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GENERALIZED MOONSHINE AND ORBIFOLD CONSTRUCTIONS

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Abstract

A brief review is given of some of our recent work on Generalised Monstrous Moonshine using abelian orbifoldings of the Moonshine Module.

1 Introduction

The Moonshine Module [FLM] is a Vertex Operator Algebra (VOA) based on an orbifolding of the Leech lattice VOA. The group of automorphisms of the Moonshine Module is the Monster group $\mathbb{M}$. Monstrous Moonshine first formulated by Conway and Norton [CN] and subsequently proved by Borcherds [B] is concerned with a correspondence between the classes of $\mathbb{M}$ and special modular functions known as hauptmoduls. This correspondence can also be understood from the point of view of orbifoldings of the Moonshine Module [T1],[T2]. Norton further postulated Generalized Moonshine [N] relating a pair of commuting Monster elements to either a hauptmodul or constant function. In this note we give a very brief overview of Moonshine from the point of view of orbifold constructions. We also review some of our recent work [IT1],[IT2],[I] on recovering the hauptmodul property in a number of Generalized Moonshine cases using Abelian orbifoldings of the Moonshine Module. Although this is an informal review the notation and terminology is aimed at a mathematical audience.

1.1 $C = 24$ Holomorphic VOAs

We begin with some relevant aspects of Vertex Operator Algebra (VOA) theory but for more details see e.g. [FLM], [Ka], [MN]. The closest related concept in physics is that of a meromorphic conformal field theory [Go]. A VOA is a quadruple $(V, Y, \mathbf{1}, \omega)$ consisting of an integer graded complex vector space $V = \bigoplus_{n\geq 0} V(n)$, a linear map $Y : V \rightarrow (\mathrm{End}V)[[z, z^{-1}]]$, and a pair of distinguished vectors (states): the vacuum $\mathbf{1}$ which spans $V(0)$ and the conformal vector $\omega \in V(2)$. The image under the Y map of a vector $v \in V$ is denoted by the vertex operator

$$Y(v, z) = \sum_{n\in\mathbf{Z}} v(n) z^{-n-1}, \tag{1}$$

with modes $v(n) \in \mathrm{End}V$ and where the creation axiom $Y(v, z).\mathbf{1}|_{z=0} = v(-1).\mathbf{1} = v$ holds. For the vacuum and conformal vectors we define

$$\begin{aligned} Y(\mathbf{1}, z) &= \mathrm{id}_V, \\ Y(\omega, z) &= \sum_{n\in\mathbf{Z}} L(n) z^{-n-2}, \end{aligned}$$

where the modes $L(n)$ form a Virasoro algebra of central charge C:

$$[L(m), L(n)] = (m-n)L(m+n) + (m^3 - m)\frac{C}{12}\delta(m, -n).$$

We will consider in this paper VOAs of central charge $C = 24$ only. The integral grading of V is provided by the action of $L(0)$ i.e. $L(0)v = nv$ for $v \in V(n)$. One of the key properties of a VOA is the "associativity" of the Operator Product Expansion (OPE)

$$Y(a, z)Y(b, w) \sim Y(Y(a, z-w)b, w), \tag{2}$$

where we formally expand in $z - w$ for $|z| > |w|$ and where $\sim$ denotes the equality of the singular part in $z - w$ of each side e.g. [Ka].

A VOA module (M, Y_M) consists of a $\mathbf{Q}$-graded complex vector space $M = \bigoplus_{\alpha\in\mathbf{Q}} M(\alpha)$ and a linear map $Y_M : V \rightarrow (\mathrm{End}M)[[z, z^{-1}]]$ with $Y_M(\mathbf{1}, z) = \mathrm{id}_M$ where

$$Y_M(a, z)Y_M(b, w) \sim Y_M(Y(a, z-w)b, w). \tag{3}$$

Clearly (V, Y) is a module for itself. Standard notions of simplicity and complete reducibility of modules can be defined. A holomorphic VOA is one where every module is completely reducible and (V, Y) is the unique simple module. There is also the notion of intertwiner vertex operators between modules [FHL].

The graded trace (or genus one partition function) for a $C = 24$ VOA is defined by

$$Z(\tau) = \mathrm{Tr}_V(q^{L(0)-1}) = \frac{1}{q}\sum_{n\geq 0} q^n \dim V(n). \tag{4}$$

where $q=e^{2\pi i\tau}$ for $\tau\in\mathbb{H}$ the upper half complex plane. For a $C=24$ Holomorphic VOA Zhu has shown that $Z(\tau)$ is modular invariant under $SL(2,\mathbb{Z})=\left(\begin{array}{cc} a & b \\ c & d \end{array}\right), ad-bc=1\}$ where $SL(2,\mathbb{Z})\ni\gamma:\tau\rightarrow\gamma\tau=\frac{a\tau+b}{c\tau+d}$ [Z]. Hence $Z(\tau)$ is given by

$$\begin{aligned} Z(\tau) &= J(\tau)+\mathrm{dim}V(1), \\ J(\tau) &= \frac{E_4^3(\tau)}{\eta^{24}(\tau)}-744=\frac{1}{q}+0+196884q+21493760q^2+\ldots \end{aligned} \tag{5}$$

where $\eta(\tau)=q^{1/24}\prod_{n>0}(1-q^n)$ and $E_4(\tau)$ is the weight four Eisenstein series. $J(\tau)$ is the hauptmodul for $SL(2,\mathbb{Z})$ i.e. the unique (up to an additive constant) modular invariant function with a simple pole at $q=0$ and unit residue. Two $C=24$ Holomorphic VOAs are of particular interest to us: the Leech lattice VOA V_Λ for which $\mathrm{dim}V_\Lambda(1)=24$ and the FLM Moonshine Module $V^\natural$ [FLM] for which $\mathrm{dim}V^\natural(1)=0$.

1.2 Automorphisms and Twisted Sectors for Holomorphic VOAs

The automorphism group $\mathrm{Aut}(V)$ of a VOA is the group of linear transformations preserving the OPE (2) and which act trivially on $\mathbf{1}$ and ω. Thus for $g\in\mathrm{Aut}(V)$ we have $gY(a,z)g^{-1}=Y(ga,z)$. Since $\mathrm{Aut}(V)$ acts separately on the graded spaces $V(n)$ we can define the graded trace

$$Z\left[\begin{array}{c} g \\ 1 \end{array}\right](\tau)=\mathrm{Tr}_V(gq^{L(0)-1})=\frac{1}{q}\sum_{n\geq 0}q^n\mathrm{Tr}_{V(n)}(g), \tag{6}$$

where the coefficients are characters for some representation of $\mathrm{Aut}(V)$. Note that for $g=1$, the identity element, $Z\left[\begin{array}{c} 1 \\ 1 \end{array}\right]=Z$. For the Moonshine Module $V^\natural$, the automorphism group is the finite Monster group $\mathbb{M}$, the largest finite sporadic simple group [FLM] and each $V^\natural(n)$ is a representation space for $\mathbb{M}$. Thus $V^\natural(2)$ is the direct sum of Monster irreducible representations of dimension 1 and 196883. Monstrous Moonshine is concerned with the modular properties of $Z\left[\begin{array}{c} g \\ 1 \end{array}\right]$ as discussed later on.

Suppose that g is of finite order n. Let $\langle g\rangle$ denote the abelian group generated by g and let $C_g=C(g,\mathrm{Aut}(V))$ be the centraliser of g. We further denote by $\mathcal{P}_{\langle g\rangle}V$ the g-invariant subVOA of V where $\mathcal{P}_{\langle g\rangle}=\frac{1}{n}(1+g+\ldots g^{n-1})$ denotes a "projection operator". It is clear that $\mathrm{Aut}(\mathcal{P}_{\langle g\rangle}V)\supseteq G_g=C_g/\langle g\rangle$. For a holomorphic VOA and g of finite order n the modules of $\mathcal{P}_{\langle g\rangle}V$ are given by so-called "g^k-twisted sector" modules (M_{g^k},Y_{g^k}) for $k=0\ldots n-1$ [DLM] and where $M_1=V$ and $Y_1=Y$. The g-twisted module M_g, which is uniquely defined for a holomorphic VOA, has rational grading $M_g=\bigoplus_{\alpha\in\frac{1}{n}\mathbb{N}}M_g(\alpha)$ where $M_g(0)$ is the highest weight space or g-twisted vacuum space. The grading is related to the Virasoro level by $L(0)u = (\alpha + E_g^0 + 1)u$ for $u \in M_g(\alpha)$ where E_g^0 is known as the vacuum energy.

The twisted sector automorphism group preserving (3) is given by $\mathrm{Aut}(M_g) = \mathbb{C}^*.C_g$ in general but if $\dim M_g(0) = 1$ then $\mathrm{Aut}(M_g) = \mathbb{C}^* \times C_g$. Hence for $\hat{h} \in \mathrm{Aut}(M_g)$ lifted from $h \in C_g$ we may define the generalised graded trace

$$Z\left[\begin{array}{c} \hat{h} \\ g \end{array}\right](\tau) = \mathrm{Tr}_{M_g}(\hat{h}q^{L(0)-1}) = q^{E_g^0} \sum_{\alpha \in \frac{1}{n}\mathbf{N}} q^n \mathrm{Tr}_{M_g(\alpha)}(\hat{h}). \tag{7}$$

where the coefficients are now characters for $\mathrm{Aut}(M_g)$. In the case of the Moonshine Module VOA $V^\natural$, such generalised traces are the subject of Norton's Generalized Moonshine Conjectures as discussed later on.

The generalised graded traces transform amongst themselves under the modular group as follows [DLM]:

$$Z\left[\begin{array}{c} \hat{h} \\ g \end{array}\right](\gamma\tau) = \varepsilon(\hat{h}, g; \gamma) Z\left[\begin{array}{c} g^{-\hat{b}}h^{d} \\ g^{a}h^{-c} \end{array}\right](\tau), \tag{8}$$

for $\varepsilon(\hat{h}, g; \gamma) \in \mathbb{C}^*$. This modular property was also discussed in earlier work on orbifold constructions by physicists [DHVW]. M_g is also naturally isomorphic to $M_{xgx^{-1}}$ under conjugation by $x \in \mathrm{Aut}(V)$ so that

$$Z\left[\begin{array}{c} \hat{h} \\ g \end{array}\right] = \theta(\hat{h}, g; x) Z\left[\begin{array}{c} x\hat{h}x^{-1} \\ xgx^{-1} \end{array}\right], \tag{9}$$

for $\theta(\hat{h}, g; x) \in \mathbb{C}^*$.

We may distinguish one lifting of g on M_g, which we also denote by g, defined by

$$gu = \exp(-2\pi i(E_g^0 + \alpha))u, \tag{10}$$

for $u \in M_g(\alpha)$. Then M_g can be decomposed into n simple modules determined by this g action. We say that g is a normal automorphism if the twisted vacuum energy obeys

$$nE_g^0 = 0 \text{ mod } 1.$$

Otherwise we say it is an anomalous automorphism. If g is a normal automorphism it acts as an n^{th} root of unity on M_g and the g invariant part of M_g defines a simple module $\mathcal{P}_{\langle g\rangle}M_g$. Modules of this type are utilized in orbifold constructions.

1.3 Abelian Orbifoldings of a Holomorphic VOA

Let g be of order n and assume all elements of $\langle g\rangle$ are normal. The orbifold VOA vector space $V_{\mathrm{orb}}^{\langle g\rangle}$ is formed by adjoining to $\mathcal{P}_{\langle g\rangle}V$ the g-invariant modules:

$$V_{\mathrm{orb}}^{\langle g\rangle} = \bigoplus_{k=0}^{n-1} \mathcal{P}_{\langle g\rangle} M_{g^k},$$

with graded trace

$$Z^{\langle g\rangle}_{\rm orb}(\tau)=\frac{1}{n}\sum_{k,l=0}^{n-1} Z\left[\begin{array}{c} g^l \\ g^k \end{array}\right]. \tag{11}$$

We assume that a consistent choice of lifting can be made for g^l acting on M_{g^k} (denoted again by g^l) where the various ε-multipliers of (8) are unity so that the trace is modular invariant. Hence $Z^{\langle g\rangle}_{\rm orb}(\tau)=J(\tau)+\dim V^{\langle g\rangle}_{\rm orb}(1)$ from (5). We further assume that the various vertex intertwiner vertex operators close to form a holomorphic VOA. The Moonshine Module $V^\natural$ is an example of such a construction where g is an involution lifted from the reflection involution of the Leech lattice. Other possible constructions are briefly mentioned below.

We may similarly consider the orbifolding of V with respect to a finite abelian group $G\subset \mathrm{Aut}(V)$ where all the elements of G are normal. The orbifold vector space is then $V^G_{\rm orb}=\oplus_{g\in G}\mathcal{P}_G M_g$ where $\mathcal{P}_G=\frac{1}{|G|}\sum_{g\in G} g$ is the G projection operator and the graded trace is $Z^G_{\rm orb}(\tau)=\frac{1}{|G|}\sum_{g,g_2\in G} Z\left[\begin{array}{c} g_1 \\ g_2 \end{array}\right]$. This trace is modular invariant provided the ε-multipliers of (8) are unity for some appropriate choice of lifting of each g_1 (which we also denote by g_1) acting on V_{g_2}.

We are particularly interested in $G=\langle g,h\rangle$ generated by a pair of commuting elements $g,h\in \mathrm{Aut}(V)$ with $o(g)=m$ and $o(h)=n$. We call such generators independent iff $\langle g\rangle\cap\langle h\rangle=1$ i.e. $|\langle g,h\rangle|=mn$. For independent generators we have $\mathcal{P}_{\langle g,h\rangle}=\mathcal{P}_{\langle g\rangle}\mathcal{P}_{\langle h\rangle}$ and

$$V^{\langle g,h\rangle}_{\rm orb}=\bigoplus_{k=1}^{m}\mathcal{P}_{\langle g\rangle}(\bigoplus_{l=1}^{n}\mathcal{P}_{\langle h\rangle}M_{g^k h^l}).$$

Thus $V^{\langle g,h\rangle}_{\rm orb}$ may be interpreted as a composition of orbifoldings i.e. for any independent normal commuting generators g,h:

$$V^{\langle g,h\rangle}_{\rm orb}=\left(V^{\langle h\rangle}_{\rm orb}\right)^{\langle g\rangle}_{\rm orb}. \tag{12}$$

For such generators, the twisted trace $Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau)$ is fixed by the modular group

$$\Gamma(m,n)=\{\left(\begin{array}{cc} a & b \\ c & d\end{array}\right)|a-1=c=0 \bmod m, b=d-1=0 \bmod n, ad-bc=1\}, \tag{13}$$

from (8). Hence $Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau)$ is a meromorphic function on $\mathbb{H}/\Gamma(m,n)$ with possible singularities at $\tau=i\infty$ and a finite number of inequivalent rational cusps. Lastly, $Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau)$ is singular at the rational cusp $\tau=a/c$ with $(a,c)=1$ if and only if $E^0_{g^a h^{-c}}<0$, the $g^a h^{-c}$-twisted vacuum energy. In particular, if

$E^0_{g^a h^{-c}} \geq 0$ for all $(a,c)=1$ then $Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau)$ is holomorphic on $\mathbb{H}/\Gamma(m,n)$ and is therefore constant.

We now apply these various ideas in order to describe and understand aspects of Monstrous and Generalized Moonshine.

2 Monstrous Moonshine

The Moonshine Module $V^\natural$ is an orbifold VOA formed from the Leech lattice VOA V_Λ and is based on an involution lifted from the reflection involution of the Leech lattice [FLM]. FLM have also conjectured that $V^\natural$ is the unique holomorphic $C=24$ VOA with graded trace $J(\tau)$ [FLM]. There is considerable evidence to support this conjecture. In particular, there are 38 specific automorphism classes g of the Conway simple group (whose double cover is the Leech lattice automorphism group) for which it is believed $V^\natural = (V_\Lambda)^{\langle g\rangle}_{\mathrm{orb}}$ [T2]. Furthermore, for each such Conway automorphism g there is a natural "dual automorphism" g^* of $(V_\Lambda)^{\langle g\rangle}_{\mathrm{orb}}$ for which $V_\Lambda = ((V_\Lambda)^{\langle g\rangle}_{\mathrm{orb}})^{\langle g^*\rangle}_{\mathrm{orb}}$ [T2]. For such g of prime order $(p=2,3,5,7,13)$ these constructions are discussed in far greater detail in [DM].

The automorphism group of $V^\natural$ is the Monster group $\mathbb{M}$ of order

$$|\mathbb{M}| = 2^{46}.3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.41.47.59.71. \tag{14}$$

Conway and Norton [CN] conjectured and Borcherds rigorously proved [B] that for each $g\in\mathbb{M}$ the Thompson series $T_g(\tau) = Z\left[\begin{array}{c} g \\ 1 \end{array}\right](\tau)$ of (6) is a hauptmodul for a genus zero fixing modular group Γ_g. This means that $T_g(\tau)$ is invariant under some modular group Γ_g and $T_g(\tau)$ is the unique modular function (up to an additive constant) on the quotient space $\mathbb{H}/\Gamma_g$ with a simple pole at $q=0$ and unit residue. $T_g(\tau)$ then defines a 1-1 mapping between $\mathbb{H}/\Gamma_g$ and the genus zero Riemann sphere. The Thompson series for the identity element is thus $J(\tau)$ of (5). In general, for g of order n, $T_g(\tau)$ is found to be $\Gamma_0(n)$ invariant up to m^{th} roots of unity where $m|n$ and $m|24$ where $\Gamma_0(n) = \{\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2,\mathbb{Z})|c = 0 \bmod n\}$. In the language of the last section, the Monster elements with $m=1$ are normal automorphisms while those with $m\neq 1$ are anomalous. $T_g(\tau)$ is then fixed by Γ_g where $\Gamma_0(N)\subseteq\Gamma_g$ and Γ_g is contained in the normalizer of $\Gamma_0(N)$ in $SL(2,\mathbb{R})$ where $N=nm$ [CN]. This normalizer contains the so-called Fricke involution $W_N: \tau\to -1/N\tau$ so that all the classes of $\mathbb{M}$ can be divided into Fricke and non-Fricke classes according to whether or not $T_g(\tau)$ is invariant under W_N. There are a total of 51 non-Fricke classes of which 38 are normal and there are a total of 120 Fricke classes of which 82 are normal.

To illustrate and understand some of these ideas consider g normal of prime order p. Then one finds two general cases: (i) g is non-Fricke and $\Gamma_g = \Gamma_0(p)$ which is of genus zero for $(p-1)|24$ i.e. $p=2,3,5,7,13$ (ii) g is Fricke and $\Gamma_g = \Gamma_0(p)+$ where $\Gamma_0(p)+ = \langle\Gamma_0(p), W_p\rangle$ which is of genus zero for exactly the primes $p \mid |\mathbb{M}|$ i.e. the primes appearing in (14). Many of these properties can be understood in terms of the orbifolding of $V^\natural$ with respect to $\langle g\rangle$ [T1], [T2]. Firstly from above $T_g(\tau)$ is invariant under $\Gamma(p,1)$. In particular, $T_g(\tau) = T_{g^{-1}}(\tau)$ so that the coefficients of $T_g(\tau)$ are real Monster characters. But the Monster has no real irrational characters and so the coefficients are rational characters implying that $T_g(\tau) = T_{g^a}(\tau)$ since g and g^a are conjugate for $a \neq 0 \bmod p$. Hence $T_g(\tau)$ is $\Gamma_0(p)$ invariant and can be singular on $\mathbb{H}/\Gamma_0(p)$ only at the inequivalent cusps $\tau = i\infty$ and 0. $T_g(\tau)$ has a simple pole by definition at $q = 0$ and is singular at $\tau = 0$ if and only if the g-twisted vacuum energy $E_g^0 < 0$. If $E_g^0 \geq 0$ then $T_g(\tau)$ cannot be Fricke invariant, has a unique simple pole and is therefore the hauptmodul for a genus zero fixing group of type $\Gamma_0(p)$. Explicitly one finds $T_g(\tau) = \frac{24}{p-1} + [\eta(\tau)/\eta(p\tau)]^{\frac{24}{p-1}}$ where p is restricted to $p = 2,3,5,7,13$ aas above.

If $T_g(\tau)$ is Fricke invariant then clearly $T_g(\tau) = Z\begin{bmatrix} g \\ 1 \end{bmatrix}(\tau) = Z\begin{bmatrix} 1 \\ g \end{bmatrix}(p\tau)$ and so

$$E_g^0 = -\frac{1}{p}, \quad \dim M_g(0) = 1. \tag{15}$$

The converse is also true as follows: Consider the $\Gamma_0(p)$ invariant $f(\tau) = T_g(\tau) - T_g(W_p(\tau))$. $f(\tau)$ is holomorphic and therefore constant on $\mathbb{H}/\Gamma_0(p)$. But $f(\tau)$ is odd under the action of W_p and therefore vanishes. Hence invariance under the Fricke involution results from (15). Furthermore, $T_g(\tau)$ has a unique simple pole at $q = 0$ on $\mathbb{H}/\Gamma_0(p)+$ and is therefore the hauptmodul for a genus zero fixing group.

Let us next consider the nature of the orbifold VOA $(V^\natural)^{\langle g\rangle}_{\mathrm{orb}}$ obtained for g normal and of order p. For g non-Fricke $Z\begin{bmatrix} 1 \\ g \end{bmatrix}(\tau) = \frac{24}{p-1} + O(q^{1/p})$ from (11) so that $Z^{\langle g\rangle}_{\mathrm{orb}} = J(\tau) + 24 = Z_\Lambda$. In these cases g is dual to one of the Conway group prime ordered automorphisms discussed above so that we expect $(V^\natural)^{\langle g\rangle}_{\mathrm{orb}} = V_\Lambda$. Considering the orbifolding of the Moonshine Module with respect to a Fricke element it follows from (11) that since $Z\begin{bmatrix} 1 \\ g \end{bmatrix}(\tau) = q^{-1/p} + 0 + O(q^{1/p})$ then $Z^{\langle g\rangle}_{\mathrm{orb}} = J(\tau)$ i.e. we expect $(V^\natural)^{\langle g\rangle}_{\mathrm{orb}} = V^\natural$ again provided the FLM uniqueness conjecture holds.

All of these concepts can be suitably generalized to include all normal Monster automorphisms [T1],[T2]. Thus, subject to assumptions like those made above, orbifolding $V^\natural$ with respect to any of the 38 normal non-Fricke Monster classes results in the Leech theory whereas orbifolding with respect to any of the 82 normal Fricke Monster classes results in $V^\natural$ again assuming the FLM uniqueness conjecture. Furthermore, assuming a number of other important properties it can also be shown that for g normal then if $(V^\natural)^{\langle g\rangle}_{\mathrm{orb}} = V_\Lambda$ then $T_g(\tau)$ is a hauptmodul and g is non-Fricke whereas if $(V^\natural)^{\langle g\rangle}_{\mathrm{orb}} = V^\natural$ then conditions such as (15) and others must hold and $T_g(\tau)$ is a Fricke invariant hauptmodul. (That the anomolous classes are hauptmoduls follows from the so-called power map formula for Thompson series [CN],[T1]). This all leads to the following general principle for explaining Monstrous Moonshine: Monstrous Moonshine is equivalent to the property that for all normal elements g of $\mathbb{M}$ either $(V^\natural)^{\langle g\rangle}_{\text{orb}} = V^\natural$ or $(V^\natural)^{\langle g\rangle}_{\text{orb}} = V_\Lambda$ [T2].

2.1 Generalized Moonshine

Consider the orbifolding of $V^\natural$ with respect to $\langle g,h\rangle$ for commuting $g,h \in \mathbb{M}$ [T3],[IT1],[IT2]. Norton's Generalized Moonshine conjecture [N] states that the twisted graded trace $Z\left[\begin{array}{c} h \\ g \end{array}\right]$ is either constant or is a hauptmodul for some genus zero fixing group. In general, a far wider variety of genus zero modular groups arise in Generalized Moonshine than those appearing in Monstrous Moonshine.

Some parts of Norton's conjecture follow directly from the orbifold analysis of Section 1. For example, the twisted trace is a constant if $E^0_{g^a h^{-c}} \geq 0$ for all $(a,c)=1$ i.e. when all the elements of $\{g^a h^{-c}, (a,c)=1\}$ are non-Fricke and so the trace is holomorphic and therefore constant. Another set of immediately understood examples arises when $\langle g,h\rangle = \langle u\rangle$ for some $u \in \mathbb{M}$. Then the corresponding graded trace can be transformed to the Thompson series $T_u(\tau)$ via an appropriate modular transformation (8) so that in these cases, the hauptmodul property follows directly from Monstrous Moonshine [T3],[DLM]. For example, if g and h have co-prime orders then this situation holds.

We will consider from now on g normal Fricke and of prime order p and $h \in C_g = C(g,\mathbb{M})$ normal of order pk for $k \geq 1$ or k prime. Such commuting pairs occur only for $p \leq 13$. Since $\dim M_g(0) = 1$ the coefficients of the twisted graded trace are characters for C_g (up to a possible overall trivial factor). In most cases these characters are rational but in some others they are irrational. The centraliser groups that arise for $p \leq 13$ are

p	2	3	5	7	11	13
C_g	2.B	3.Fi	$5 \times$ HN	$7 \times$ He	$11 \times M_{12}$	$5 \times L_3(3)$

for the Baby Monster, Fischer, Harada-Norton, Held, Mathieu and $L_3(3)$ finite simple groups [CCNPW]. For all such normal generators g,h that are also independent we further have that

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](p\tau) = \frac{1}{q} + 0 + O(q), \tag{16}$$

i.e. h acts as unity on $M_g(0)$. Consider $(V^\natural)^{\langle g,h\rangle}_{\text{orb}}$ as a composition of orbifoldings with respect to any pair of independent generators u,v of $\langle g,h\rangle$ as in (12). For $u^a v^b$ Fricke let $\phi_{u^a v^b}(u)$ denote the action of u on $M_{u^a v^b}(0)$. Then one can show [IT1]:

1. If h is Fricke then $(V^\natural)^{\langle g,h\rangle}_{\text{orb}} = V^\natural$. If u is Fricke then $u^a v$ is also Fricke for a unique a mod $o(u)$ with $o(u^a v) = o(v)$ and $\phi_{u^a v}(u) = 1$.

2. If h is non-Fricke then $(V^\natural)^{\langle g,h\rangle}_{\text{orb}} = V_\Lambda$.

Hence the orbifold VOA $(V^\natural)_{\rm orb}^{\langle g,h\rangle}$ is either the Moonshine Module or the Leech lattice theory again and properties of the singularities of the twisted trace (16) can be determined. Other constraints on the singularities can also be proved [IT1]. In [IT1] and [IT2] we have shown that these results together with the other properties of twisted traces reviewed above are sufficient to determine the twisted trace pole structure for those traces with rational coefficients for $k=1$ and k prime and for those traces with irrational coefficients for $k=1$. Furthermore, in each case the twisted trace is the hauptmodul for a genus zero-fixing group generated by $\Gamma(p,pk)$ and some other modular symmetries under which all of the twisted trace singularities are identified. This accounts for approximately 130 non-trivial cases not directly related to Monstrous Moonshine. This analysis also provides many restrictions on the possible Monster classes to which the elements of $\langle g,h\rangle$ may belong. An analysis for rational traces for $o(g)=p^a k$, k prime has now been completed in [I].

2.2 Some Examples and Conclusion

We illustrate these results with two basic examples where the twisted trace (16) has rational coefficients. Then one can show in general that (16) is $\Gamma_0(p^2k)$ invariant [IT1]. We then have the following examples:

Example 1. Suppose that g and its Monster conjugates g^a, for $a\not\equiv 0 \bmod p$, are the only Fricke elements in $\{g^a h^{-c},(a,c)=1\}$. Then (16) has a unique simple pole at $q=0$ and is therefore a hauptmodul for a genus zero group $\Gamma_0(p^2k)$. This restricts the possible values of p^2k to $4,8,9,12,18$ and 25. In practice $p^2k=25$ does not occur. Thus very specific conditions are placed on the possible Monster classes to which the elements of $\{g^a h^{-c},(a,c)=1\}$ may belong.

Example 2. Suppose that g and h (and their Monster conjugates) are the only Fricke elements in $\{g^a h^{-c},(a,c)=1\}$. Then from 1 above $(V^\natural)_{\rm orb}^{\langle g,h\rangle}=V^\natural$. It also follows with $u=g^{-1}$ and $v=h$ that $g^{-a}h$ is Fricke for a unique a mod p with $o(g^{-a}h)=pk$ and $\phi_{g^{-a}h}(g^{-1})=1$. By assumption this implies $a=0$ and hence $\phi_h(g^{-1})=1$. Then acting with the Fricke involution $W_{p^2k}:\tau\to -1/p^2k\tau$ we find

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](pW_{p^2k}(\tau))=Z\left[\begin{array}{c} g^{-1} \\ h \end{array}\right](pk\tau)=\frac{1}{q}+0+O(q).$$

Following a similar argument to that given earlier we find that (16) is a hauptmodul for $\langle\Gamma_0(p^2k),W_{p^2k}\rangle$. Such groups only exist when $p^2k=4$, 8, 9, 12, 18, 20, 25, 27, 49 or 50. In practice $p^2k=49$ and 50 do not occur. Thus again specific conditions are placed on the possible Monster classes to which the elements of $\{g^a h^{-c},(a,c)=1\}$ may belong.

We conclude with a remark on another constraint on the Monster group that arises from Moonshine (and a little group theory). Suppose that $p^2\mid|\mathbb{M}|$. Then $\mathbb{M}$ must contain an Abelian subgroup H of order p^2. Either (a) $H=\langle g\rangle$

with $o(g) = p^2$ or (b) $H = \langle g, h \rangle$ with $o(g) = o(h) = p$. If (a) then T_g is a hauptmodul for either $\Gamma_0(p^2)$ or $\langle \Gamma_0(p^2), W_{p^2} \rangle$ for which $p = 2, 3, 5$ only. If (b) then the genus zero groups found only occur $p \leq 13$. Hence $p^2 \nmid |\mathbb{M}|$ for $p > 13$ as can indeed be observed from (14).

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