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Terwilliger Algebras and Substructures of a Distance-Regular Graph

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1 Introduction

In [5], P. Terwilliger defined subconstituent algebras of a distance-regular graph, which are also called Terwilliger algebras. The algebra $\mathcal{T}(x)$ is defined with respect to a base vertex x, and it is semisimple. Since then much work has been done investigating the module structures of Terwilliger algebras.

We introduce a generalization of the definition of the algebra by replacing the base vertex by a base subset. We give an exposition of the advantages of this generalization.

Distance-regular graphs: Let $\Gamma = (X, R)$ be a connected graph with the vertex set X and the edge set R. Let $\partial(x, y)$ denote the distance between vertices x and y in X. $D = \max\{\partial(x, y) \mid x, y \in X\}$ is called the diameter of Γ.

Definition 1 A connected graph $\Gamma = (X, R)$ is said to be a *distance-regular graph* (DRG) if the number

$$p^h_{i,j} = |\{z \in X \mid \partial(x, z) = i, \text{ and } \partial(z, y) = j\}|$$

is independent of $x, y \in X$ with $\partial(x, y) = h$.

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Let $\Gamma = (X, R)$ be a DRG. Let $\mathrm{Mat}_X(\boldsymbol{C})$ denote the set of matrices whose rows and columns are indexed by X. The i-th adjacency matrix $A_i \in \mathrm{Mat}_X(\boldsymbol{C})$ is a symmetric matrix defined as follows.

$$(A_i)_{x,y} = \begin{cases} 1, & \text{if } \partial(x,y) = i \\ 0, & \text{otherwise} \end{cases}.$$

Then the following hold.

$$A_i A_j = \sum_{h=0}^{D} p_{i,j}^h A_h. \tag{1}$$

$A = A_1$ is called the adjacency matrix of Γ. Let $c_h = p_{h-1,1}^h$, $a_h = p_{1,h}^h$, and $b_h = p_{1,h+1}^h$. Then by (1) we have that

$$A_i A = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}.$$

Hence by induction we can show that there exists a polynomial $v_i(t)$ of degree i such that $A_i = v_i(A)$.

Bose-Mesner Algebra: Let $\mathcal{M} = \mathrm{Span}(A_0, A_1, \ldots, A_D)$. Then $\mathcal{M} = \boldsymbol{C}[A]$ and is called the *Bose-Mesner algebra* of Γ. By (1) it becomes a commutative semisimple algebra.

Let $E_0, E_1, E_2, \ldots, E_D \in \mathrm{Mat}_X(\boldsymbol{C})$ be orthogonal primitive idempotents of $\mathcal{M}$. Then there exist real numbers $p_i(j)$ and $q_i(j)$ for $i, j \in \{0, 1, \ldots, D\}$ and the following hold.

$$A_i = \sum_{j=0}^{D} p_i(j) E_j, \text{ and } E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j.$$

Let $\theta_i = p_1(i)$. Then $\theta_0, \theta_1, \ldots, \theta_D$ are distinct eigenvalues of $A = A_1$. We order $E_1, E_2, \ldots, E_D$ so that

$$k = \theta_0 > \theta_1 > \cdots > \theta_D.$$

Let $m(\theta_i) = \mathrm{rank} E_i > 0$ be the multiplicity of θ_i in A.

Subset and Subconstituents: Let Y be a nonempty subset of X. Let $\Gamma_i(Y) = \{x \in X \mid \partial(x, Y) = i\}$, where $\partial(x, Y) = \min\{\partial(x, y) \mid y \in Y\}$. $\Gamma_i(Y)$ is called the i-th subconstituent with respect to Y. Note that $Y = \Gamma_0(Y)$. Now we have

$$X = \Gamma_0(Y) \cup \Gamma_1(Y) \cup \cdots \cup \Gamma_D(Y) \text{ (disjoint union).}$$

We note that some of the subconstituents may be empty.

Let $V = \boldsymbol{C}^X$. For $x \in X$, $\hat{x}$ is the unit vector in V with 1 at x-entry and 0 otherwise.

For each $i \in \{0, 1, \ldots, D\}$ a diagonal matrix $E_i^* = E_i^*(Y) \in \mathrm{Mat}_X(\boldsymbol{C})$ is defined as follows.

$$(E_i^*)_{x,y} = \begin{cases} 1, & \text{if } x = y \text{ and } \partial(x, Y) = i \\ 0, & \text{otherwise} \end{cases} .$$

E_i^* is the projection onto the subspace $E_i^* V = \mathrm{Span}(\hat{x} \mid x \in \Gamma_i(Y))$.

Definition 2 The subalgebra

$$\mathcal{T} = \mathcal{T}(Y) = \langle A, E_0^*, E_1^*, \ldots, E_D^* \rangle_{\mathrm{alg}}$$

of $\mathrm{Mat}_X(\boldsymbol{C})$ is called the *Terwilliger algebra of* Γ *with respect to* Y.

A $\mathcal{T}$-module is a $\mathcal{T}$-invariant subspace of V. Since $\mathcal{T}$ is generated by real symmetric matrices, $\mathcal{T}$ is semisimple.

Let W be an irreducible $\mathcal{T}$-module. W is said to be *thin* if

$$\dim E_i^* W \leq 1 \text{ for all } i \in \{0, 1, \ldots, D\}.$$

2 Terwilliger algebra and its modules

In the rest of this article, let $\Gamma = (X, R)$ be a DRG of diameter D, and $\emptyset \neq Y \subset X$. Let $E_i^* = E_i^*(Y)$, $\mathcal{T} = \mathcal{T}(Y)$ and $w(Y) = \max\{\partial(x, y) \mid x, y \in Y\}$. $w(Y)$ is called the *width* of Y. We adopt the convention that $E_i^* = O$ if $i < 0$ or $i > D$.

Lemma 1 *Let W be an irreducible module of $\mathcal{T}$. Then the following hold.*

(1) $AE_j^* W \subset E_{j-1}^* W + E_j^* W + E_{j+1}^* W \quad (0 \leq j \leq D)$.

(2) *There exist indices ν and $\nu + \delta$ with $0 \leq \nu \leq \nu + \delta \leq D$ such that*

$$E_j^* W \neq 0 \textit{ if and only if } \nu \leq j \leq \nu + \delta,$$

(3) *Suppose W is thin. Let $E_\nu^* W = \mathrm{Span}(\boldsymbol{v})$. Then $W = \mathcal{T}\boldsymbol{v} = \mathcal{M}\boldsymbol{v}$.*

Remarks.

1. The original definition of Terwilliger algebra is confined to the case $Y = \{x\}$. Lemma 1 for the case $Y = \{x\}$ can be found in [5].

2. The index ν is called the endpoint of W. Every irreducible $\mathcal{T}(Y)$-module of endpoint ν can be regarded as an irreducible $\mathcal{T}(\Gamma_\nu(Y))$-module of endpoint 0. This is one of the advantages of our generalization. Hence as far as we are concerned with one irreducible module, we may assume by taking an appropriate base subset that it is of endpoint 0. Thus it is sufficient to take a nonzero vector $\boldsymbol{v} \in E_0^*V$ and study the module $\mathcal{T}\boldsymbol{v}$.

The following proposition suggests that whether $\mathcal{T}\boldsymbol{v}$ is a thin irreducible module or not can be determined by investigating the vectors $E_i^* A_j \boldsymbol{v}$.

Proposition 2 *Let* $\mathbf{0} \neq \boldsymbol{v} \in E_0^*V$. *Then the following are equivalent.*

(i) $\mathcal{T}\boldsymbol{v}$ *is a thin irreducible* $\mathcal{T}$*-module.*

(ii) $\dim E_i^* \mathcal{M}\boldsymbol{v} \leq 1$ *for every* $i \in \{0, 1, \ldots, D\}$.

(iii) $E_i^* A_{i+h}\boldsymbol{v} \in \mathrm{Span}(E_i^* A_i \boldsymbol{v})$ *for every* $i \in \{0, 1, \ldots, D\}$ *and* $h \in \{0, 1, 2\}$.

3 Shortest Module

Let $\mathbf{0} \neq \boldsymbol{v} \in E_0^*V$. Since $E_0\boldsymbol{v}, E_1\boldsymbol{v}, \ldots, E_D\boldsymbol{v}$ are mutually orthogonal, we have that

$$\dim \mathcal{T}\boldsymbol{v} \geq \dim \mathcal{M}\boldsymbol{v} = |\{\, i \mid E_i\boldsymbol{v} \neq \mathbf{0},\ i \in \{0, 1, \ldots, D\}\}|. \qquad (2)$$

Moreover, if $\mathcal{T}\boldsymbol{v}$ is thin, equality holds above by Lemma 1 (3). What is the lower bound of the right hand side of (2). Let

$$r(\boldsymbol{v}) = |\{\, i \mid E_i\boldsymbol{v} \neq \mathbf{0},\ i \in \{0, 1, \ldots, D\}\}| - 1, \text{ and}$$

$$\rho_{\boldsymbol{v}}(t) = \frac{1}{|X|} \sum_{j=0}^{D} \frac{1}{k_j} \frac{{}^t\boldsymbol{v} A_j \overline{\boldsymbol{v}}}{{}^t\boldsymbol{v}\overline{\boldsymbol{v}}} v_j(t) \in \boldsymbol{R}[t]. \qquad (3)$$

We obtain the following formula by a simple computation.

Lemma 3 *The following hold.*

$$\frac{\|E_i\boldsymbol{v}\|^2}{\|\boldsymbol{v}\|^2} = m(\theta_i)\rho_{\boldsymbol{v}}(\theta_i).$$

Note that ${}^t\boldsymbol{v}A_j\overline{\boldsymbol{v}} = \mathbf{0}$ if $j > w(Y)$. Hence the polynomial $\rho_{\boldsymbol{v}}(t)$ in (3) is of degree at most $w(Y)$. Thus the first part of the following theorem follows from Lemma 3.

Theorem 4 *Let $\Gamma = (X, R)$ be a DRG of diameter D, and $\emptyset \neq Y \subset X$. Let $\mathbf{0} \neq \boldsymbol{v} \in E_0^*(Y)V$. Let $\mathcal{T} = \mathcal{T}(Y)$. Then the following hold.*

(1) $|\{\, i \mid E_i\boldsymbol{v} = \mathbf{0},\ i \in \{0, 1, \ldots, D\}\}| \leq w(Y)$, *i.e.,* $r(\boldsymbol{v}) \geq D - w(Y)$.

(2) *Equality holds in* (1) *if and only if $\mathcal{T}\boldsymbol{v}$ is a thin irreducible module of dimension $r(\boldsymbol{v}) + 1 = D + 1 - w(Y)$.*

Definition 3 $\mathbf{0} \neq \boldsymbol{v} \in E_0^*(Y)V$ with $Y \subset X$ is said to be *tight* with respect to Y, if $|\{i \mid \rho_{\boldsymbol{v}}(\theta_i) = 0\}| = w(Y)$, i.e., if $r(\boldsymbol{v}) \geq D - w(Y)$.

The first main theorem states that each tight vector generates a thin irreducible module with an extremal condition.

4 Thin Modules

Let $\Theta = \{\theta_0, \theta_1, \ldots, \theta_D\}$, $\mathbf{0} \neq \boldsymbol{v} \in E_0^*V$, $r = r(\boldsymbol{v})$ and

$$\Theta(\boldsymbol{v}) = \{\theta \in \Theta \mid \rho_{\boldsymbol{v}}(\theta) \neq 0\}.$$

of cardinality $r + 1$. Let $\omega = m \cdot \rho_{\boldsymbol{v}}$. Then

$$\omega : \Theta(\boldsymbol{v}) \longrightarrow \boldsymbol{R}^{>0} \ :\ (\theta \mapsto m(\theta)\rho_{\boldsymbol{v}}(\theta)).$$

Now there is a system of orthogonal polynomials $\{g_0(t), g_1(t), \ldots, g_r(t)\}$ associated with the weight function ω. We may assume that the leading coefficient of $g_i(t)$ coincides with that of $v_i(t)$, where $v_i(t)$ is a polynomial of degree i such that $v_i(A) = A_i$. Then such a system of polynomials is unique.

Theorem 5 *Let $\boldsymbol{u}_i = g_i(A)\boldsymbol{v}$ for $i \in \{0, 1, \ldots, r+1\}$. Then*

$$\mathcal{T}\boldsymbol{v} \supset \mathcal{M}\boldsymbol{v} = \boldsymbol{C}[A]\boldsymbol{v} = \mathrm{Span}(\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r)$$

and the following hold.

(1) *For every $i \in \{0, 1, \ldots, r\}$,*

$$\frac{||E_i^*A_i\boldsymbol{v}||^2}{||\boldsymbol{v}||^2} \leq \langle g_i(t), g_i(t)\rangle_\omega = ||g_i(t)||_\omega^2. \tag{4}$$

Moreover, equality holds above if and only if $\boldsymbol{u}_i = E_i^*A_i\boldsymbol{v}$.

(2) *The following are equivalent.*

(i) $\mathcal{T}$*-module* $\mathcal{T}\boldsymbol{v}$ *is thin and irreducible.*

(ii) *Equality holds in* (4) *for every* $i \in \{0, 1, \dots, r\}$.

(iii) *Equality holds in* (4) *for* $i = r$.

Definition 4 A nonzero vector $\boldsymbol{v} \in E_0^*(Y)V$ with $\emptyset \neq Y \subset X$ is said to be a T-vector with respect to Y, if $\mathcal{T}(Y)\boldsymbol{v}$ is a thin irreducible $\mathcal{T}(Y)$-module.

Theorem 4 states that this bound is automatically attained if $r(\boldsymbol{v}) = D - w(Y)$, i.e., tight vectors are T-vectors.

5 Applications and Examples

For $z \in \Gamma_i(Y)$, let

$$\pi_j^i = \pi_j^i(z) = |\{y \in Y \mid \partial(z, y) = j\}|.$$

Definition 5 A nonempty subset Y of the vertex set X of a DRG is called a *completely regular code* if $\pi_j^i(z)$ is independent of $z \in \Gamma_i(Y)$.

Proposition 6 *Let* $\Gamma = (X, R)$ *be a DRG and let* $\emptyset \neq Y \subset X$. *Let* $\mathcal{T} = \mathcal{T}(Y)$. *Then the following are equivalent.*

(i) Y *is a completely regular code of* Γ.

(ii) *The characteristic vector* $\mathbf{1}_Y$ *of* Y *is a* T*-vector.*

Hence Theorem 5 gives an algebraic characterization of a completely regular code, and T-vector can be viewed as a generalization of a completely regular code. The investigation of $T(Y)$-modules concerns not only the characteristic vector of a subset Y, or a code, but also the vectors whose supports lie in Y. For algebraic characterization of completely regular codes, see [1, 4].

Tight Subgraphs and T-Subgraphs:

Definition 6 1. An induced subgraph on Y is said to be a *tight subgraph*, if E_0^*V is spanned by tight vectors.

2. An induced subgraph on Y is said to be a T*-subgraph*, if E_0^*V is spanned by T-vectors with respect to Y.

Example 1 1. The tight graphs defined in [3] can be characterized as follows:

> A nonbipartite distance-regular graph is tight (in the sense defined in [3]) if and only if $E^*(Y)V \cap \mathrm{Span}(\mathbf{1}_Y)^\perp$ is spanned by tight vectors with $Y = \Gamma_1(x)$ for some vertex x.

See [2].

2. Let $Y = \Gamma_D(x)$. Then Y is a completely regular code. Since $\mathcal{T}(x) = \mathcal{T}(Y)$, if Γ is thin in the sense of Terwilliger, i.e., if every $\mathcal{T}(x)$-module is thin, then Y is a T-subgraph. There are many DRGs Γ such that $\Gamma_D(x)$ is a T-subgraph; $J(n, D)$, $H(D, q)$, all bipartite Q-polynomial DRGs. In these cases Y has rich structure.

3. $H(d, q)$ is embedded in $H(D, q)$ if $d < D$. This subgraph is a tight subgraph.

4. All dual polar graphs have many tight subgraphs. The natural embeddings of dual polar graphs of smaller ranks seem to be tight.

5. Tightness can be defined in Hecke algebras or C-algebras level as well.

To close this exposition we give two more propositions, which can be regarded as generalizations of results on tight graphs in [3].

In the following for $z \in \boldsymbol{C} \setminus \{-1\}$, let

$$\tilde{z} = -1 - \frac{b_1}{1+z}.$$

Proposition 7 *Let Γ be a DRG of diameter $D \geq 3$ such that $\Gamma_D(x)$ is a clique.*

(1) *The following hold.*

$$\begin{aligned} &k(a_1 + \tilde{\theta}_1\tilde{\theta}_D) \\ &\leq\ (a_1 - \tilde{\theta}_1)(a_1 - \tilde{\theta}_D) + a_D(a_D - b_{D-1} - \tilde{\theta}_1)(a_D - b_{D-1} - \tilde{\theta}_D) \end{aligned}$$

(2) *The following are equivalent.*

(i) *Every irreducible $\mathcal{T}(x)$-module of endpoint 1 is thin.*

(ii) *Equality holds in* (1)

Proposition 8 *Let $\Delta = (Y, R_{|Y\times Y})$ with $Y \subset X$ is a κ-regular subgraph of size $m = |Y|$ such that $d(Y) = 2$. Let $\eta_1 = \kappa \geq \eta_2 \geq \cdots \geq \eta_m$ be eigenvalues of Δ. Then the following hold.*

(1) $(\theta_1 - \tilde{\kappa})(\theta_D - \tilde{\kappa}) + \dfrac{m\kappa b_1^2}{(\kappa+1)^2(m-\kappa-1)} \geq 0.$

(2) *The equality holds in* (1) *if and only if the following holds.*

$$\{\eta_2, \ldots, \eta_m\} \subset \{\tilde{\theta}_1, \tilde{\theta}_D\},$$

i.e., $E_0^*(Y)V \cap \mathrm{Span}(\mathbf{1}_Y)^{\perp}$ *is spanned by tight vectors.*

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