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A new stage of vertex operator algebra

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1 Introduction

Historically, a conformal field theory is a mathematical method for physical phenomena, for example, a string theory. In a string theory, a particle (or a string) is expressed by a simple module. Physicists constructed many examples of conformal field theories whose simple modules were explicitly determined. Such conformal field theories are called solvable models. It is natural to divide theories into two types. One has infinitely many kinds of particles or simple modules, the other has only finitely many simple modules. For example, a conformal field theory for free bosons has infinitely many simple modules and lattice theories are of finite type. In 1988, in connection with the moonshine conjecture (a mysterious relation between the largest sporadic finite simple group "Monster" and the classical elliptic modular function $j(\tau)=q^{-1}+744+196884q+\cdots$), a concept of vertex operator algebra was introduced as a conformal field theory with a rigorous axiom. In this paper, we will treat a vertex operator algebra of finite type.

Until 1992, in the known theories of finite type, all modules were completely reducible. This fact sounds natural for physicists, because a module in the string theory was thought as a bunch of strings, which should be a direct sum of simple modules. At this stage, one of the most important methods for conformal field theory, modular invariance or $SL_2(\mathbb{Z})$-invariance of the set of characters, was proved by Zhu for vertex operator algebra under the assumptions of the completely reducibility of modules and some technical condition which is now called C_2-finiteness. He has also introduced a powerful method, Zhu algebra $A(V)$, which determines all simple modules.

However, after that, physicists have constructed more examples and found strange VOAs. The first one was found in 1992, but went unnoticed. The second was found in

1995 and several examples succeeded and then have begun to make a mark. Physicists are trying to understand physical meanings of these models (for example, gravitationally dressed conformal field theory, etc). Anyway, these models are vertex operator algebras of finite type, but some module is not completely reducible and the set of characters is not $SL_2(\mathbb{Z})$-invariant. I will show you an example later. (Garberdiel, etc.)

The results in this paper are, roughly speaking, in any finite models, it doesn't matter whether modules are completely reducible or not:
(1) Modular invariance property holds.
(2) C_2-finite condition is a natural condition.
(3) Extend Zhu algebra $A_n(V)$, but not Zhu algebra, plays an important role.

2 Notation

Let me explain notation and terms. Let V be a VOA $(V, Y, \mathbf{1}, \omega)$. If the reader is not familiar with VOA, just consider it as a $\mathbb{Z}_+$-graded vector space

$$V = \oplus_{n=0}^{\infty} V_n$$

with infinitely many products $\times_n$ $(n \in \mathbb{Z})$ and $\mathbf{1}$ and ω are two special elements of V.

The following is a short introduction of VOA. For any $v \in V$, we have infinitely many endomorphism $v_n := v \times_n$ of V and we denote them $Y(v, z) = \sum v_n z^{-n-1}$ by using formal variable z and call it a **vertex operator** of v on V. One of axiom of VOA is locality:

$$\forall v, u \in V, \exists N \in \mathbb{Z} \text{ s.t. } (z - x)^N \{Y(v, z)Y(u, x) - Y(u, x)Y(v, z)\} = 0.$$

Moreover, V has two special elements $\mathbf{1} \in V_0$ and $\omega \in V_2$. $\mathbf{1}$ (called **Vacuum**) is corresponding to identity $Y(\mathbf{1}, z) = 1_V$, and the coefficients of vertex operator $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ satisfy Virasoro algebra relation

$$[L(m), L(n)] = (m - n)L(m + n) + \delta_{m+n,0} \binom{m+1}{3} \frac{c}{2}$$

with $c \in \mathbb{C}$ (called central charge of V) such that $\omega_1 = L(0)$ defines a grading and $\omega_0 = L(-1)$ is a differential operator

$$[L(-1), Y(v, z)] = \frac{d}{dz} Y(v, z).$$

Similarly, a module W is a Z_+-graded vector space $W = \oplus_{m=0}^{\infty} W(m)$ and for each $v \in V$, operator v_m on W satisfies the similar conditions as the operators on V do.

A V-module is a Z_+-graded vector space $W = \oplus_{m=0}^{\infty} W(m)$ such that for any $v \in V$, we have infinitely many endomorphisms v_n^W of W ($n \in \mathbb{Z}$) and $Y^W(v,z) = \sum_{n\in\mathbb{Z}} v_n^W z^{-n-1}$ satisfies the same properties as $Y(v,z)$ does. Our finiteness condition assures that if W is simple, then $\dim W(m) < \infty$ and the grading $L(0) := \omega_1^W$ is a scalar $r+m$ on $W(m)$ with some $r \in \mathbb{C}$. Among the endomorphisms $\langle v_n^W \mid n \in \mathbb{Z}\rangle$, there is a grade preserving operator $o(v)$ of W. Then trace function is given by

$$S^W(v,\tau) = \sum_{m=0}^{\infty} (\mathrm{tr}_{|W(m)} o(v)) q^{r+m-c/24}$$

where $q = exp(2\pi i \tau)$. In particular, using $o(\mathbf{1}) = 1_W$, we have a character of W

$$S^W(\mathbf{1},\tau) = \sum_{m=0}^{\infty} (\dim(W(m)) q^{r+m-c/24}.$$

For example, character of the moonshine VOA $V^\natural$,

$$S^{V^\natural}(\mathbf{1},\tau) = J(\tau) = q^{-1} + 196884q + \cdots$$

is J-function. If we have any even positive lattice L and a coset $x + L \subseteq \mathbb{Q}L$, then there is a lattice VOA V_L and its (twisted) module V_{L+x} whose character is

$$S^{V_{L+x}}(\mathbf{1},\tau) = (\frac{1}{\eta(\tau)})^c \theta_{L+x}(\tau),$$

where $\eta(\tau)$ is Dedekind eta-function and $\theta_{L+x}(\tau)$ is the theta-function of $L+x$.

2.1 Zhu algebra

We next explain n-th Zhu algebra $A_n(V)$ and its role. For V-module W, $o(v)$ acts on n-th homogeneous part $W(n)$. Define $O_n(V) \subseteq V$ by

$$v \in O_n(V) \Leftrightarrow o(v) = 0 \text{ on } W(n) \text{ for all } V\text{-modules } W = \oplus_{m=0}^{\infty} W(m).$$

We are able to define a product $*$ by

$$o(v * u) = o(v) \times o(u) \text{ on } W(n) \text{ for all } V\text{-modules}$$

(Axioms of VOA assures the existence of such an element $v * u$). These are all well-defined and the factor space

$$A_n(V) = V/O_n(V)$$

becomes an algebra. It is clear that the n-th piece $W(n)$ of W is a module of n-th Zhu algebra $A_n(V)$. We should note that the real definition of n-th Zhu algebra is given by V itself, but not modules.

An excellent property of n-th Zhu algebra is the converse, that is, if T is an $A_n(V)$-module, then by generating from T by the formal actions of V and divided by relations (truncations, locality and associativity), we have a V-module W whose n-th piece is T. In particular, if V has only finitely many simple modules, then there is one to one correspondence between $A_n(V)$-modules and V-modules for a sufficiently large n.

Originally, Zhu introduced $A(V) = A_0(V)$ and Dong, Li and Mason extended it to n-th Zhu algebras.

Definition 1 *Define*

$$C_2(V) = \langle v \times_{-2} u \mid v, u \in V \rangle.$$

*V is called C_2-***cofinite** *if* $\dim V/C_2(V) < \infty$.

A modular invariance property that Zhu proved is:

Theorem 1 *If all modules are completely reducible (or $A(V)$ is semisimple) and V is C_2-cofinite, then trace functions are all holomorphic function on the upper half plane and the set of all trace function of v is $SL_2(Z)$-invariant, that is,*

$$\langle S^W(v,\tau) \mid W \text{ all simple modules} \rangle$$

is $SL_2(\mathbb{Z})$-invariant for $v \in V$.

Here modular transformation is given by

$$S^W|\begin{pmatrix} ab \\ cd \end{pmatrix}(v,\tau) = \frac{1}{(c\tau+d)^n} S^W(v, \frac{a\tau+b}{c\tau+d})$$

if the weight of v is n.

2.2 Conformal block of torus and Zhu's result

Zhu introduced **conformal block $\mathcal{C}_1(V)$ on torus** by the set of family of functions $S(*,*) : V \times \mathcal{H} \to \mathbb{C}$ satisfying

(1) $S(v,\tau)$ is a holomorphic function on $\mathcal{H}$ for $v \in V$

(2) $u \in O_q(V) \Rightarrow S(u,\tau) = 0$

(3) $S(L(-2)u,\tau) = \frac{1}{2\pi i}\frac{d}{d\tau}S(u,\tau) + \sum_{k=2}^{\infty} E_{2k}(\tau)S(L(2k-2)u,\tau)$.

Here $O_q(V) = \langle v \times_0 u, v \times_{-2} u + \sum_{k=2}^{\infty}(2k-2)E_{2k}(\tau) \mid v, u \in V\rangle$ and $E_{2k}(\tau)$ denotes Eisenstein series.

It is easy to see:

Proposition 1 *$C_1(V)$ is $SL_2(\mathbb{Z})$-invariant*

Zhu showed the space spanned by trace functions on simple modules is equal to conformal block.

Theorem 2 (Zhu) $\langle S^W(*,\tau) | W$ *simple modules*$\rangle = \mathcal{C}_1(V)$

3 General Case

If some module is not completely reducible, the space of trace function is not necessary to be equal to conformal block.

$$\langle S^W(*,\tau) | W \text{ simple}\rangle \subsetneq \mathcal{C}_1(V)$$

What is the difference?
This is my motivation of today's result. To fill a gap, I introduce "interlocked module" and pseudo-trace function pstr, which are defined by a symmetric linear function of n-th Zhu algebra $A_n(V)$. My main result is that if V is of finite type, then the conformal block is equal to the space of pseudo-trace functions on interlocked modules (including all simple modules). In particular, the conformal block is isomorphic to the space of symmetric linear functions of n-th Zhu algebra for some n.

Namely, setting

$$S^W(v,\tau) = \mathbf{pstr}_W o(v) q^{L(0)-c/24}$$

for an interlocked module W, we have:

Theorem 3 *(Main Theorem) If V is of finite type, then*

$$\langle S^W(*,\tau) | W \textit{ interlocked modules}\rangle = \mathcal{C}_1(V)$$

In particular, $\mathcal{C}_1(V) \cong$ space of symm. func. of $A_n(V)$ for a sufficiently large n.

In order to get these results, the assumptions we need are :
(1) trace function should be well-defined and
(2) $\dim \mathcal{C}_1(V) < \infty$
It is already known that C_2-finiteness means the both by Zhu, DLM and G.

Although C_2-finiteness was introduced by Zhu as a technical condition to obtain a differential equation, it is a natural condition in order to consider trace functions on all (weak) modules.

Theorem 4 *The followings are equivalent.*
1) V satisfies C_2-cofiniteness
2) V is finitely generated and all weak modules are $\mathbb{Z}_+$-graded
3) $S^W(\mathbf{1}, \tau)$ is well-defined on finitely generated weak modules.

3.1 Logarithmic forms

The well known examples of trace functions are rational power sum of q. However, in our case, $L(0)$ may not act semisimply. So we devide $L(0)$ into

$$L(0) = \overset{\text{semisimple}}{L^{se}(0)} + \overset{\text{nilpotent}}{L^{nl}(0)}$$

Then we have

$$q^{L(0)} = q^{L^{se}(0)}\left(\sum_{m=0}^{r} \frac{(2\pi i L^{nil}(0))^m}{m!}(\tau)^m\right)$$

which contains τ-terms, that is, logarithmic form ($\ln q = 2\pi i\tau$). However, if we take a trace $\mathrm{tr}_W q^{L(0)}$, then there is no τ-terms since $L^{nl}(0)$ is a nilpotent operator. So we need a new kind of "trace". What does "trace" mean in our setting? It is just a symmetric linear function of the ring generated by grade preserving operators of V. So we have a question.

Is there another suitable symmetric linear function?

The answer is "**Yes**" and I will introduced a new trace function "pseudo-trace".

3.2 Pseudo-trace

Consider

$$R_m = \left\{ g = \begin{pmatrix} A_g & B_g \\ O & A_g \end{pmatrix} \mid A_g, B_g \in M_{m,m}(\mathbb{C}) \right\}.$$

Then $\mathrm{pstr}(g) = \mathrm{tr}B_g$ is a symmetric linear map. We will show that these symmetric linear function of the ring of graded preserving operators of V is defined by a symmetric linear function of $A_n(V)$, which is also given by a conformal block.

Let me explain the image of pseudo-trace function. Let R be a ring and W a left R-module. Set $P = \mathrm{End}_R(W)$. Assume that there is an R-isomorphism $\phi : W/WJ(P) \to W\mathrm{soc}(P)$. Consider $\alpha \in R$ and $\alpha : W \to W$. An oridinary trace $\mathrm{tr}\alpha$ is the trace of a matrix (m_{ij}) given by $\alpha(w^i) = \sum m_{ij}w^j$ for some basis $\{w^i\}$ of W. In our case, we have a matrix representation

$$\alpha = \begin{pmatrix} A & * & B \\ \hline 0 & C & * \\ \hline O & 0 & A \end{pmatrix}$$

and we define $\mathrm{pstr}^{\phi}\alpha = \mathrm{tr}B$.

4 Finite dimensional algebra

Now let start the proof of the main theorem.

If we choose $S(*,\tau) \in \mathcal{C}_1(V)$, then since $\dim \mathcal{C}_1(V) < \infty$, $S(v,\tau)$ is a solution of some differential equation of regular singularity type and so $S(v,\tau)$ has a form

$$\sum_{t=0}^{p}\sum_{s=0}^{q}\sum_{i=0}^{\infty} \lambda_{t,s,i}(v) q^i q^{r_s} \tau^t$$

The first result we have is

Lemma 1 *$\phi = \lambda_{t,s,n} : V \to \mathbb{C}$ is symmetric linear function of $A_n(V)$*

We use ordinary finite dimensional ring theoretic arguments since $A_n(V)$ is a finite dimensional ordinary algebra and $A_n(V)/\mathrm{Rad}\phi$ is symmetric algebra, where $\mathrm{Rad}\phi = \{a \in A_n(V) \mid \phi(A_n(V)aA_n(V)) = 0\}$.

4.1 Definition of Symmetric algebra.

Let A be a finite dimensional algebra over complex number field.

Definition 2 *A is called Frobenius if the left module ${}_AA$ is isomorphic to the dual $\mathrm{Hom}_A(A_A, \mathbb{C})$ of right module A_A. If we denote the regular action of $a \in A$ on A from the right and left by $R(a)$ and $L(a)$, then A is Frobenius algebra means that there is a non-singular matrix Q such that $Q^{-1}R(a)Q = L(a)$. If we can take Q as a symmetric matrix, then A is called a symmetric algebra.*

Lemma 2 *A is symmetric if and only if A has a symmetric linear map ϕ with zero radical. It is also equivalent to that A has an associative, symmetric nondegenerated bilinear form*

4.2 Result by C.Nesbitt, W.Scott (1943)

(A short proof was given by Oshima (1952))

There is a classical result for symmetric algebra given by Nesbitt and Scott. This is a good method to explain a strategy of my proof. They showed:

Theorem 5 (Nesbitt and Scott) *A is symmetric algebra if and only if so is its basic algebra.*

4.3 Definition of basic algebra.

This is the last definition I will introduce here.

Definition 3 *Decompose $A/J(A)$ into the direct sum of simple components.*

$$A/J(A) = A_1 \oplus \cdots \oplus A_k$$

where A_i is a matrix algebra $M_{n_i}(\mathbb{C})$. Take a primitive idempotent $e_i \in A_i$, say $e_i = \begin{pmatrix} 1 & 0\cdots 0 \\ 0 & 0\cdots 0 \\ & \cdots \\ 0 & 0\cdots 0 \end{pmatrix}$. Set $e = e_1 + \cdots + e_k$ and we can consider that e is an idempotent of A. Then eAe is called a basic algebra of A. The important properties of basic algebra is that the semisimple factor is commutative and Ae is a faithful $A \times eAe$-module and $eAe = \mathrm{End}_A(Ae)$.

5 Outline of proof of the main theorem

Now let me explain my strategy. Take a function $S(*, \tau)$ from conformal block. As I explained, we have a symmetric function ϕ of $A_n(V)$. So we have a symmetric algebra $A = A_n(V)/\mathrm{Rad}(\phi)$. As we explained, there is an idempotent e of A such that $P = eAe$ is a basic algebra, which is also a symmetric algebra by Nesbitt and Scott. Since Ae is an $A_n(V)$-module, we can construct V-module W by multiplying the actions of V from the left side. So their actions commute with the actions of P from the right side. Consider the endomorphism ring $R = End_P(W)$, which contains all actions of V. I proved that P is the basic algebra of R if n is sufficiently large. We should note that since the actions of V generate infinite dimensional ring, we always have to consider the actions on finite dimensional parts $\oplus_{m=0}^{k} W(m)$. This is the definition of interlocked module. Then again by Nesbitt and Scott, R is a symmetric algebra with a symmetric linear map, which we will call pseudo-trace. Then define pseudo-trace function, which almost coincides with the originial one. Actually, we have to construct a symmetric function explicitly.

6 Example

6.1 logarithmic form

Let's show you one example. Assume that W is a V-module such that $L(0)^2$ is zero on the top module $W(0)$. We also assume:

$$W \cong W^1 \oplus L^{nil}(0)W, \text{ as vector spaces}$$
$$L^{nil}(0)W \cong W/WJ(P) \cong W^1 \text{ as } V\text{-modules}$$

Then consider a pseudo-trace function.

$$\begin{aligned} S^W(1,\tau) =& \sum_{n=0}^{\infty} \mathrm{tr}^{\phi}_{W(n)} \left(1 + 2\pi i(L^{nil}(0) - \tfrac{c}{24})\tau\right) q^{n-c/24} \\ =& \ (\mathrm{ch}_{(L^{nil}(0)W)}(\tau) 2\pi i\tau \end{aligned}$$

where $L^{nil}(0) \Leftrightarrow \begin{pmatrix} O & I \\ O & O \end{pmatrix}$ and $1 + 2\pi i(L^{nil}(0) - \frac{c}{24})\tau \Leftrightarrow \begin{pmatrix} I - \frac{\pi i c}{12}\tau I & 2\pi i\tau I \\ O & I - \frac{\pi i c}{12}\tau I \end{pmatrix}$

6.2 Triplet algebra with central charge $c = -2$.

It is generated by ω and three vectors $v^a \in V_3$ $(a = 1,2,3)$. $o_m(v)$ denotes $v_{\mathrm{wt}(v)-1+m}$

$$\begin{cases} [L_m, L_n] = (m-n)L_{m+n} - \frac{m(m^2-1)}{6}\delta_{m+n,0} \\ [L_m, o_n(v^a)] = (2m-n)o_{m+n}(v^a) \\ [o_m(v^a), o_n(v^b)] = \delta_{a,b}\left(2(m-n)o_{m+n}(L_{-2}\omega - \frac{3}{10}L_{-1}L_{-1}\omega)\right. \\ \quad \left. + \frac{(m-n)(2m^2+2n^2-mn-8)}{20}L_{m+n} - \delta_{m+n,0}\binom{m+2}{5}\right) \\ \sqrt{-1}\epsilon^{abc}\left(\frac{5(2m^2+2n^2-3mn-4)}{14}o_{m+n}(v^c) + \frac{12}{5}o(L(-2)v^c) - \frac{18}{35}o_{m+n}(L(-1)^2v^c)\right) \end{cases}$$

It has six irreducible modules

$$V,\ M^1,\ M^{-1/8},\ M^{3/8},\ X^0,\ X^1$$

Theire characters are

$$\begin{cases} S^1(\tau) = \frac{1}{2}(\eta(\tau)^{-1}\theta_{1,2}(\tau) + \eta(\tau)^2) \\ S^2(\tau) = \frac{1}{2}(\eta(\tau)^{-1}\theta_{1,2}(\tau) - \eta(\tau)^2) \\ S^3(\tau) = \eta(\tau)^{-1}\theta_{0,2}(\tau) \\ S^4(\tau) = \eta(\tau)^{-1}\theta_{2,2}(\tau) \\ S^5(\tau) = 2\eta(\tau)^{-1}\theta_{1,2}(\tau) \\ S^6(\tau) = 2\eta(\tau)^{-1}\theta_{1,2}(\tau) \end{cases}$$

The space spanned by last four characters is invariant under $SL(2,\mathbb{Z})$, but

$$\begin{cases} S^1(-1/\tau) = \frac{1}{4}S^3(\tau) - \frac{1}{4}S^4(\tau) - \frac{i}{2}\eta(\tau)^2\tau \\ S^2(-1/\tau) = \frac{1}{4}S^3(\tau) - \frac{1}{4}S^4(\tau) + \frac{i}{2}\eta(\tau)^2\tau \end{cases}$$

It is not a linear sum of characters, but there exists an interlocked module W such that a pseudo-trace function is

$$S^7(\tau) = (S^1(\tau) - S^2(\tau))2i\pi\tau) = \eta(\tau)^2(2\pi i\tau).$$

$$\begin{aligned} S^1(\tfrac{-1}{\tau}) &= \tfrac{1}{4}(S^3(\tau) - S^4(\tau) + 2i\tau\eta(\tau)^2 \\ &= \tfrac{1}{4}(S^3(\tau) - S^4(\tau) + \tfrac{1}{\pi}S^7(\tau) \\ S^2(\tfrac{-1}{\tau}) &= \tfrac{1}{4}(S^3(\tau) - S^4(\tau) - 2i\tau\eta(\tau)^2) \\ &= \tfrac{1}{4}(S^3(\tau) - S^4(\tau) - \tfrac{1}{\pi}S^7(\tau)) \\ S^7(\tfrac{-1}{\tau}) &= (S^1(\tfrac{-1}{\tau}) - S^2(\tfrac{-1}{\tau}))(\tfrac{-2\pi i}{\tau}) \\ &= (-i\tau)\eta(\tau)^2(-2\pi i/\tau) = -2\eta(\tau)^2 \\ &= 2\pi(-S^1(\tau) + S^2(\tau)) \end{aligned}$$

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