{4.4}$$

for some continuous ξ in Ω that satisfies

$$\xi(x_i) = m_i^0,\ i = 1, 2, \dots, \ell - 1. \tag{4.5}$$

Then $f \in \partial\varphi(u_0)$. Conversely, if $f \in \partial\varphi(u_0)$, then f is of the form (4.4) with (4.5).

Proof. The proof is similar to that of [GGK, §3.2, Lemma 1]. We shall check

$$\langle v - u_0, f\rangle \le \varphi(v) - \varphi(u_0)$$

for all $v \in D(\varphi) = \{v; \varphi(v) < \infty\}$. By definition

$$\langle v - u_0, f\rangle = -\int_\Omega (v - u_0, \xi_x)dx. \tag{4.6}$$

Since $|\xi| \le 1$, integrating by parts we see

$$-\int_\Omega (v, \xi_x)dx = \int_\Omega (v_x, \xi)dx - (u_0, \xi)|_{z_0}^{z_1} \le \varphi(v) - u_0\xi|_{z_0}^{z_1}, \tag{4.7}$$

where v_x is regarded as a Radon measure; $\varphi(v)$ equals the total variation of $(\tilde{v})_x$. For example

$$\varphi(u_0) = \sum_{i=1}^{\ell-1} |h_i^0 - h_{i-1}^0|.$$

Since $\xi(x_i) = m_i$, we see that

$$\begin{aligned} \int_\Omega (u_0, \xi_x)dx &= u_0\xi|_{z_0}^{z_1} - \sum_{i=1}^{\ell-1}(h_i^0 - h_{i-1}^0)m_i^0 \\ &= u_0\xi|_{z_0}^{z_1} - \varphi(u_0). \end{aligned} \tag{4.8}$$

The formula (4.6)-(4.8) now yields

$$\begin{aligned} \langle v - u_0, f\rangle &\le \varphi(v) - u_0\xi|_{z_0}^{z_1} + u_0\xi|_{z_0}^{z_1} - \varphi(u_0) \\ &= \varphi(v) - \varphi(u_0), \end{aligned}$$

which implies $f \in \partial\varphi(u_0)$.

Conversely, assume that $f \in \partial\varphi(u_0)$. Let ζ be a primitive of $-f$. Since $f \in L^2(\Omega;\mathbf{R}^N)$, ζ must be absolutely continuous on Ω. The condition $f \in \partial\varphi(u_0)$ is equivalent to

$$\int_\Omega (v - u_0,\ \zeta_x)dx \le \varphi(v) - \varphi(u_0). \tag{4.9}$$

We test various v in this inequality to derive properties of ζ.

We plug

$$v(x) = u_0(x) - \lambda m_i \int_{z_0}^{x} \delta(\tau - x_i)d\tau,\ \lambda \in \mathbf{R},\ |\lambda| < |h_i^0 - h_{i-1}^0|$$

in (4.9) and integrate by parts to get

$$-\lambda(m_i, \zeta(x_i)) \le -\lambda.$$

for $i = 1, \dots, \ell - 1$. Since this inequality holds for both positive and negative λ, we conclude that

$$(m_i, \zeta(x_i)) = 1,\ i = 1, \dots, \ell - 1.$$

For $\hat{x} \in (z_0, z_1) \setminus \{x_i\}_{j=1}^{\ell-1}$ and $i \in \{1, \dots, \ell - 1\}$ we set

$$v(x) = u_0(x) + \lambda \int_{z_0}^{x} m\delta(\tau - \hat{x})d\tau,\ \lambda \in \mathbf{R},\ m \in S^{N-1}.$$

We plug this v in (4.9) and integrate by parts to get

$$\lambda(m, \zeta(\hat{x})) \le |\lambda|$$

Since this inequality holds for both positive and negative λ, we observe that

$$|(m, \zeta(\hat{x}))| \le 1$$

Since $m \in S^{N-1}$ is arbitrary, this implies $|\zeta(\hat{x}))| \le 1$. Be continuity of ζ we see that

$$|\zeta(x)| \le 1 \quad \text{for all} \quad x \in \Omega.$$

Since $(m_i, \zeta(x_i)) = 1$, the inequality $|\zeta(x)| \le 1$ implies that $\zeta(x_i) = m_i$. We have thus proved that $f \in \partial\varphi(u_0)$ must have the form (4.4)-(4.5). □

4.2 Minimal section

We shall calculate ${}^0P_{u_0}\partial\varphi(u_0)$ for a piecewise constant function u_0 in (3.7). In general it is not clear that ${}^0P_v\partial\varphi(v) = P_v\partial^0\varphi(v)$ but for our u_0 this property holds.

Lemma 4.2. *Let L_i be the length of the interval (x_i, x_{i+1}), i.e., $L_i = x_{i+1} - x_i$. Then*

$$-{}^0P_{u_0}((\partial\varphi)(u_0))(x) = \begin{cases} L_i^{-1}\pi_{h_i^0}(m_{i+1}^0 - m_i^0) & \text{for} \quad x \in (x_i, x_{i+1}), \\ & \qquad i = 1, \dots, \ell-2, \\ 0 & \text{for} \quad x \in (x_0, x_1) \cup (x_{\ell-1}, x_\ell). \end{cases}$$

Moreover, ${}^0P_{u_0}((\partial\varphi)u_0) = P_{u_0}(\partial^0\varphi)(u_0))$.

Proof. By Lemma 4.1 we already know the explicit form of $\partial\varphi(u_0)$. If $q = {}^0P_{u_0}(\partial\varphi)(u_0)$, it must be

$$q = -P_{u_0}(\eta_x)$$

with η minimizing

$$\|q\|^2 = \sum_{i=0}^{\ell-1}\int_{x_i}^{x_{i+1}} |\pi_{h_i^0}\eta_x|^2 dx$$

with constraints $\eta(x_i) = m_i^0$ $(i = 1, 2, \dots, \ell-1)$ and $|\eta(x)| \le 1$ for $x \in \Omega$. It suffices to minimize

$$\int_{x_i}^{x_{i+1}} |\pi_{h_i^0}\eta_x|^2 dx$$

with above constraint. The answer is easy. The minimum is attained when η is linear

$$\eta(x) = \{(x - x_i)m_{i+1}^0 + (x_{i+1} - x)m_i^0\}L_i^{-1} \quad \text{for} \quad x \in (x_i, x_{i+1})$$

for $i = 1, 2 \dots, \ell-1$ and

$$\eta(x) = \begin{cases} m_1^0 & \text{for} \quad x \in (x_0, x_1), \\ m_{\ell-1}^0 & \text{for} \quad x \in (x_{\ell-1}, x_\ell). \end{cases}$$

Since $q = -P_{u_0}(\eta_x)$, we have an expression of ${}^0P_{u_0}(\partial\varphi)(u_0)$ in Lemma 4.2. □

Since $\partial^0\varphi(u_0)$ is also computable and

$$\partial^0\varphi(u_0) = \begin{cases} L_i^{-1}(m_{i+1}^0 - m_i^0) & \text{for} \quad x \in (x_i,\ x_{i+1}),\ i = 1, 2, \dots, \ell-2, \\ 0 & \text{for} \quad x \in (x_0, x_1) \cup (x_{\ell-1}, x_\ell), \end{cases}$$

we obtain ${}^0P_{u_0}(\partial\varphi)(u_0) = P_{u_0}(\partial^0\varphi(u_0))$.

4.3 Dynamics

We consider (4.1)-(4.2) assuming that

$$u(x,t) = h_i(t) \in \mathbf{R}^N \quad \text{on} \quad (x_i, x_{i+1}),\ i = 0, 1, \dots, \ell - 1,\ t > 0 \tag{4.10}$$

with $h_0(t) = g(z_0)$ and $h_{\ell-1}(t) = g(z_1)$. The values $h_i(t)$ and $h_{i+1}(t)$ may agree for some $t > 0$. It turns out that the problem (4.1)-(4.2) is reduced to an ODE system for h_i's. Moreover, there exists a unique global solution.

Theorem 4.3. *Assume that M is compact. There exists a unique*

$$h(t) = (h_1(t), \dots, h_{\ell-2}(t))$$

such that $h_i (1 \le i \le \ell - 2)$ is Lipschitz continuous from $[0, \infty)$ to M which is smooth except finitely many points such that (4.10) solves (4.1)-(4.2). Moreover, h_i solves

$$\frac{dh_i(t)}{dt} = \frac{1}{L_i}\pi_{h_i(t)}(m_{i+1}(t) - m_i(t)) \quad \text{for} \quad \begin{array}{l} x \in (x_i, x_{i+1}), \\ i = 1, \dots, \ell - 2 \end{array} \tag{4.11}$$

for sufficiently small $t > 0$, where

$$m_i(t) = (h_i(t) - h_{i-1}(t))/|h_i(t) - h_{i-1}(t)|, i = 1, \dots, \ell - 1. \tag{4.12}$$

Proof. If h_i's are Lipschitz on $[0, \infty)$ and smooth except finitely many points, u given by (4.10) fulfills the regularity assumptions of Theorem 3.1. Then by Theorem 3.1 and Lemma 4.2 h_i must solve (4.11) until the first *merging time* when $h_i = h_{i+1}$ for some i.

Of course, (4.11) is uniquely solvable until the first merging time. If h_i's merges at some time t_0, we removes some x_i's and renumber jumps x_i's such that $h_i(t_0) \ne h_{i+1}(t_0)$ for $i = 0, 1, \dots, \ell_0 - 2$ with $\ell_0 < \ell$, Again we are able to solves (4.11). Repeating this procedure finitely many times, one is able to solve (4.1)-(4.2) uniquely and globally-in-time. (Since h_i's are bounded, the solution of (4.11) can be extended unless some h_i's merge.) Since the right hand side of (4.11) is bounded (independent of t), the solution h_i's must be globally Lipschitz continuous in time. □

4.4 Constrained gradient system of ordinary differential equations

If $u = u(x,t)$ is of the form (4.10), then

$$\varphi(u(t)) = \psi(h_1(t),\dots,h_d(t)),\ d = \ell - 2\ (\ell \geq 3)$$
$$\psi(h_1,\dots,h_d) = \sum_{j=1}^{d+1} |h_j - h_{j-1}|,\ h_0 = g(z_0),\ h_{\ell-1} = g(z_1).$$

(If $\ell = 2$, $\varphi(u(t)) = |h_1 - h_0|$ and is independent of t.) Using this $\psi : \mathbf{R}^{Nd} \to \mathbf{R}$, we are able to rewrite (4.11) as

$$\frac{dh}{dt} = -\pi_h \operatorname{grad}_* \psi(h),\ h(t) = (h_1(t),\dots,h_d(t)), \tag{4.13}$$

where grad_* is the gradient of ψ in $\mathbf{R}^{Nd}$ with respect to the inner product

$$(h,g)_* = \sum_{i=1}^{d} L_i(h_i, g_i)$$

for $g = (g_1,\dots,g_d)$ and $\pi_h = (\pi_{h_1},\dots,\pi_{h_d})$. Indeed, by definition,

$$\operatorname{grad}_*\psi(h) = \left(L_i^{-1}\frac{\partial\psi}{\partial h_i}\right)_{i=1}^{d}.$$

Since $\dfrac{\partial\psi}{\partial h_i}(t) = -(m_{i+1}(t) - m_i(t))$, (4.11) is the same as (4.13). This weight is very natural since our subdifferential of φ is taken with respect to $L^2(\Omega)$-inner product. Let us summarize what we obtained here.

Proposition 4.4. *Assume that M is compact. Let $h(t)$ be a function defined in Theorem 4.3. Then h solves (4.13) for t before the first merging time.*

We expect that in finite time our solution u stops moving. We shall prove such a phenomena when $M = S^1$. For this purpose we study the large time behabiour of (4,13) assuming that there is no merging of h_i's.

Proposition 4.5 *Assume that M is compact. Let h be a global solution of (4.13) for $t \in [t_*, \infty)$ such that no h_i's merge for $t \in [t_*, \infty)$. Then*

$$\int_{t_0}^{\infty} (h_t, h_t)_*\, dt \leq \psi(h(t_*)) \quad \text{and} \quad \frac{d\psi(h(t))}{dt} \leq 0 \quad \text{for} \quad t > t_*.$$

Moreover, there is a subsequence of $\{u(x, t + t_* + k)\}_{k=1}^{\infty}$ converges in L^2 $(\Omega \times (0,1); M)$ to a piecewise constant stationary solution u_∞ of (4.1) in the sense that ${}^0P_{u_\infty}(\partial\varphi(u_\infty)) = 0$. Here $u(x,t)$ is defined by

Proof. We observe that h is smooth for (t_*, ∞). We take inner product of (4.13) and h_t and observe that

$$(h_t, h_t)_* = -\frac{d\psi}{dt}(h(t))$$

which yields $d\psi(h(t))/dt \le 0$ for all $t \in (t_*, \infty)$. We integrate over (t_*, s) and send s to infinity to get

$$\int_{t_*}^{\infty} (h_t, h_t)_* dt \le \psi(h(t_*))$$

since $\psi \ge 0$. In particular, $(h_k)_t(t) = h_t(t + t_k + k)$ converges in $L^2(0,1)$ to zero. Since $\{h_k(t)\} \subset M$ is bounded for $t \in (0,1]$ $\{h_k(t)\}$ has a convergent subsequence. Since $(h_k)_t \to 0$ in $L^2(0,1)$, the limit of $\{u_k\}$ (defined by (4.10) with h_i replaced by h_{ki}) converges to u_∞ (by taking a subsequence) which is a stationary solution. (In this argument there might be a chance that $(h_i - h_{i-1})(t) \to 0$ as $t \to \infty$ so we rather use u instead of h). □

4.5 S^1-valued problem

We shall study a more detailed dynamics when the set of constraint M equals the unit circle S^1 in $\mathbf{R}^2$. We first characterize all stationary piecewise constant solutions. For two vectors in $p, q \in M$ we define $\arg(p,q) = \arg p - \arg q$. The value is taken so that $\arg(p,q) \in (-\pi, \pi]$.

Lemma 4.6. *Let u_0 be of the form (4.2) with $h_i^0 \ne h_{i+1}^0$ for $i = 0, 1, \dots \ell - 2, \ell \ge 2$ and $h_0^0 = g(z_0)$ and $h_{\ell-1}^0 = g(z_1)$. Then u_0 is a stationary solution of (4.1) (in the sense that ${}^0P_{u_0}(\partial\varphi(u_0)) = 0$) if and only of $\arg(h_i^0, h_{i-1}^0)$ is independent of $i = 1, 2, \dots, \ell - 1$.*

Proof. We may assume $\ell \ge 3$. By elementary geometry we observe that

$$\pi_{h_i^0}(m_{i+1}^0 - m_i^0) = 0$$

is equivalent to say that $\arg(h_i^0, h_{i-1}^0) = \arg(h_{i+1}^0, h_i^0)$ for $i = 1, \dots, \ell - 2$. □

We next study the stability of stationary solutions. For u_0 in (4.2) we observe that

$$\varphi(u_0) = \sum_{i=1}^{\ell-1} |h_i^0 - h_{i-1}^0| = \sum_{i=1}^{\ell-1} 2|\sin \xi_i|, \quad \xi_i = \frac{1}{2}\arg(h_i^0, h_{i-1}^0).$$

Since h_0^0 and $h_{\ell-1}^0$ are fixed by the Dirichlet data, the sum $\sum_{i=1}^{\ell-1} \xi_i =: \lambda$ is constant independent of $(\xi_1 \dots, \xi_{\ell-1})$ (at least small perturbation of $(\xi_i, \dots, \xi_d)$). We set $E(\xi_1, \dots, \xi_d) = \sum_{i=1}^{d-1} |\sin \xi_i| + |\sin(\lambda - \sum_{j=1}^{d} \xi_j)|$, $d = \ell - 2$. By definition $E(\xi_1, \dots, \xi_d) = \varphi(u_0)/2$. If u_0 is a stationary solution of (4.1), then by Lemma 4.6 we see that $\xi_1 = \xi_2 = \dots = \xi_d = \lambda - \sum_{i=1}^{d} \xi_i$. The next lemma shows that such a stationary solution is local maximum of E so in particular it is unstable in all direction. Note that when we discuss the stability it suffices to check Hesse matrix for grad $(= \nabla)$ instead of grad_*.

Lemma 4.7. *Assume that $d = \ell - 2 \geq 1$. Assume that $\lambda \neq 0$ and $\lambda/(\ell - 1) \in (-\pi/2, \pi/2]$. Then the Hesse metrix $\nabla^2 E$ at $\xi_0 = (\lambda/(\ell-1), \dots, \lambda/(\ell-1))$ is negative definite.*

Proof We may assume that $\lambda > 0$. We differentiate E and observe that

$$\nabla E = (\cos \xi_i - \cos(\lambda - \sum_{j=1}^{d} \xi_j))_{i=1}^{d} \quad \text{near} \quad \xi_0 \quad \text{and}$$
$$-\nabla^2 E(\xi_0) = (\delta_{ij} a + a)_{1 \leq i,j \leq d}, \; a = \sin(\lambda/(\ell - 1)),$$

where δ_{ij} is Kronecker's delta. Since

$$(\delta_{ij} a + a) = a(\delta_{ij} + \sigma_i \sigma_j) \quad \text{with} \quad \sigma = (\sigma_1, \dots, \sigma_d) = (1, \dots, 1),$$

its determinant is easy to calculate. Indeed,

$$\det(\delta_{ij} a + a) = a^d \det(\delta_{ij} + \sigma_i \sigma_j) = a^d (1 + |\sigma|^2) = a^d (1 + d).$$

Thus we conclude that

$$\det((\delta_{ij} a + a)_{1 \leq i,j \leq r}) > 0$$

for all $r = 1, 2, \dots, d$, which implies that $-\nabla^2 E(\xi_0)$ is positive definite. □

By Lemma 4.7 all piecewise constant stationary solution (except one jump or no jump solution) are local maximum in a class of piecewise constant functions having the same location of jump discontinuities. Of course all one jump and no jump solutions are isolated global minimizers since each stationary solution has a different value of energy φ. Combining Proposition 4.5 and Lemmas 4.6, 4.7, we obtain a full convergence result.

Proposition 4.8. *Assume that $M = S^1$ and $N = 2$. Let u be of the form and $h = (h_1, \dots, h_{\ell-2})$ solves (4.13) for $t \in [t_*, \infty)$ such that no h_i's merges for $t \in [t_*, \infty)$. Assume that $u(x, t_*)$ is not a stationary solution of (4.1). Then $u(x,t)$ converges to a (piecewise constant) stationary solution with jump discontinuities strictly contained in $\{x_i\}_{i=1}^{\ell-1}$. In particular, $h_i - h_{i-1} \to 0$ as $t \to \infty$ for some $i = 1, \dots, \ell - 1$, as $t \to \infty$.*

4.6 Stopping in finite time

We continue to study the case when $M = S^1$ with $N = 2$. We shall prove that our piecewise constant solution $u = u(t)$ actually stops moving after finite time and it becomes a stationary solution. For this purpose we shall rewrite (4.11) by using argument $\theta_i(t)$ of $h_i(t)$. Since

$$\begin{aligned} m_{i+1} &= (\cos\theta_{i+1} - 1, \sin\theta_{i+1})/A_{i+1}, \\ m_i &= (1 - \cos\theta_{i-1}, -\sin\theta_{i-1})/A_i \end{aligned}$$

with $A_i = ((\cos\theta_i - 1)^2 + \sin^2\theta_i)^{\frac{1}{2}}$ if $h_i = (1, 0)$, we see that

$$\pi_{h_i}(m_{i+1} - m_i) = \tau(\sin\theta_{i+1}/A_{i+1} + \sin\theta_{i-1}/A_i)$$

with $\tau = (0, 1)$. Since $A_i^2 = 4\sin^2(\theta_i/2)$, we see that

$$\pi_{h_i}(m_{i+1} - m_i) = \tau\left(\frac{\sin\theta_{i+1}}{2|\sin(\theta_{i+1}/2)|} + \frac{\sin\theta_{i-1}}{2|\sin(\theta_{i-1}/2)|}\right).$$

For a general $h_i = (\cos\theta_i, \sin\theta_i)$ our calculation shows that

$$\begin{aligned} \pi_{h_i}(m_{i+1} - m_i) &= \tau\left\{\frac{\sin(\theta_{i+1} - \theta_i)}{2|\sin(\frac{\theta_{i+1}-\theta_i}{2})|} + \frac{\sin(\theta_{i-1} - \theta_i)}{2|\sin(\frac{\theta_{i-1}-\theta_i}{2})|}\right\} \\ &= \tau\left\{\operatorname{sgn}\left(\sin\frac{\theta_{i+1} - \theta_i}{2}\right)\right. \end{aligned}$$

$$\cos\frac{\theta_{i+1} - \theta_i}{2}$$

$$\left. + \operatorname{sgn}\left(\sin\frac{\theta_{i-1} - \theta_i}{2}\right)\cos\frac{\theta_{i-1} - \theta_i}{2}\right\}$$ with $\tau = (-\sin\theta_i, \ \cos\theta_i)$. Since

$$\frac{dh_i}{dt} = \tau\frac{d\theta_i}{dt},$$

(4.11) becomes

$$\begin{aligned} \frac{d\theta_i}{dt} = \ & L_i^{-1}\left[\operatorname{sgn}\left(\sin\frac{\theta_{i+1} - \theta_i}{2}\right)\cos\frac{\theta_{i+1} - \theta_i}{2}\right. \\ & \left. + \operatorname{sgn}\left(\sin\frac{\theta_{i-1} - \theta_i}{2}\right)\cos\frac{\theta_{i-1} - \theta_i}{2}\right] \end{aligned} \tag{4.14}$$

for $i = 1, \ldots, \ell - 2$. If we consider the evolution of u, (4.14) holds until the first merging times of h_i's. At the merging time we renumber jumps so that renumbered θ_i's solves (4.14) until the next merging time. We set $\xi_i = (\theta_i - \theta_{i-1})/2$ as before.

Theorem 4.9 (Stopping in finite time). *Assume that $N = 2$ and $M = S^1$. Let u be the solution of (4.1)-(4.2) of the form (4.10). Then there exists $t_* \geq 0$ such that $u(x,t) = U(x)$ for $t \geq t_*$ with some (piecewise constant) stationary solution of (4.1).*

Proof. We may assume that the initial data is not a stationary solution. Then there are finitely many times $t_0 < t_1 < \cdots < t_s$, $t_0 > 0$ such that the set of jump discontinuous decreases at $t_j, j = 0, \ldots, s$ while in $[0, t_0)$, $[t_j, t_{j+1})$, $(j = 0, \ldots, s-1)$ and $[t_s, \infty)$ the set of jump discontinuities is independent of time. (At each t_j some h_i merges.) We claim that $u(x, t_s) = U(x)$— some stationary solution so that $u(x,t) = U(x)$ for $t > t_s$. If $u(x, t_s)$ is not a stationary solution, then we have a situation of Proposition 4.8 with $t_* = t_s$. By Proposition 4.8 there exists an nonempty set $I \subset \Lambda = \{1, \ldots, \ell - 1\}$ that satisfies

$$\lim_{t \to \infty} (\theta_i(t) - \theta_{i-1}(t)) = 0 \quad \text{for} \quad i \in I.$$

(i)If $I \neq \Lambda$, then there is $i_0 \in I$ such that either $i_0 + 1$ or $i_0 - 1$ does not belong to I. If $i_0 + 1 \notin I$, then $|d\theta_{i_0}/dt|$ is bounded away from zero for sufficiently large t by (4.14) since $\theta_{i_0} - \theta_{i_0 - 1} \to 0$ while $\theta_{i_0+1} - \theta_{i_0}$ is bounded away from zero. Similarly, if $i_0 - 1 \notin I$ then $|d\theta_{i_0-1}/dt|$ is bounded away from zero for sufficiently large t. In both cases these properties contradict the convergence of h_{i_0} or h_{i_0-1} as $t \to \infty$. So this case does not occur.

(ii)If $I = \Lambda$, then $g(z_0) = g(z_1)$. Then there is some $i_0 \in \Lambda$ such that $\operatorname{sgn}\sin\xi_{i_0} > 0$ and either $\operatorname{sgn}\sin\xi_{i_0+1} > 0$ or $\operatorname{sgn}\sin\xi_{i_0-1} > 0$. Note that $\operatorname{sgn}\sin\xi_{i_0+1}(t)$ is independent of $t \geq t_*$. By (4.14) either $|d\theta_{i_0}/dt|$ or $|d\theta_{i_0-1}/dt|$ is bounded away from zero for large t. This property contradicts the convergence of h_{i_0} or h_{i_0-1} as $t \to \infty$. So this case does not occur neither.

We thus conclude that $u(x, t_x) = U(x)$. □

Remark 4.10. The stationary solution $U(x)$ we obtain at t_s is not necessarily one jump or no jump solution. Here is a simple example. We set

$$h_0^0 = (0, -1),\ h_3^0 = (0, 1),\ h_1^0 = (\cos\theta_0, \sin\theta_0),\ h_2^0 = (\cos\theta_0, \sin\theta_0)$$

with $\ell = 4$ and $\theta_0 \in (0, \pi/2)$. Assume that the initial data u_0 is given by (4.2) with these h_i^0's and that $L_0 = L_1 = L_2 = L_3$. Then the solution $u(x,t)$ becomes

$$U(x) = \begin{cases} h_0^0, & x \in (x_0, x_1), \\ (1,0), & x \in (x_1, x_3), \\ h_3^0, & x \in (x_3, x_4) \end{cases}$$

at the first merging time which is a stationary solution.

Although all piecewise constant stationary solution (except one or no jump solution) are local maximum in a class of piecewise constant functions having the same location of jump discontinuities, it may be attained at the merging time of evolution as this example shows.

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