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# Analytic Truncation and Rankin-Selberg versus Algebraic Truncation and Non-Abelian Zeta

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— Dedicated to Professor S. Kobayashi for his 70th birthday

Zagier once said in his paper on Rankin-Selberg method: "one of the most fruitful ideas in the theory of automorphic forms is the observation, made independently by Rankin and Selberg around 1939, that the Mellin transform of the constant term in the Fourier development of an automorphic function can be represented as the scalar product of the automorphic functions with an Eisenstein series and hence inherits the analytic properties of the Eisenstein series." For more detailed achievements of using Rankind-Selberg methods, we recommend the reader to consult the paper of Bump on "The Rankind-Selberg Method: A Survey". In this paper, we find a way going slightly beyond the Rankind-Selberg method and constant terms so as to obtain some new terms which are essentially non-abelian.

Surprisingly enough, our starting point is also the beautiful formula of Langlands on the inner product of what I call Arthur's analytic truncated Eisenstein series. It is well-known that Langlands's formula plays a key role in the theory of Eisenstein series, and the analytic truncation is systematically (introduced and) studied by Arthur in his fundamental work on trace formula.

Put this in a simple term, we then see that Rankin-Selberg method is a kind of device, where we have the analytic truncation, Eisenstein series as input, and the constant terms of the Fourier expansion as out put. In other words, the Rankin-Selberg method may be viewed as a kind of linearization, or better, abelization process.

In this paper, we introduce a new device to obtain non-abelian terms. More precisely, instead of using analytic truncation, we use an algebraic truncation via a kind of intersection stability. While we also consider the integration of Eisenstein series, due to the fact that such an algebraic truncation is essentially a geometric one, we get finally the non-abelian aspect of automorphic functions.

This paper may be viewed as a supplementary to Part B of our Program paper, where we introduced non-abelian zeta functions for global fields.

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## I. Eisenstein Series and Non-Abelian Zeta Functions

### I.1. Epstein Zeta Functions and Non-Abelian Zeta Functions

For simplicity, assume that the number field involved is the field of rationals. A lattice $\Lambda$ over $\mathbf{Q}$ is called *semi-stable* if for any sublattice $\Lambda_1$ of $\Lambda$,

$$\left(\mathrm{Vol}\,\Lambda_1\right)^{\mathrm{rank}\,\Lambda} \geq \left(\mathrm{Vol}\,\Lambda\right)^{\mathrm{rank}\,\Lambda_1}.$$

Denote the moduli space of rank $r$ semistable lattices over $\mathbf{Q}$ by $\mathcal{M}_{\mathbf{Q},r}$. By definition, the *rank* $r$ *non-abelian zeta function* $\xi_{\mathbf{Q},r}(s)$ of $\mathbf{Q}$ is

$$\xi_{\mathbf{Q},r}(s) := \int_{\mathcal{M}_{\mathbf{Q},r}} \left(e^{h^0(\mathbf{Q},\Lambda)} - 1\right) \cdot \left(e^{-s}\right)^{\deg(\Lambda)} d\mu(\Lambda), \qquad \mathrm{Re}(s) > r,$$

where $h^0(\mathbf{Q},\Lambda) := \log\left(\sum_{x\in\Lambda} \exp\left(-\pi|x|^2\right)\right)$ and $\deg(\Lambda)$ denotes the Arakelov degree of $\Lambda$. It is known that $\xi_{\mathbf{Q},r}(s)$ coincides with the (completed) Riemann-zeta function when $r = 1$, can be meromorphically extended to the whole complex plane, satisfies the function equation

$$\xi_{\mathbf{Q},r}(s) = \xi_{\mathbf{Q},r}(1-s),$$

and has only two singularities, simple poles, at $s = 0, 1$ with residues $\mathrm{Vol}\,(\mathcal{M}_{\mathbf{Q},r}[1])$, the Tamagawa type volume of the space of rank $r$ semi-stable lattice of volume 1. For details, please see [We1,2].

Denote by $\mathcal{M}_{\mathbf{Q},r}[T]$ the moduli space of rank $r$ semi-stable lattices of volume $T$. We have a trivial decomposition

$$\mathcal{M}_{\mathbf{Q},r} = \cup_{T>0}\mathcal{M}_{\mathbf{Q},r}[T].$$

Moreover, there is a natural morphism

$$\mathcal{M}_{\mathbf{Q},r}[T] \to \mathcal{M}_{\mathbf{Q},r}[1], \qquad \Lambda \mapsto T^{\frac{1}{r}} \cdot \Lambda.$$

With this,

$$\xi_{\mathbf{Q},r}(s) = \int_{\cup_{T>0}\mathcal{M}_{\mathbf{Q},r}[T]} \left(e^{h^0(\mathbf{Q},\Lambda)} - 1\right) \cdot \left(e^{-s}\right)^{\deg(\Lambda)} d\mu(\Lambda) = \int_0^\infty T^s \frac{dT}{T} \int_{\mathcal{M}_{\mathbf{Q},r}[1]} \left(e^{h^0(\mathbf{Q},T^{\frac{1}{r}}\cdot\Lambda)} - 1\right) \cdot d\mu_1(\Lambda),$$

where $d\mu_1$ denotes the induced Tamagawa measure on $\mathcal{M}_{\mathbf{Q},r}[1]$.

Thus note that

$$h^0(\mathbf{Q}, T^{\frac{1}{r}} \cdot \Lambda) = \log\left(\sum_{x\in\Lambda} \exp\left(-\pi|x|^2 \cdot T^{\frac{2}{r}}\right)\right),$$

and

$$\int_0^\infty e^{-AT^B} T^s \frac{dT}{T} = \frac{1}{B} \cdot A^{-\frac{s}{B}} \cdot \Gamma(\frac{s}{B}),$$

we have

$$\xi_{\mathbf{Q},r}(s) = \frac{r}{2} \cdot \pi^{-\frac{r}{2}s}\Gamma(\frac{r}{2}s) \cdot \int_{\mathcal{M}_{\mathbf{Q},r}[1]} \left(\sum_{x\in\Lambda\backslash\{0\}} |x|^{-rs}\right) \cdot d\mu_1(\Lambda).$$

Set now the completed Epstein zeta function, a special kind of Eisenstein series, associated to the rank $r$ lattice $\Lambda$ over $\mathbf{Q}$ by

$$\hat{E}(\Lambda; s) := \pi^{-s}\Gamma(s) \cdot \sum_{x\in\Lambda\backslash\{0\}} |x|^{-2s},$$

then we have the following

**Proposition.** (Relation between Eisenstein series and Non-Abelian Zeta Functions) *With the same notation as above,*

$$\xi_{\mathbf{Q},r}(s) = \frac{r}{2} \int_{\mathcal{M}_{\mathbf{Q},r}[1]} \hat{E}(\Lambda, \frac{r}{2}s)\, d\mu_1(\Lambda).$$

Thus to study our non-abelian zeta functions, we need to understand Eisenstein series and algebraic(=geometric) truncations.

### I.2. Rankin-Selberg Method: An Example with $SL_2$

From the previous subsection, we know that

$$\xi_{\mathbf{Q},2}(s) = \int_{\mathcal{M}_{\mathbf{Q},2}[1]} \hat{E}(\Lambda, s)\, d\mu_1(\Lambda).$$

Thus to study $\xi_{\mathbf{Q},2}(s)$, we need to know what is the moduli space of $\mathcal{M}_{\mathbf{Q},2}$ and what is the integration of the Eisenstein series $\hat{E}(\Lambda; s)$ over this space. Before discussing this, let us take a more traditional approach.

Consider the action of SL(2, $\mathbf{Z}$) on the upper half plane $\mathcal{H}$. A standard fundamental domain of $SL(2,\mathbf{Z})$ may be described by

$$D = \{z = x + iy \in \mathcal{H} : |x| \leq \frac{1}{2}, y > 0, x^2 + y^2 \geq 1\}.$$

Associated to this is aslo the Eisenstein series

$$\hat{E}(z; s) := \pi^{-s}\Gamma(s) \cdot \sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|mz+n|^{2s}}.$$

At this stage, a natural question is to consider the integration

$$\int_D \hat{E}(z,s)\frac{dx\,dy}{y^2}. \tag{1}$$

However, this integration diverges due to the following facts: Near the only cusp $y = \infty$, $\hat{E}(z,s)$ has the Fourier expansion

$$\hat{E}(z; s) = \sum_{n=-\infty}^{\infty} a_n(y,s)e^{2\pi i n x}.$$

Here

$$a_n(y,s) = \begin{cases} \xi(2s)y^s + \xi(2-2s)y^{1-s}, & \text{if } n = 0 \\ 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y), & \text{if } n \neq 0 \end{cases}$$

where $\xi(s)$ denotes the completed Riemann zeta function,

$$\sigma_s(n) := \sum_{d|n} d^s, \quad \text{and} \quad K_s(y) := \frac{1}{2}\int_0^\infty e^{-y(t+\frac{1}{t})/2}t^s\frac{dt}{t}$$

is the K-Bessel function. Moreover,

$$|K_s(y)| \leq e^{-y/2}K_{\mathrm{Re}(s)}(2), \qquad \text{if } y > 4, \qquad \text{and} \qquad K_s = K_{-s},$$

so $a_{n\neq 0}(y,s)$ decay exponentially, and the problematic term comes from $a_0(y,s)$, which is of slow growth.

Therefore, to make the integration (1) meaningful, we need to cut-off the slow growth part. Naturally, there are two ways to do so: the analytic one and the geometric one.

#### (a) Geometric Truncation

Draw a horizontal line $y = T \geq 1$ and consider the part $D_T$ of the domain $D$ which is under the line $y = T$. (So we get a compact subset.) Denote the complement of $D_T$ in $D$ by $D^T$, the closure of a neighborhood near the only cusp $\infty$. That is to say,

$$D_T = \{z = x + iy \in D : y \leq T\}, \qquad D^T = \{z = x + iy \in D : y \geq T\}.$$

Introduce the integration

$$I_T^{\mathrm{Geo}}(s) := \int_{D_T} \hat{E}(z,s)\frac{dx\,dy}{y^2}. \tag{2}$$

(b) **Analytic Truncation**

Define a truncated Eisenstein series $\hat{E}_T(z;s)$ by

$$\hat{E}_T(z;s) := \begin{cases} \hat{E}(z;s), & \text{if } y \leq T \\ \hat{E}(z,s) - a_0(y;s), & \text{if } y > T. \end{cases}$$

Introduce the integration

$$I_T^{\mathrm{Ana}}(s) := \int_D \hat{E}_T(z;s)\,\frac{dx\,dy}{y^2}. \tag{3}$$

With this, from the Rankin-Selberg method, we finally have the following:

**Proposition.** (Analytic Truncation=Geometric Truncation in Rank 2) *With the same notation as above,*

$$I_T^{\mathrm{Geo}}(s) = \xi(2s)\frac{T^{s-1}}{s-1} - \xi(2s-1)\frac{T^{-s}}{s} = I_T^{\mathrm{Ana}}(s). \tag{4}$$

### I.3. Algebraic Truncation

Now we should justify why the above discussion has anything to do with our non-abelian zeta functions. For this, we introduce yet another truncation, the algebraic one.

So back to the moduli space of rank 2 lattices of volume 1 over $\mathbf{Q}$. There is a natural map from this space to $D$: For any lattice $\Lambda$, choose a vector $\mathbf{x}_1$ such that its length gives the first minimum $\lambda_1$ of Minkowski. Then via rotation, we may assume that $\mathbf{x}_1 = (\lambda_1, 0)$. It is well-known from the reduction theory that $\frac{1}{\lambda_1}\Lambda$ may be viewed as the lattice of the volume $\lambda_1^{-2} = y_0$ which is generated by $(1,0)$ and $\omega = x_0 + iy_0 \in D$. That is to say, the points in $D_T$ are in one-to-one corresponding to rank two lattices of volume one whose first Minkowski minimum $\lambda_1 \leq \sqrt{T}$. Set $\mathcal{M}_{\mathbf{Q},2}^{\leq -\frac{1}{2}\log T}[1]$ to be the moduli space of rank 2 lattices $\Lambda$ of volume 1 over $\mathbf{Q}$ whose sublattices $\Lambda_1$ of rank 1 have degrees $\leq -\frac{1}{2}\log T$. As a direct consequence, we have the following

**Fact.** (Geometric Truncation = Algebraic Truncation) *With the same notation as above, there is a natural one-to-one, onto morphism*

$$\mathcal{M}_{\mathbf{Q},2}^{\leq -\frac{1}{2}\log T}[1] \simeq D_T.$$

For example, $\mathcal{M}_{\mathbf{Q},2}^{\leq 0}[1] = \mathcal{M}_{\mathbf{Q},2}[1] \simeq D_1$.

With this, by Proposition I.2, we may introduce a more general type non-abelian zeta functions, parametrized by $T$, with the help of a Harder-Narasimhan type discussion on intersection stability. (See II.1 below.) As a special case, we have the following

**Corollary.** (Degeneration in Rank 2) *With the same notation as above,*

$$\xi_{\mathbf{Q},2}(s) = \xi(2s)\frac{1}{s-1} - \xi(2s-1)\frac{1}{s}. \tag{5}$$

Quite disappointed. Isn't it?! After all, what we previously claim is a non-abelian zeta, yet the calculation gives only abelian zetas. However, a positive thinking then leads to the following three observations:
(i) The special values $\zeta(2n)$ and $\zeta(2n-1)$ of the Riemann zeta function are naturally related via the rank two zeta. That is to say, non-abelian zeta could be used to understand abelian zetas;
(ii) The volume of $D_T$ may be evaluated from this formula via a residue argument;
(iii) The dependence on $T$ of the integrations (4) is quite regular: The 'main term' is simply

$$\xi(2s)\cdot\frac{1}{s-1} - \xi(2s-1)\cdot\frac{1}{s}.$$

Indeed, as the whole paper indicates, among all non-abelian zetas, rank 2 and only the rank two non-abelian zeta degenerate: The practical purpose of this paper is to justify this latest assertion.

## II. Algebraic, Geometric and Analytic Truncations

Still we need to answer the question on why non-abelian zeta degenerates to abelian zetas in rank 2, as indicated from the Rankin-Selberg method above. For this, in this chapter, we study a more general algebraic truncation for lattices over any number fields, motivated by Lafforgue's work for vector bundles over function fields [L], and discuss its relation with the analytic truncation introduced by Arthur [Ar1-6].

### II.1. Algebraic Truncation

Let $G = \mathrm{GL}_r$ be the general linear group of rank $r$. Corresponding to each partition $r = r_1+r_2+\ldots+r_k$, we have the corresponding (standard) parabolic subgroup $P_{r_1,r_2,\ldots,r_k}$ of $G$, consisting of blocked upper-triangle submatrices whose diagonals are of size $r_1, r_2, \ldots, r_k$. The natural order for these parabolic subgroups corresponds to the natural order of partitions so that the group $P_0 := P_{1,1,\ldots,1}$ (resp. $P_r = G$) is a minimal (resp. the maximal) parabolic subgroup of $G$. Moreover, we know that all parabolic subgroups $P$ are conjugations of these standard parabolic subgroups. Denote by $\mathcal{P}_0$ (resp. $\mathcal{P}$) the collection of all standard parabolic subgroups (resp. parabolic subgroups) of $G$.

For a fixed parabolic subgroup $P$, denote by $N_P$ the unipotent radical of $P$ and let $M_P$ be the unique Levi component of $P$, which is supposed also to contain $M_{P_0}$ when $P \in \mathcal{P}_0$. Denote the center of $M_P$ by $A_P$. Let $X(M_P)$ be the group of characters of $M_P$ defined over $\mathbf{Q}$. Then $a_P = \mathrm{Hom}(X(M_P), \mathbf{R})$ is the real vector space whose dimension equals that of $A_P$. (Thus if $P = P_{r_1,\ldots,r_k}$, then the dimension is simply $k-1$. For this reason, we usually also write $k$ as $|P|$.) Its dual space is $a_P^* = X(M_P)\otimes\mathbf{R}$. Denote the set of simply roots of $(P, A)$ by $\Delta_P \subset X(A_P) \subset a_P^*$. The set $\Delta_0 = \Delta_{P_0}$ is a base for a root system, which as usual we write as $\{e_1 - e_2, e_2 - e_3, \ldots, e_{r-1} - e_r\}$ [H].

Fix a number field $F$. Denote its ring of integers by $\mathcal{O}_F$. For each place $v$ of $F$, Denote by $F_v$ the $v$-completion of $F$, and if $v$ is finite, $\mathcal{O}_v$ the ring of integers of $F_v$. Denote the ring of adeles of $F$ by $\mathbf{A} = \mathbf{A}_F$, $K = \prod K_v$ the maximal compact subgroup of $G(\mathbf{A}) = \mathrm{GL}(r, \mathbf{A})$, where $K_v$ denotes $\mathrm{GL}(r, \mathcal{O}_v)$ if $v$ is finite, $O(r)$ if $v$ is real, and $U(r)$ if $v$ is complex. Then associated to each element of the quotient $G(\mathbf{A})/K$ is an $\mathcal{O}_k$-lattice of rank $r$ in $(\mathbf{R}^r)^{r_1} \times (\mathbf{C}^r)^{r_2}$. Indeed, $(g_v)_{v,\mathrm{finite}}$ first gives a locally free sheaf $\mathcal{E}$ of rank $r$ over $\mathrm{Spec}(\mathcal{O}_F)$ such that $\mathcal{E}\otimes_{\mathcal{O}_F} F \simeq F^r$ which under the natural embedding $F \hookrightarrow \mathbf{R}^{r_1}\times\mathbf{C}^{r_2}$ yields a lattice above equipped with the metrics induced by $g_\infty := (g_\sigma)_{\sigma:\mathrm{infinite}}$ from the standard one. For simplicity, write this lattice by $(\mathcal{E}, g_\infty) = \mathcal{E}^g$ so that $\mathcal{E}^g \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v} = g_v(\mathcal{O}_{F_v})$ if $v$ is finite. As noted by Weil, this association gives a one-to-one, onto correspondence between the quotient $G(F)\backslash G(\mathbf{A})/K$ and the moduli of all rank $r$ $\mathcal{O}_K$-lattices over $F$. As usual, define the degree of $\mathcal{E}^g$ associated to $g \in G(\mathbf{A})$ to be $-\log N(\det g)$, where $N : \mathbf{I}_F = \mathrm{GL}(1,\mathbf{A}) \to \mathbf{R}$ denotes the norm of the ideles of $F$, and the slope of $\mathcal{E}^g$ by $\mu(\mathcal{E}^g) = \frac{\deg(\mathcal{E}^g)}{\mathrm{rank}(\mathcal{E}^g)}$.

Let $P \in \mathcal{P}_0$ be the parabolic subgroup corresponding to the partition $r = r_1 + r_2 + \ldots + r_{|P|}$. Then the map $\delta \mapsto \delta^{-1}P\delta$ gives a one-to-one correspondence between the quotient $P(F)\backslash G(F)$ and the subset $\mathcal{P}_P$ of $\mathcal{P}$ whose associated filtrations have successive simple quotient factors of sizes $r_1, r_2, \ldots, r_{|P|}$. For $Q \in \mathcal{P}_P$, denote by $\mathcal{E}_\cdot^{g,Q}$ the filtration of $\mathcal{E}^g$ which is stablized by $Q$. In this way, $g \mapsto (\mathcal{E}^g, \mathcal{E}_\cdot^{g,P})$ gives a natural identification between $P(F)\backslash G(\mathbf{A})/K$ and collection of pairs consisting of $\mathcal{O}_F$-lattices of rank $r$ and their filtrations with the associated graded quotient ranks $r_1, r_2, \ldots, r_{|P|}$. Clearly, $\mathcal{E}_\cdot^{\delta g,P} = \mathcal{E}_\cdot^{g,\delta^{-1}P\delta}$ for all $g \in G(\mathbf{A})$ and $\delta \in G(F)$.

Let $p, q : [0, r] \to \mathbf{R}$ be two polygons. For any $P \in \mathcal{P}$, if

$$q(\mathrm{rank}E_i^P) > p(\mathrm{rank}E_i^P), \qquad i = 1, \ldots, |P|,$$

we say $p$ is bigger than $q$ with respect to $P$, and denote this as $q >_P p$. Moreover, introduce a canonical polygon $p_Q^g : [0, r] \to \mathbf{R}$ associated to $g \in G(\mathbf{A})$ and $Q \in \mathcal{P}$ as follows: Divide the interval $[0, r]$ into subintervals consisting of $[\mathrm{rank}\mathcal{E}_i^{g,Q}, \mathrm{rank}\mathcal{E}_i^{g,Q}]$ according to the partition of $r$ corresponding to $Q$; Then $p_Q^g$ is affine over all subintervals $[\mathrm{rank}\mathcal{E}i^{g,Q}, \mathrm{rank}\mathcal{E}_i^{g,Q}]$, and at the ends of the subintervals,

$$p_Q^g\Big(\mathrm{rank}(\mathcal{E}_i^{g,Q})\Big) := \deg(\mathcal{E}_i^{g,Q}) - \frac{\mathrm{rank}(\mathcal{E}_i^{g,Q})}{r}\cdot\deg(\mathcal{E}^g).$$

Also as usual, denote the characteristic function of $S$ by $\mathbf{1}_S$ for a subset $S$.

With this, we may list the fundamental properties of algebraic truncation as follows:

**Key Facts.** ($\alpha$) (Partial Canonical Polygon) *For all $g \in G(\mathbf{A})$, and $P \in \mathcal{P}$, the collection of polygons $p_Q^g$ associated to all $Q \in \mathcal{P}_P$ has a maximal element, which we denote by $\bar{p}_P^g$. Moreover, there is a parabolic subgroup in $\mathcal{P}_P$, which we denote by $\bar{Q}_P^g$, such that $p_{\bar{Q}_P^g}^g = \bar{p}_P^g$. Denote the associated filtration, the canonical filtration associated to $g$ and $P$ by $\bar{\mathcal{E}}^{.g,P}$;*
($\beta$) (Canonical Polygon) *For all $g \in G(\mathbf{A})$, the collection of polygons $p_Q^g$ associated to all $Q \in \mathcal{P}$ has a maximal element, which we denote by $\bar{p}^g$. Moreover, there is a parabolic subgroup, which we denote by $\bar{Q}^g$, such that $p_{\bar{Q}^g}^g = \bar{p}^g$. Denote the associated filtration, the canonical filtration associated to $g$, by $\bar{\mathcal{E}}^{.g}$;*
($\gamma$) (Compactness) *For any $t \in \mathbf{R}$ and polygon $p : [0, r] \to \mathbf{R}_+$, the subset*

$$\{g \in G(F)\backslash G(\mathbf{A})/K : \bar{p}^g \leq p\} \qquad \text{and hence} \qquad \{g \in G(F)\backslash G(\mathbf{A}) : \bar{p}^g \leq p\}$$

*are compact;*
($\delta$) (Partial Algebraic Truncation versus Geometric Truncation) *For any real cocharacter $T$ of $M_0$, introduce an associated polygon $p_T : [0, r] \to \mathbf{R}$ such that it is affine over $[r', r' + 1]$ for all $r = 0, \ldots, r - 1$ and $\left(\Delta^2\right) p_T(r' - 1) = (e_{r'} - e_{r'+1})(T)$ where $\Delta f(x) := f(x + 1) - f(x)$. Then*

$$\mathbf{1}\left(\Delta^2(p_P^g) >_P \Delta^2(p_T)\right) = \hat{\tau}_P(H_P(g) - T),$$

*where $\hat{\tau}_P(H(g) - T)$ is Arthur's truncation as recalled in II.2 below;*
($\psi$) (Global Algebraic Truncation versus Partial Algebraic Truncation) *For any polygon $p : [0, r] \to \mathbf{R}_+$,*

$$\mathbf{1}(\bar{p}^g \leq p) = \sum_{P \in \mathcal{P}_0} (-1)^{|P|-1} \sum_{\delta \in P(F)\backslash G(F)} \mathbf{1}(p_P^{\delta g} >_P p).$$

*Sketch of the proof.* ($\alpha$),($\beta$) come from the fact that for a fixed lattice, the collection of $\mu$-invariants of all its sublattices is discrete in $\mathbf{R}$. ($\gamma$) is clear as the volume one condition gives a fundamental domain via the ordinary reduction theory, while the stability is simply a finite closed bounded condition. Finally ($\delta$) is from the definition while the proof of $\psi$ for function fields of Lafforgue [Laf] works for number fields as well.

With the above discussion, we may introduce the following more general non-abelian zeta function for number field $F$: Let $p : [0, r] \to \mathbf{R}_{\geq 0}$ be a convex polygon which is symmetric with respect to the line $x = \frac{r}{2}$. Set

$$\xi_{F,r}^p(s) := \int_{\mathcal{M}_{\mathbf{A}_F,r}^{\leq p}} \left(e^{h^0(\mathbf{Q},g)} - 1\right) \cdot \left(e^{-s}\right)^{\deg(g)} d\mu(g), \qquad \mathrm{Re}(s) > r,$$

where $\mathcal{M}_{\mathbf{A}_F,r}^{\leq p}$ denotes the *adelic* moduli space of rank r lattices whose canonical polygons are bounded from above by $p$. One chects that $\xi_{F,r}^{\leq p}(s)$ is well-defined and satisfies all the fundamental properties of our non-abelian zeta functions. It is in fact a very interesting problem to understand such generalized non-abelian zeta functions: For example, *the structure of an analogy for function fields when p is sufficiently large is rather simple since the constant terms of Eisenstein series along parabolic subgroups coincide with the Eisenstein series itself.* (See e.g., [MW].) (In a certain sense, for the purpose of trace formula, what we care is the asymptotic behaviors of integrations for more general test; while for the non-abelian zetas, what we care is precise expressions for integrations of Eisenstein series.)

### II.2. Rankin-Selberg Method and Arthur's Analytic Truncation

Following Arthur [Ar1-6], consider only Arthur's analytic truncation over $\mathbf{Q}$ (but) for general reductive algebraic groups. For a more general discussion over arbitrary number fields, see, e.g., [MW].

Let $G$ be a reducitive algebraic group defined over $\mathbf{Q}$. Let $A_G$ be the split component of the center of $G$ and set $\mathfrak{a}_G = \mathrm{Hom}(X(G), \mathbf{R})$ where $X(G)$ is the group of characters of $G$ defined over $\mathbf{Q}$. Let $G(\mathbf{A})^1$ be the kernel of the map $H_G : G(\mathbf{A}) \to a_G$ defined by $< H_G(x), \xi >:= \log|\xi(x)|, x \in G(\mathbf{A}), \xi \in X(G)$. Then $G(\mathbf{Q})$ embeds diagonally as a discrete subgroup of $G(\mathbf{A})^1$.

Fix a minimal parabolic subgroup $P_0$ of $G$ with Levi component $M_0$ and unipotent radical $N_0$. Fix also a maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbf{A})^1$.

Associated to each standard parabolic subgroup $P$, i.e., those parabolic subgroups which contains $P_0$, is the geometric truncation $\hat{\tau}_P$: Write $a_P = a_{M_P}$ and $A_P = A_{M_P}$. If $Q$ is a parabolic subgroup that contains $P$, there is a natural map from $a_P$ onto $a_Q$. Denote its kernel by $a_P^Q \subset a_P$. Let $\Delta_P$ denote the set of simple roots of $(P, A_P)$. Naturally, $\Delta_P \subset a_P^* = X(M_P) \otimes \mathbf{R}$, the dual of $a_P$. To each $\alpha \in \Delta_P$, we have the associated co-root $\alpha^\vee \in a_P^G$. Let $\hat{\Delta}_P$ be the dual basis of $a_P^*/a_G^*$ of $\{\alpha^\vee : \alpha \in \Delta_P\}$. Then by definition $\hat{\tau}_P$ is the characteristic function of $\{H \in a_P : \omega(H) > 0, \omega \in \hat{\Delta}_P\}$.

Fix once and for all a suitably regular point $T \in a_0 = a_{P_0}$. (Recall that $T$ is suitably regular if $\alpha(T)$ is sufficiently large for all $\alpha \in \Delta_0 = \Delta_{P_0}$.) If $\phi$ is a continuous function on $G(\mathbf{Q})\backslash G(\mathbf{A})^1$, define Arthur's analytic truncation $(\Lambda^T\phi)(x)$ to be the function

$$\sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \phi(n\delta x)dn \cdot \hat{\tau}_P(H(\delta x) - T).$$

where $H_P$ is the continuous function from $G(\mathbf{A})$ to $a_P$ defined by $H_P(nmk) = H_{M_P}(m), n \in N_P(\mathbf{A}), m \in M_P(\mathbf{A}), k \in K$, the sum over $P$ is over all parabolic subgroups. One checks that if $\phi(x)$ is a cusp form, then $\Lambda^T\phi = \phi$ and if $\phi(x)$ is of slow growth in the sense that $|\phi(x)| \leq C\|x\|^N$ for some $C$ and $N$, then so is $\Lambda^T\phi(x)$. More generally, for a fixed $P_1$ and $\phi \in C(G(\mathbf{Q})\backslash G(\mathbf{A})^1)$, $\int_{N_1(\mathbf{Q})\backslash N_1(\mathbf{A})} \Lambda^T\phi(n_1 x)dn_1 = 0$ unless $\omega(H_0(x) - T) < 0$ for each $\omega \in \hat{\Delta}_1$. As direct consequences, we have $\Lambda^T\Lambda^T = \Lambda^T$ and $\Lambda^T$ is a self-dual operator.

Now recall some fact from the theory of Eisenstein series. Let $W = W_0$ be the restricted Weil group of $G$. Set $X$ to be the set of $W$-orbits of pairs $(M_B, r_B)$ where $B$ are standard parabolic subgroups of $G$ and $r_B$ are irreducible cuspidal automorphic representations of $M_B(\mathbf{A})^1$. For any given $\chi \in X$ let $\mathcal{P}_\chi$, an associated class of standard parabolic subgroups, be the set of groups $B$ appeared in the orbit $\chi$.

Suppose that $\chi \in X$ and $P \subset P_0$ are given. Let $L^2(N_P(\mathbf{A})M_P(\mathbf{Q})\backslash G(\mathbf{A})^1)_\chi$ be the space of functions $\phi \in L^2(N_P(\mathbf{A})M_P(\mathbf{Q})\backslash G(\mathbf{A})^1)$ with the following property: For every standard parabolic subgroup $B \subset P$, and almost all $x \in G(\mathbf{A})^1$, the projection of the function

$$\phi_{B,x}(m) := \int_{N_B(\mathbf{Q})\backslash N_B(\mathbf{A})} \phi(nmx)dn, \qquad m \in M_B(\mathbf{A})^1$$

onto the space of cusp forms in $L^2(M_B(\mathbf{Q})\backslash M_B(\mathbf{A})^1)$ transforms under $M_B(\mathbf{A})^1$ as a sum of representations $r_B$, in which the pair $(M_B, r_B)$ is in $X$. (If there is no such pairs in $X$, $\phi_{B,x}$ will be orthogonal to the space of cusp forms on $M_B(\mathbf{Q})\backslash M_B(\mathbf{A})^1$.)

**Facts.** (Langlands [La2]) (a) $L^2(N_P(\mathbf{A}M_P(\mathbf{Q})\backslash G(\mathbf{A})^1)_\chi = \{0\}$ *if there is no groups in* $P\chi$ *which are contained in* $P$*;*
(b) $L^2(N_P(\mathbf{A})M_P(\mathbf{Q})\backslash G(\mathbf{A})^1) = \oplus_{\chi \in X} L^2(N_P(\mathbf{A})M_P(\mathbf{Q})\backslash G(\mathbf{A})^1)_\chi$.

Denote by $F(M_0)$ the collection of parabolic subgroups of $G$ defined over $\mathbf{Q}$ and containing $M_0$. For any $P \in F(M_0)$, denote by $A^2(P)$ the space of $L^2$-automorphic forms on $N_P(\mathbf{A})M_P(\mathbf{Q})\backslash G(\mathbf{A})$ whose retriction to $M_P(\mathbf{A})^1$ is $L^2$ as well. For any $\phi \in A^2(P)$, define the associated Eisenstein series by

$$E(x, \phi, \lambda) := \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \phi(\delta x) e^{(\lambda + \rho_P)(H_P(\delta x))}, \qquad x \in G(\mathbf{A}).$$

Here $\rho_P \in a_P$ is the element such that the modular function $\delta_P(p) = |\det(\mathrm{Ad}p)_{n_P(A)}|, p \in P(\mathbf{A})$ on $P(\mathbf{A})$ equals $e^{2\rho_P(H_P(p))}$, where $n_P$ stands for the Lie algebra of $N_P$. $E(x, \phi, \lambda)$ converges for $\lambda$ in a certain chamber, and continuous analytically to a meromorphic function of $\lambda \in a_{P,\mathbf{C}}^*$. If $\chi \in X$ and $\pi \in \Pi(M_P(\mathbf{A}))$, the collection of equivalence classes of all irreducible unitary representations of $M_P(\mathbf{A})$, let $A^2_{\chi,\pi}(P)$ be the space of vectors $\phi \in A^2(P)$ such that:

(i) The restriction of $\phi$ to $G(\mathbf{A})^1$ is in $L^2(N_P(\mathbf{A}M_P(\mathbf{Q})\backslash G(\mathbf{A})^1)_\chi$;
(ii) For every $x \in G(\mathbf{A})$, the function $m \mapsto \phi(mx), m \in M_P(\mathbf{A})$ transforms under $M_P(\mathbf{A})$ according to $\pi$.

Let $\bar{A}^2_{\chi,\pi}(P)$ be the completion of $A^2_{\chi,\pi}(P)$ with respect to the inner product

$$(\phi,\psi) = \int_K \int_{M_P(\mathbf{Q})\backslash M_P(\mathbf{A})^1} \phi(mk)\overline{\psi(mk)}dm\,dk.$$

For each $\lambda \in a^*_{P,\mathbf{C}}$ there is an induced representation $\rho_{\chi,\pi}(P,\lambda)$ of $G(\mathbf{A})$ on $\bar{A}^2_{\chi,\pi}(P)$, defined by

$$\Big(\rho_{\chi,\pi}(P,\lambda)\,\phi\Big)(x) := \phi(xy)e^{(\lambda+\rho_P)(H_P(xy)-H_P(x))}.$$

One checks that $\rho_{\chi,\pi}$ is unitary if $\lambda$ is purely imaginary.

Given $P \subset P_0, \pi \in \Pi(M_P(\mathbf{A})), \lambda \in ia^*_P$ and a suitably regular $T \in a_0$, define an operator $\Omega^T_{\chi,\pi}(P,\lambda)$ on $A^2_{\chi,\pi}(P)$ by

$$(\Omega^T_{\chi,\pi}(P,\lambda)\phi,\psi) := \int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \Lambda^T E(x,\phi,\lambda)\overline{\Lambda^T E(x,\psi,\lambda)}dx$$

for any pair of vectors $\phi,\psi \in A^2_{\chi,\pi}(P)$. Naturally, we want to know how to evaluate the above inner product of Eisenstein series. As the formula for $SL_2$ suggests, this is a kind of Rankin-Selberg type calculation, for which a special case is derived by Arthur and Langlands.

More precisely, Langlands' case is for $P \in \mathcal{P}_\chi$. That is to say, when the Eisenstein series are *cuspidal.* To describe it, recall that if $P, P_1 \in F(M_0)$, $s \in W(a_P, a_{P_1})$, the set of isomorphisms from $a_P$ onto $a_{P_1}$ obtained by restricting elements in $W$ to $a_P$, and $\phi \in A^2(P)$, define the functional $M_{P_1|P}(s,\lambda)$ by

$$(M_{P_1|P}(s,\lambda)\phi)(x) := \int_{N_{P_1}(\mathbf{A})\cap w_s N_P(\mathbf{A})w_s^{-1}\backslash N_{P_1}(\mathbf{A})} \phi(w_s^{-1}nx)e^{(\lambda+\rho_P)(H_P(w_s^{-1}nx))-(s\lambda+\rho_{P_1})(H_{P_1}(x))}dn.$$

Here $w_s$ denotes the element in $G$ corresponding to $s$. This integral converges only for the real part of $\lambda$ in a certain chamber, but $M_{P_1|P}(s,\lambda)$ can be analytically continued to a meromorphic function of $\lambda \in a^*_{P,\mathbf{C}}$ with values in the space of linear maps from $A^2(P)$ to $A^2(P_1)$. Indeed, suppose $\pi \in \Pi(M_P(\mathbf{A}))$, $M_{P_1|P}(s,\lambda)$ maps $A^2_{\chi,\pi}(P)$ to $A^2_{\chi,s\pi}(P_1)$.

Now for $\lambda \in ia^*_P$, define $\omega^T_{\chi,\pi}(P,\lambda)$ to be the value at $\lambda = \lambda'$ of

$$\sum_{P_1\supset P_0}\ \sum_{t,t'\in W(a_P,a_{P_1})} M_{P_1|P}(t,\lambda)^{-1}M_{P_1|P}(t',\lambda')e^{(t'\lambda'-t\lambda)(T)}\theta_{P_1}(t'\lambda'-t\lambda)^{-1},$$

where

$$\theta_{P_1}(t'\lambda'-t\lambda)^{-1} = \mathrm{Vol}(a^G_p/\mathbf{Z}(\Delta^\vee_{P_1}))^{-1}\prod_{\alpha\in\Delta_{P_1}}(t'\lambda'-t\lambda)(\alpha^\vee).$$

Here $\mathbf{Z}(\Delta^\vee_{P_1})$ is the lattice in $a^G_P$ generated by $\{\alpha^\vee : \alpha \in \Delta_{P_1}\}$. Then $\omega^T_{\chi,\pi}(P,\lambda)$ is an operator on $A^2_{\chi,\pi}(P)$.

**Fact.** (Langlands [La1,2] and [Ar 3]) *If $P \in \mathcal{P}_\chi$,*

$$\Omega^T_{\chi,\pi}(P,\lambda) = \omega^T_{\chi,\pi}(P,\lambda). \tag{6}$$

*That is, we have an explicit formula for the inner product of the truncated Eisenstein series when $P \in \mathcal{P}_\chi$.*

Unfortunately, if $P \notin \mathcal{P}_\chi$, we may not have the above beautiful formula, as Arthur notices. However, Arthur, for the purpose of trace formula, proves the following elegant results.

Recall that $T \in a^*_{P,\mathbf{C}}$ is said to approach infinity strongly with respect to $P_0$ if $\|T\|$ approaches infinity but $T$ remains within a region $\{T \in a_0 : \min\{\alpha(T) : \alpha \in \Delta_0\} > \delta\|T\|\}$, for some $\delta > 0$.

**Fact.** (Arthur [Ar 4,5]) *If $\phi,\psi \in A^2_{\chi,\pi}(P)$, then $(\Omega^T_{\chi,\pi}(P,\lambda)\phi,\psi) - (\omega^T_{\chi,\pi}(P,\lambda)\phi,\psi)$ approaches zero as $T$ approaches infinity strongly with respect to $P_0$. The convergence is uniform for $\lambda$ in compact subset of $ia^*_P$. Moreover, by the analytic continuation, the above facts actually hold for all well-defined $\lambda a^*_{P,\mathbf{C}}$.*

## III. Where Non-Abelian Contributions Come

In Chapter II, we show that the rank two non-abelian zeta functions degenerate. In this chapter, we explain why this happens and use the example of rank three zeta functions to indicate where the non-abelian contributions come. Moreover, we show that at least to find the special values of rank 3 zeta functions, a Kronecker limit type formula using all terms of Fourier expansions is needed. As such the discussion here is rather practical. I hope I would come back to this point later together with a more theoretical approach.

### III.1. The Group $SL_3$

As indicated in II, the moduli of all rank three lattices of volume one may be viewed as the space $SL(3,\mathbf{Z})\backslash SL(3,\mathbf{R})/SO(3,\mathbf{R})$. We start with a description of several coordinates for $SL(3,\mathbf{R})/SO(3,\mathbf{R})$. For this, consider the following standard parabolic subgroups of $G = SL(3,\mathbf{R})$.

$P_0 = P_{1,1,1}$: the subgroup of $G$ consisting of all matrices of the form $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$;

$P_1 = P_{2,1}$: the subgroup of $G$ consisting of all matrices of the form $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$; and

$P_2 = P_{1,2}$: the subgroup of $G$ consisting of all matrices of the form $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$.

Write the corresponding Langlands decompositions as $P_i = N_i A_i M_i, i = 0,1,2$ where $N_i$ is the unipotent radical of $P_i$, $A_i$ is reducible and $M_i$ is simple. So,

$$M_0 = \left\{ I_3, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

More generally, if we denote the matrices of each subgroup by the corresponding lower-case letters. The subgroups above consists of the following elements:

$$n_0 = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix}; \quad a_0 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}; \qquad m_0 \in M_0;$$

$$n_1 = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & t_1 \\ 0 & 0 & 1 \end{pmatrix}; \quad a_1 = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1^{-1} \end{pmatrix}; \quad m_1 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot m_0;$$

$$n_2 = \begin{pmatrix} 1 & x_2 & t_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad a_2 = \begin{pmatrix} \alpha_2^{-2} & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}; \quad m_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \cdot m_0,$$

where $a_{ij}, x_i, t_i \in \mathbf{R}, a_{ii}, \alpha_i > 0$.

Note that by the Iwasawa decomposition with respect to $P_0$, we have $G = A_0^+ N_0 K$. Thus choose a coset $G/K$ amounts to choosing an element of $N_0$ and one of $A_0^+$, the identity component of $A_0$. Hence, identify $G/K$ with

$$\left\{ Y := \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & (y_1y_2)^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} : y_1, y_2 > 0, x_1, x_2, x_3 \in \mathbf{R} \right\}.$$

As such it is then convenient to introduce two coordinate systems according to the parabolic subgroups $P_1$ and $P_2$. In fact, notice that $M_1/M_1 \cap K \simeq SL(2,\mathbf{R})/SO(2,\mathbf{R})$ so natural coordinates for $G/K$ are given by

$$\begin{pmatrix} u_1^{1/2} & v_1 u_1^{-1/2} & 0 \\ 0 & u_1^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1^{-2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & t_1 \\ 0 & 0 & 1 \end{pmatrix}$$

where $z_1 = v_1 + iu_1$ can be regarded as a point in the Poincare upper half plane. Similarly, consideration of $P_2$ yields coordinates

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & u_2^{1/2} & v_2 u_1^{-1/2} \\ 0 & 0 & u_1^{-1/2} \end{pmatrix} \cdot \begin{pmatrix} \alpha_2^{-2} & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & t_2 & x_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $y_i = \alpha_i^6$, $i = 1, 2$ then a Haar measure on $G/K$ may be given in terms of Langlands coordinates as follows

$$d\mu = \frac{dy_1}{y_1^2}\frac{dz_1}{u_1^2}\,dx_1\,dt_1 = \frac{dy_2}{y_2^2}\frac{dz_2}{u_2^2}\,dx_2\,dt_2$$

where $z_1 = v_1 + iu_1$ and $z_2 = v_2 + iu_2$.

Let $\Gamma = \mathrm{SL}(3, \mathbf{Z})$ acting on $G/K$, and $\mathcal{D}$ be a fundamental domain for $\Gamma$. Then by the theory of Eisenstein series,

$$L^2(\Gamma\backslash G/K) = \mathbf{H}_0 \oplus \Theta_0^{(1)} \oplus \Theta_0^{(2)} \oplus \Theta_{1,2}^{(2)}$$

where $H_0$ denotes the cusp forms of $\Gamma$, while the Theta's may be defined as follows using Eisenstein series:

Associated to minimal parabolic subgroup $P_0$ we have the Eisenstein series

$$E^0(Y; s, t) := \sum_{\gamma\in P_0\cap\Gamma\backslash\Gamma} y_1(\gamma Y)^s u_1(\gamma Y)^t. \tag{7}$$

It is known that this series converges when $3\mathrm{Re}(s) - \mathrm{Re}(t) > 2, \mathrm{Re}(t) > 1$ and admits a meromorphic to the whole $(s, t)$-space. Despite that there are many poles, but these which are of some interests to us are on the lines $t = 1, 3s - t = 2, 3s + t = 3$. The residues at these poles are meromorphically continued Eisenstein series of one variable and generate the closed subspace $\Theta^{(1)}$. One checks that $\Theta_0^{(2)}$ is simply the span of $E^0(Y, 1/2 + ir_1, 1/2 + ir_2)$.

Now, let $\phi$ be an even cusp forms for $SL(2, \mathbf{Z})$ on the upper half-plans. Set

$$E_i(Y; \phi; s) := \sum_{P_i\cap\Gamma\backslash\Gamma} y_i(\gamma Y)^s \cdot \phi(z_i(\gamma Y)), \qquad i = 1, 2. \tag{8}$$

These series converge for $\mathrm{Re}(s) > 1$ and have meromorphic extension on the whole $s$-plane which has no poles on the line $(1/2, 1]$. So the space $\Theta_{1,2}$ generated by $E_i(Y; \phi; s), i = 1, 2$ for all $\phi$ coincides with $\Theta_{1,2}^{(2)}$, the closed space spanned by $E_i$ along the line $\mathrm{Re}(s) = 1/2$. Indeed, one may also hav a refined orthogonal decomposition of $\Theta_{1,2}^{(2)}$ according to that of $\phi$. For details, see [Venkov].

### III.2. Fourier Expansions

To go further, we need to understand the Fourier expansion of Eisenstein series near cusps. However, before that let us briefly discuss the relation between the above general theory of Eisenstein series and the Epstein zeta function used in our construction of non-abelian zeta functions. (In fact, to have a completed theory, we should equally use the algebraic truncation and general Eisenstein series to define a more general type of non-abelian $L$-functions.) The main references are [IT], [T] and [V]. A parallel discussion may also be carried out by using Whittaker functions (see e.g. [Bu]).

It is the space $\Theta_0^{(1)}$ which is of interests to us. In fact, two types of functions are used: the constant functions and the Epstein zeta functions. It is quite clear why Epstein zeta function is needed: the integration of a single Epstein zeta function may be viewed as an inner product of it with the constant functions.

Thus it suffices to study the Eisenstein series $E^0(Y; s, t)$ of the highest level. Recall that $E^0(Y; s, t)$ as in (7) may also be written in the style of (8) as follows:

$$E^0(Y; s; t) = E^0(Y; E(z_1, t); s) := \sum_{\gamma\in\mathrm{SL}(3,\mathbf{Z})\cap P_1\backslash\mathrm{SL}(3,\mathbf{Z})} E(v_1(\gamma Y) + iu_1(\gamma Y); t) \cdot y_1(\gamma Y)^s,$$

where $E(z,s)$ denotes the standard Eisenstein series appeared in I.3. So after taking the residues on either $s$ or $t$ (resp. on $s$ and $t$), we get naturally the Epstein zeta function (resp. constant functions).

Let us start with the simplest terms, i.e., the so-called constant terms appeared in the Fourier expansion for the cusps. As the cusps correspond to parabolic subgroups of $G$. Thus for an automorphic function $f(Y)$, set

$$f_{P_j}(Y) := \int_{\Gamma\cap N_j\backslash N_j} f(nY)dn, \qquad j=0,1,2,$$

the constant term along $P_j$. Set also

$$c(s) := \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\cdot\frac{\zeta(2s-1)}{\zeta(2s)}.$$

**Proposition.** (Venkov[V]) *With the same notation as above,*

$$\begin{aligned}
E^0_{P_0}(Y;s,t) =& y_1^s u_1^t + y_1^s u_1^{1-t} c(\frac{3s-t}{2}) + y_1^{1-s/2-t/2} u_1^{1-3/2s+t/2} c(t)c((\frac{3s-t}{2})c(\frac{3s+t-1}{2})\\
&+ y_1^{1/2(1-s-t} u_1^{1/2(3-3s-t)} c(\frac{3s-t}{2})c(\frac{3s+t-1}{2}) + y_1^{1-1/2s-1/2t} u_1^{3s/2-t/2} c(t)c(\frac{3s+t-1}{2});\\
E^0_{P_1}(Y;s,t) =& y_1^s E(z_1,t) + y_1^{1/2(1-s-t)} c(\frac{3s-t}{2})E(z_1,\frac{3s+t-1}{2})\\
&+ y_1^{1-s/2-t/2} c(t)c(\frac{3s+t-1}{2})E(z_1,\frac{3-3s-t}{2});\\
E^0_{P_2}(Y;s,t) =& y_2^s E(z_2,t) + y_2^{1/2(1-s-t)} c(\frac{3s-t}{2})E(z_2,\frac{3s+t-1}{2})\\
&+ y_2^{1-s/2-t/2} c(t)c(\frac{3s+t-1}{2})E(z_2,\frac{3-3s-t}{2}).
\end{aligned}$$

For the proof see, e.g., that of Lemmas 2 and 8 of [Venkov].

Next, let us recall the Fourier expansions of $E^0(Y;s,t)$ along the parabolic subgroups $P_1$ and $P_2$ due to Imai and Terras. (In theory, we should also know the Fourier expansion along $P_0$. However, as the later calculation shows, by an induction on the rank, to see the non-abelian contributions, we need not to have detailed information about such an expansion: the terms involved will finally lead to a combination of abelian zeta functions by reducing to the case discussed in Chapter I.) For this, view the rank lattice of volume one as positive quadratic forms of determinant 1, write

$$Y = \begin{pmatrix} U & 0\\ 0 & w\end{pmatrix}\begin{bmatrix} I_2 & x\\ 0 & 1\end{bmatrix} = \begin{pmatrix} I_2 & 0\\ x^t & 1\end{pmatrix}\cdot\begin{pmatrix} U & 0\\ 0 & w\end{pmatrix}\cdot\begin{pmatrix} I_2 & x\\ 0 & 1\end{pmatrix}$$

and define the first type of matrix k-Bessel function to be

$$k_{2,1}(Y;s_1,s_2;A) := \int_{X\in\mathbf{R}^{2\times1}} p_{-s_1,-s_2}\left(Y^{-1}\begin{bmatrix} 1 & 0\\ x^t & I_2\end{bmatrix}\right)\exp\left(2\pi i\mathrm{Tr}(A^t\cdot X)\right)dX$$

for $(s_1,s_2)\in\mathbf{C}^2$, $Y\in\mathcal{SP}_3, A\in\mathbf{R}^{2\times1}$ and $p_{s_1,s_2}(Y) := |Y_1|^{s_1}|Y_2|^{s_2}$ where $Y_j\in\mathcal{SP}_j$ is the $j\times j$ upper left hand corner in $Y$, $j=1,2$. Here as usual, we denote by $\mathcal{SP}_n$ the collection of rank $n$ positive quadratic forms of determinant 1. Set also

$$\alpha_0 = \frac{\Lambda(s,r)}{B(\frac12,\frac12-r)},\qquad \alpha_0' = \frac{\Lambda(s,r)}{B(\frac12,r-\frac12)},\qquad \alpha_{k\neq0} = \Lambda(s,r)\frac{\sigma_{1-2r}(k)}{\zeta(2r)},$$

$$\begin{aligned}
\Lambda(s,r) =& \pi^{-(s-\frac{r}{2})}\Gamma(s-\frac{r}{2})\pi^{-(s-\frac{1-r}{2})}\Gamma(s-\frac{1-r}{2}),\\
c(s,r) =& \xi(2r)\xi(2s-r)\xi(2s-1+r)\cdot E(\frac{U}{\sqrt{|U|}};r)|U|^{-s}; \qquad \text{with}
\end{aligned}$$

$$E(V;r) = \frac12\sum_{\gcd(\mathrm{a})=1} V[\mathrm{a}]^{-r}, \qquad \mathrm{Re}(r)>1.$$

**Proposition.** ([IT]) *With the same notation as above, we have*

$$
\begin{aligned}
&\Lambda(s,r)\cdot E^0\Big(\begin{pmatrix}U&0\\0&w\end{pmatrix}\begin{bmatrix}I_2&x\\0&1\end{bmatrix};r,s\Big)\\
=&c(s,r)+c(\frac{6-2s-3r}{4},s-\frac{r}{2})+c(\frac{3+3r-2s}{4},s-\frac{1-r}{2})\\
&+\sum_{A\in \mathrm{SL}(2,\mathbf{Z})/P(1,1)}\Big(\sum_{c,d_2\in\mathbf{Z}_{>0},d_1\in\mathbf{Z}\backslash\{0\}}\\
&\Big[\alpha_0' c^{2-2s-r}d_2^{r-2s}\exp\Big(2\pi i x^tA\cdot\begin{pmatrix}cd_1\\0\end{pmatrix}\Big)\cdot k_{2,1}(\begin{pmatrix}A^{-1}UA^{-t}&0\\0&w\end{pmatrix};s-\frac{r}{2},r;\pi\begin{pmatrix}cd_1\\0\end{pmatrix})\\
&+\alpha_0 c^{1-2s+r}d_2^{1-r-2s}\exp\Big(2\pi i x^tA\begin{pmatrix}cd_1\\0\end{pmatrix}\Big)\cdot k_{2,1}\Big(\begin{pmatrix}A^{-1}UA^{-t}&0\\0&w\end{pmatrix};s-\frac{1-r}{2},1-r;\pi\begin{pmatrix}cd_1\\0\end{pmatrix}\Big)\Big]\\
&+\sum_{k\neq0}\sum_{c,d_2\in\mathbf{Z}_{>0},d_2|k,d_1\in\mathbf{Z}\backslash\{0\}}\\
&\alpha_k c^{2-2s-r}d_2^{r-2s}\exp\Big(2\pi i x^tA\begin{pmatrix}cd_1\\ck/d_2\end{pmatrix}\Big)\cdot k_{2,1}\Big(\begin{pmatrix}A^{-1}UA^{-t}&0\\0&w\end{pmatrix};s-\frac{r}{2},r;\pi\begin{pmatrix}cd_1\\ck/d_2\end{pmatrix}\Big)\Big),
\end{aligned}
$$

*where $P(1,1)$ is the subgroup of upper triangle matrices of determinant 1. Similar Fourier expansion holds for $E^0(E(z;s),t)$ with respect to $P_2$.*

### III.3. Non-abelian Contributions

To give a precise expression for the rank 3 non-abelian zeta functions for $\mathbf{Q}$, by definition, what we need to do is the follows:
1) Give a concrete description of $\mathcal{M}_{\mathbf{Q},3}[1]$ as a closed subset of a certain fundamental domain of $\mathrm{SL}(3,\mathbf{Z})$; and
2) Calculate the integration of the Epstein zeta function over $\mathcal{M}_{\mathbf{Q},3}[1]$.

However, as the details are much more complicated, we in this paper only indicate the key points for doing so. (The reader who wants to know how complicated it would be may turn to the paper of Venkov on the Trace Formula for $\mathrm{SL}(3,\mathbf{Z})$, where only the so-called dominate terms, i.e, the principal asymptotic terms nearing the cusps of type $P_{2,1}$ are calculated: the formulas run pages even there.)

First, for simplicity, consider the geometric truncated fundamental domain of $\Gamma:=\mathrm{SL}(3,\mathbf{Z})$ obtained by cutting off the cusp regions corresponding to $P_1,P_2$ and $P_0$.

More precisely, put $\Gamma_j=\Gamma\cap P_j, j=0,1,2$ and $\Gamma_{N_0}=\Gamma\cap N_0$. Then the fundamental domain $F_*$ in $S:=\mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})$ for the groups $*=\Gamma_0,\Gamma_1,\Gamma_2,\Gamma_{N_0}$ may be choosen to be

$$
\begin{aligned}
F_{N_0}:=&\{Y\in S: y_1>0,u_1>0,-1/2<v_1,x_1,t_1<1/2\};\\
F_0:=&\{Y\in F_{N_0}: v_1+x_1>0,v_1+t_1>0,x_1+t_1>0\};\\
F_j:=&\{Y\in F_0: v_j^2+u_j^2\geq 1\},\qquad j=1,2.
\end{aligned}
$$

With this, by a discussion following Selberg, (see e.g. Thm 7 in [V],) we know that there exists a compact set $F^0\subset S$ such that

$$F_1\cap F_2=F^0\cup F$$

where $F$ denotes the fundamental domain of $\Gamma$. That is to say, the cusp regions for $P_j$, $j=0,1,2$ in the fundamental region $F$ of $\mathrm{SL}(3,\mathbf{Z})$ may be read from $F_1$ and $F_2$.

As usual, we may then introduce a geometric truncated compact subset in $F$ by cutting off the neighborhood of cusps along $F_1$ and $F_2$, so as to get $F_T=F\backslash(D_1^T\cup D_2^T)$. Note that $D_0^T:=D_1^T\cap D_2^T$ gives a neighborhood for the cusps with respect to $P_0$. Thus, we may analytically understand this geometric truncation as

$$\mathbf{1}_{F_T}=\mathbf{1}_F-\mathbf{1}_{D_1^T}-\mathbf{1}_{D_2^T}+\mathbf{1}_{D_0^T},$$

which is compactible with the truncations in Chapter II.

Secondly, let us simply look at the contributions of standard parabolic subgroups so as to get the analytic truncation

$$\begin{aligned}
&\Lambda_T E^0(Y;s,t): \\
=&E^0(Y;s,t) - E^0_{P_1}(Y;s,t)\cdot 1_{D_1^T} - E^0_{P_2}(Y;s,t)\cdot 1_{D_2^T} + E^0_{P_0}(Y;s,t)\cdot 1_{D_0^T} \\
=&\Big(\big(E^0(Y;s,t) - E^0_{P_1}(Y;s,t)\cdot 1_{D_1^T}\big) + \big(E^0(Y;s,t) - E^0_{P_2}(Y;s,t)\cdot 1_{D_2^T}\big)\Big) \\
&\quad - \Big(E^0(Y;s,t) - E^0_{P_0}(Y;s,t)\cdot 1_{D_0^T}\Big) \\
=&H^0_{P_1}(Y;s,t) + H^0_{P_2}(Y;s,t) - H^0_{P_0}(Y;s,t).
\end{aligned}$$

Here

$$H^0_{P_j}(Y;s,t) := E^0(Y;s,t) - E^0_{P_j}(Y;s,t)\cdot 1_{D_j^T}, \qquad j=1,2,0$$

denotes the non-constant part of the corresponding Fourier expansion.

Thirdly, we want to know the integration $\int_{F_T} E^0(Y;s,t)d\mu(Y)$. For this, we go as follows:

$$\begin{aligned}
\int_{F_T} E^0(Y;s,t)d\mu(Y) &= \int_F \Lambda^T E^0(Y;s,t)d\mu(Y) - \int_{F\backslash F_T} \Lambda^T E^0(Y;s,t)d\mu(Y) \\
&= \int_F \Lambda^T E^0(Y;s,t)d\mu(Y) - \int_{F^T} \Lambda^T E^0(Y;s,t)d\mu(Y),
\end{aligned}$$

where $F^T := F\backslash F_T = D_1^T \cup D_2^T$.

Finally, let us look at the structure of this latest expression:

(A) (*Abelian Term: Application of Rankin-Selberg Method*) By the Rankin-Selberg method, in particular, the version generalized by Langlands and Arthur recalled in II, the part $\int_F \Lambda^T E^0(Y;s,t)d\mu(Y)$, being the integration of analytic truncation of Eisenstein series on the whole fundamental domain of $\mathrm{SL}(3,\mathbf{Z})$, is essentially abelian;

Thus, it suffices to know the structure of $\int_{F^T} \Lambda^T E^0(Y;s,t)d\mu(Y)$. Clearly,

$$\begin{aligned}
&\int_{F^T} \Lambda^T E^0(Y;s,t)d\mu(Y) \\
=&\int_{D_1^T} \Lambda^T E^0(Y;s,t)d\mu(Y) + \int_{D_2^T} \Lambda^T E^0(Y;s,t)d\mu(Y) - \int_{D_0^T} \Lambda^T E^0(Y;s,t)d\mu(Y) \\
=&\int_{D_1^T} \Big(H^0_{P_1}(Y;s,t) + H^0_{P_2}(Y;s,t) - H^0_{P_0}(Y;s,t)\Big)d\mu(Y) \\
&\quad + \int_{D_2^T} \Big(H^0_{P_1}(Y;s,t) + H^0_{P_2}(Y;s,t) - H^0_{P_0}(Y;s,t)\Big)d\mu(Y) \\
&\qquad - \int_{D_0^T} \Big(H^0_{P_1}(Y;s,t) + H^0_{P_2}(Y;s,t) - H^0_{P_0}(Y;s,t)\Big)d\mu(Y) \\
=&I_1^T(s,t) + I_2^T(s,t) - I_0^T(s,t),
\end{aligned}$$

where

$$I_j^T(s,t) := \int_{D_j^T} \Big(H^0_{P_1}(Y;s,t) + H^0_{P_2}(Y;s,t) - H^0_{P_0}(Y;s,t)\Big)d\mu(Y), j=0,1,2.$$

(B) (*Terms obtained from Lower Rank Non-Abelian Zeta: Induction on the Rank*) Consider the integrations

$$I_j^T(s,t) := \int_{D_1^T} \Big(H^0_{P_1}(Y;s,t) + H^0_{P_2}(Y;s,t) - H^0_{P_0}(Y;s,t)\Big)d\mu(Y), j=0,1,2.$$

If the fundamental domain $F$ is chosen so that $F$ is of exact box shape as $Y$ approaches to all levels of cusps, we have

$$\int_{D_j^T} H_{P_j}(Y;s,t)=0.$$

(This is possible by a result of Grenier [G] as also recalled in [T]. From now on, we always assume this condition for the fundamental domain.) Then what left is to consider the following integrations:

$$\mathrm{II}_1^T(s,t):=\int_{D_1^T}\Big(H_{P_2}^0(Y;s,t)-H_{P_0}^0(Y;s,t)\Big)d\mu(Y);$$

$$\mathrm{II}_2^T(s,t):=\int_{D_2^T}\Big(H_{P_1}^0(Y;s,t)-H_{P_0}^0(Y;s,t)\Big)d\mu(Y);$$

$$\mathrm{III}^T(s,t):=\int_{D_0^T}\Big(H_{P_1}^0(Y;s,t)+H_{P_2}^0(Y;s,t)\Big)d\mu(Y).$$

With this, we see that $\mathrm{II}_i^T(s,t)$ is in fact essentially a rank two zeta functions, which may be understood via an induction argument. So we are left with only

$$\mathrm{III}^T(s,t):=\int_{D_0^T}\Big(H_{P_1}^0(Y;s,t)+H_{P_2}^0(Y;s,t)\Big)d\mu(Y),$$

which in the case of rank 3, is the only essential non-abelian contribution.

(C) (*Essential Non-abelain Contributions: New Ingredients*) The evaluation of the integration $\mathrm{III}^T(s,t)$ is rather difficult: what we should do is to calculate the integration of all non-constant terms of the Fourier expansion of $E^0(Y;s,t)$ with respect to $P_1$ and $P_2$ for the cusp region corresponding to that for $P_0=P_1\cap P_2$. By the result of Imai and Terras cited above, these coefficients consist of matrix version of k-Bessal functions. So an impossible mission.

On the other hand, to finally get our non-abelian zeta functions, what we need is not the integration of $E^0(Y;s,t)$, we still need to take residues with respect to the $t$ variable. Indeed, what we discuss here is the integration for the Eisenstein series $E^0(Y;s,t)$, while what is used in Prop. I.1 for non-abelain zeta functions is the integration for the Epstein zeta functions associated to maximal parabolic subgroups. So at this level of discussion, it is then much better to directly use the Fourier expansion of Epstein zeta function, a special kind of Eisenstein series:

For any $Y\in\mathcal{P}_n$, set

$$E_n(Y;s):=\frac{1}{2}\sum_{\mathbf{a}\in\mathbf{Z}\backslash\{0\}}(Y[\mathbf{a}])^{-s},\qquad \mathrm{Re}(s)>\frac{n}{2}.$$

Then we have the following result of Berndt and Terras:

**Proposition.** ([B] & [T]) *With the same notation as above, if* $Y=\begin{pmatrix}V&0\\0&W\end{pmatrix}\begin{bmatrix}I&X\\0&I\end{bmatrix}$ *with* $V\in\mathcal{P}_m, W\in\mathcal{P}_{n-m}$*, then*

$$\begin{aligned}&\pi^{-s}\Gamma(s)E(Y;s)\\=&\pi^{-s}\Gamma(s)E_m(V;s)+\pi^{-s}\Gamma(s)\cdot|V|^{-1/2}E_{n-m}(W;s-m/2)\\&+|V|^{-1/2}\sum_{\mathbf{b}\in\mathbf{Z}^m\backslash\{0\},\mathbf{c}\in\mathbf{Z}^{n-m}\backslash\{0\}}\exp\left(2\pi i\mathbf{b}^tX\mathbf{c}\right)\cdot\Big(\frac{V^{-1}[\mathbf{b}]}{W[\mathbf{c}]}\Big)^{2s-m)/4}\cdot K_{s-m/2}(2\pi\sqrt{V^{-1}[\mathbf{b}]\cdot W[\mathbf{c}]}),\end{aligned}$$

*where $K_s$ denotes the K-Bessel function.*

Thus, by taking $n=3$ and $m=1,2$, we see that the non-constant terms of the Fourier expansions of $E_3(Y;s)$ are given in terms of $K$-Bessel functions $K_{s-\frac{1}{2}}$ and $K_{s-1}$. It is the integration of these terms over $D_0^T$ that gives the essential non-abelian contribution to our rank three zeta functions. From here we also expect that a kind of Kronecker limit formula holds for our non-abelian zeta functions.

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