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Engel structures and Legendrian foliations

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0 Introduction

The goals of this short article are to introduce the notion and properties of an Engel structures and to announce the results in [A]. For the precise proofs, see [A].

In any category of geometry, exotic structures are interesting objects to study. In addition, the classification is an important problem. Engel structures are largely dominated by the characteristic foliations. This paper is devoted to the characterization of Engel structures on $M\times S^1$ and $M\times I$, where M is a 3-dimensional manifold, with the same characteristic foliation as the standard Engel structure. The results are described in Section 2.

In the following section, we define an Engel structure and introduce some properties. The notions, Prolongations of contactstructures (Section 1.2), twisting property and Development mappings (Section 1.3), are important for the proof of the results in this article.

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1 Introduction to Engel structures

In this section, we define an Engel structure, and introduce some important properties.

1.1 Basic definitions

A maximally non-integrable distribution of rank 2 on a 4-dimensional manifold is called an *Engel structure*. More precisely, it is defined as follows. Let W be a 4–dimensional manifold. A *distribution* of rank 2 or a 2–*plane field* $\mathcal{D}$ on W is a distribution of 2–dimensional tangent planes $\mathcal{D}_p \subset T_pW$, $p \in W$. It is considered as a rank 2 subbundle of the tangent bundle TW. We can think of $\mathcal{D}$ as a locally free sheaf of smooth vector fields on M. Let $[\mathcal{D}, \mathcal{D}]$ denote the sheaf generated by all Lie brackets $[X, Y]$ of vector fields X, Y on M, which are cross sections of $\mathcal{D}$. We set $\mathcal{D}^2 := \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ and $\mathcal{D}^3 := \mathcal{D}^2 + [\mathcal{D}, \mathcal{D}^2]$.

Definition. A distribution $\mathcal{D}$ of rank 2 on a 4–dimensional manifold W is called an *Engel structure* if it satisfies,

$$\dim \mathcal{D}^2 = 3, \qquad \dim \mathcal{D}^3 = 4, \tag{1.1}$$

at any point $p \in W$.

We note that $\mathcal{D}^2$ is a distribution of rank 3 and $\mathcal{D}^3$ is the tangent bundle TW itself, if $\mathcal{D}$ is an Engel structure. This corank 1 distribution $\mathcal{D}^2$ is an even-contact structure on W. Let $\mathcal{E}$ denote it. An *even-contact structure* is, by definition, a corank 1 distribution on an even-dimensional manifold, which is defined, at least locally, by 1–form θ with a property that $\theta \wedge (d\theta)^{n/2-1}$ is never-vanishing $(n-1)$–form, where n is the dimension of the manifold. An even-contact structure $\mathcal{E}$ on a 4–manifold W has a characteristic 1–dimension. We define a rank 1 subdistribution $\mathcal{L}(\mathcal{E})$ of $\mathcal{E}$ by $[\mathcal{L}(\mathcal{E}), \mathcal{E}] \subset \mathcal{E}$. In this case, its rank is 1. It is called the *characteristic line field* of $\mathcal{E} = \mathcal{D}^2$ or sometimes of $\mathcal{D}$. We call the 1–dimensional foliation obtained by integrating the characteristic line field the *characteristic* *foliation* of $\mathcal{D}^2$ or $\mathcal{D}$. We note that an Engel structure $\mathcal{D}$ should include its characteristic line field to satisfy the Engel condition (1.1).

Contact structures are sometimes described in terms of contact forms. Similarly, Engel structures are described in terms of pairs of differential 1-forms. A pair of 1-forms (α, β) on a 4-manifold W is called an *Engel pair of* 1-*forms* if it satisfies the following three conditions,

(1) $\alpha \wedge \beta \wedge d\alpha$ never vanishes,

(2) $\alpha \wedge \beta \wedge d\beta \equiv 0$,

(3) $\beta \wedge d\beta$ is a never-vanishing 3-form.

It is known that the distribution $\mathcal{D} := \{\alpha = 0,\ \beta = 0\}$ defined by an Engel pair of 1-forms is an Engel structure (see [Ge]). Under these conditions above, the 1-form β defines an even-contact distribution as $\mathcal{D}^2 = \{\beta = 0\}$, and the characteristic line field $\mathcal{L}(\mathcal{D}^2)$ is defined as a kernel of the 3-form $\beta \wedge d\beta$. For an even-contact distribution $\mathcal{E} = \{\beta = 0\}$, a vector field X_0 is called the *characteristic vector field* of $\mathcal{E}$, if it satisfies $X_0 \lrcorner \Omega = \beta \wedge d\beta$ for some volume form Ω on W. We note that it generates the characteristic line field and foliation of $\mathcal{E}$.

Engel structures have no local invariant, similarly to contact and symplectic structures. There exists a local normal form, written as a kernel of two differential 1-forms,

$$dy - z{\cdot}dx = 0, \qquad dz - w{\cdot}dx = 0, \tag{1.2}$$

where (x, y, z, w) is a coordinate system. Further, it is known that this property occurs only on line fields, contact structures, even-contact structures, and Engel structures, among regular tangent distributions on manifolds (see [M1], [VG]). The fact above indicates the importance of the study of Engel structures. However, different from contact structures, global stability does not hold for Engel structures, that is, the Gray type theorem does not hold. The moduli of the characteristic foliation are the obstructions to global stability. It is proved by A. Golubev and R. Montgomery in [Go] and [M2] that a deformation of an Engel structure is realized by an isotopy if it fix the characteristic line field. If the above two differential 1-forms (1.2) are defined globally on $\mathbb{R}^4$, the obtained Engel structure is called the *standard* Engel structure on $\mathbb{R}^4$. Let $\mathcal{D}_{st}$ denote it. The standard Engel structure has its characteristic line field spanned by a vector field in the w direction, $\mathcal{L}({\mathcal{D}_{st}}^2) = \mathrm{Span}\langle \partial/\partial w \rangle$. V. Gershkovich constructs in [Ge] examples of Exotic Engel structures on $\mathbb{R}^4$ with non-trivial characteristic line fields. To study exotic Engel structures with the same characteristic line field as the standard Engel structure is a motivation for this paper.

1.2 Prolongation procedures of contact manifolds.

The notion of prolongation is introduced by E. Cartan in the theory of exterior differential systems (see [C], [BCG3]). We consider the prolongations of contact structures on 3–manifolds. Let ξ be a contact structure on a 3–manifold M, namely a certain 2–plane field. We construct a new 4–dimensional manifold from ξ by fibrewise projectivizations,

$$\mathbb{P}(\xi) := \bigcup_{p \in M} \mathbb{P}(\xi_p),$$

where $\mathbb{P}(\xi_p)$ is a projectivization of a tangent plane ξ_p. A point of $\mathbb{P}(\xi)$ can be regarded as a line l in the contact plane ξ_p through the origin. The constructed 4–manifold $\mathbb{P}(\xi)$ has a structure of a circle bundle over M. Let $\pi\colon \mathbb{P}(\xi) \to M$ be its projection. This 4–manifold is endowed with a 2–plane field $\mathcal{D}(\xi)$ induced naturally as follows. We define 2–plane $\mathcal{D}(\xi)_q \subset T_q(\mathbb{P}(\xi))$ at $q = (p, l) \in \mathbb{P}(\xi)$. A point $q = (p, l) \in \mathbb{P}(\xi)$ is regarded as a pair of a point $p \in M$ and a tangent line $l \subset \xi_p \subset T_pM$. Then we set $\mathcal{D}(\xi)_q := (d\pi^{-1})_q l$. Thus we obtain a 2–plane field $\mathcal{D}(\xi)$ on a 4–manifold $\mathbb{P}(\xi)$. We call this $(\mathbb{P}(\xi), \mathcal{D}(\xi))$ the *prolongation* of a contact structure ξ on a 3–manifold M. It is known that the prolongation $(\mathbb{P}(\xi), \mathcal{D}(\xi))$ is an Engel manifold (see [M2]). We note that the prolonged manifold $\mathbb{P}(\xi)$ is diffeomorphic to $M \times S^1$ if the contact structure ξ belongs to the trivial class as plane fields. In this paper we consider this case especially.

Further, we consider some variants of prolongations. Let $(\mathbb{P}(\xi), \mathcal{D}(\xi))$ be the prolongation of a contact structure ξ on a 3–manifold M. We define a new 4–manifold $\mathbb{P}_n(\xi)$ by fibrewise n–fold covering of $\mathbb{P}(\xi)$,

$$\mathbb{P}_n(\xi) := \bigcup_{p \in M} \mathbb{P}_n(\xi_p),$$

where $\mathbb{P}_n(\xi_p)$ is an n–fold covering space of $\mathbb{P}(\xi_p)$. Let $\varphi_n \colon \mathbb{P}_n(\xi) \to \mathbb{P}(\xi)$ be a fibrewise covering bundle mapping. We obtain a corresponding Engel structure $\mathcal{D}_n(\xi)$ on $\mathbb{P}_n(\xi)$: $(\varphi_n)_* \mathcal{D}_n(\xi) = \mathcal{D}(\xi)$. This pair $(\mathbb{P}_n(\xi), \mathcal{D}_n(\xi))$ is called the *n–fold prolongation* of a contact structure on a 3–manifold M. According to this notation, we have $(\mathbb{P}(\xi), \mathcal{D}(\xi)) = (\mathbb{P}_1(\xi), \mathcal{D}_1(\xi))$. When the given contact structure ξ on a 3–manifold M is trivial as plane fields, $\mathbb{P}_n(\xi)$ is diffeomorphic to $M \times S^1$ for any $n \in \mathbb{N}$. Then we obtain a corresponding Engel structure $\bar{\mathcal{D}}_n(\xi)$, $n = 1, 2, 3, \ldots$ on $M \times S^1$. In addition, we consider a fibrewise universal covering of a prolongation $\mathbb{P}(\xi)$, and let $(\tilde{\mathbb{P}}(\xi), \tilde{\mathcal{D}}(\xi))$ denote it.

Next, we consider the deprolongation procedure, the inverse of the prolongation, in a sense. Similarly to the above, we consider deprolongation of Engel structures especially. Let $\mathcal{D}$ be an Engel structure on a 4–manifold W, $\mathcal{E} := \mathcal{D}^2$ its even-contact structure, and $L(\mathcal{E})$ its characteristic foliation. We consider the leaf space $W/L(\mathcal{E})$ and its projection $\pi \colon W \to W/L(\mathcal{E})$. The foliation $L(\mathcal{E})$ is said to be *nice*, according to [M2], if $W/L(\mathcal{E})$ is a smooth 3–manifold and π is a submersion. We suppose here that $L(\mathcal{E})$ is nice. We set $\xi(\mathcal{E}) := \pi_* \mathcal{E}$. It is a 2–plane field on $W/L(\mathcal{E})$, which is well defined because the characteristic vector field X_0, along $L(\mathcal{E})$, preserves the even-contact structure $\mathcal{E}$. In fact, the even-contact structure $\mathcal{E} = \mathcal{D}^2$ is determined by the second 1–form β of the Engel pair of 1–forms (α, β). Since the characteristic vector field X_0 is defined so that $X_0 \lrcorner (\beta \wedge d\beta) = 0$ (see Section 1.1), we have $L_{X_0}\beta = f \cdot \beta$, for some function f. This implies that X_0 preserves $\mathcal{E}$. Therefore, we have $\xi(\mathcal{E})_{\pi(p)} := (d\pi)_p(\mathcal{E}_p) = (d\pi)_q(\mathcal{E}_q)$ for any point q on the same leaf of $L(\mathcal{E})$ as p because π is the projection along $L(\mathcal{E})$ or X_0. It is known that this distribution $\xi = \xi(\mathcal{E})$ is a contact structure on $W/L(\mathcal{E})$ (see [M2], [Ge]). We call this $(W/L(\mathcal{E}), \xi(\mathcal{E}))$ the *deprolongation* of the Engel structure $\mathcal{D}$.

Let $(W, \mathcal{D})$ be an Engel manifold with the characteristic foliation $L(\mathcal{D}^2)$, and $M \subset (W, \mathcal{D})$ an embedded 3–manifold. We assume that M is transverse to the characteristic foliation L. Then we can take a neighborhood U of M as a flow-box for L. In this neighborhood U, the foliation L is nice in the sense above. Thus we can apply the deprolongation procedure. In this case, we can identify the leaf space $U/L(\mathcal{D}^2)$ with M. Then the obtained contact structure is $\pi_*\mathcal{D}^2 = TM \cap \mathcal{D}^2$. In the case where the Engel manifold is a prolongation $(\mathbb{P}(\xi), \mathcal{D}(\xi))$ of a contact structure ξ on a 3–manifold M and embedded 3–manifold is a cross section M_θ, the characteristic foliation $L(\mathcal{D}(\xi)^2)$ is nice globally. Then, the leaf space $\mathbb{P}(\xi)/L(\mathcal{D}^2)$ is identified with $M_\theta = M$ and the obtained structure is the original ξ.

1.3 Twisting property and Development mappings.

The development mapping is a local Engel diffeomorphism or an immersion into a prolongation $(\mathbb{P}(\xi), \mathcal{D}(\xi))$ or $(\tilde{\mathbb{P}}(\xi), \tilde{\mathcal{D}}(\xi))$, introduced in [M2]. It is constructed by a property that an Engel structure is twisting along leaves of its characteristic foliation. First, we observe the twisting condition of Engel structures. Let us recall that an Engel structure $\mathcal{D}$ contains its characteristic line field $\mathcal{L} = \mathcal{L}(\mathcal{D}^2) \subset \mathcal{D}$ of $\mathcal{D}^2 =: \mathcal{E}$. Let $\tilde{\mathcal{D}}$ be another rank 2 distribution which is contained in $\mathcal{E}$ and contains $\mathcal{L} = \mathcal{L}(\mathcal{E})$: $\mathcal{L} \subset \tilde{\mathcal{D}} \subset \mathcal{E}$. Then the *twisting condition*:

$$\tilde{\mathcal{D}} + [\mathcal{L}, \tilde{\mathcal{D}}] = \mathcal{E} \tag{1.3}$$

implies that $\tilde{\mathcal{D}}$ is Engel, that is, the Engel condition (1.1). Let X_0 be a characteristic vector field of $\mathcal{E}$, and V a vector field which forms a basis of

$\tilde{\mathcal{D}}$ with X_0. We note $[V,\mathcal{E}] \not\subset \mathcal{E}$ because V is independent of the integrable subdistribution $\mathcal{L}(\mathcal{E}) \subset \mathcal{E}$. Then the derived distributions are

$$\begin{aligned}\tilde{\mathcal{D}}^2 &= \tilde{\mathcal{D}} + [\mathcal{L},\tilde{\mathcal{D}}] + [V,\tilde{\mathcal{D}}] = \mathcal{E},\\ \tilde{\mathcal{D}}^3 &= \mathcal{E} + [\mathcal{L},\mathcal{E}] + [V,\mathcal{E}] = \mathcal{E} + [V,\mathcal{E}],\end{aligned}$$

where we use the fact that the rank of $\tilde{\mathcal{D}}$ is 2. Therefore we obtain $\dim \tilde{\mathcal{D}}^2 = 3$, $\dim \tilde{\mathcal{D}}^2 = 4$, namely, the Engel condition (1.1).

Using this twisting condition, we construct a mapping from a domain in an Engel manifold with a nice characteristic foliation to a prolongation of some contact structure. Let $(W,\mathcal{D})$ be an Engel manifold, and $L = L(\mathcal{D}^2)$ the characteristic foliation. Let $U \subset (W,\mathcal{D})$ be a domain where the characteristic foliation L is nice in the sense above, and $\pi\colon U \to U/L$ the projection to the leaf space. We set $\xi := \pi_*(\mathcal{D}^2)$. It is a deprolonged contact structure on U/L from $\mathcal{D}$. Let $l \subset W$ be a leaf of L, and $q \in l \cap U$ a point. From another point of view, l is a point of the leaf space U/L. For a point $q \in l$, there corresponds a tangent 2–plane $\mathcal{D}_q \subset T_qU$. Since an Engel distribution $\mathcal{D}$ contains the characteristic line field $\mathcal{L} = \mathcal{L}(\mathcal{D}^2)$, and is contained in $\mathcal{D}^2$: $\mathcal{L} \subset \mathcal{D} \subset \mathcal{D}^2$, there corresponds a tangent line $d\pi_q(\mathcal{D}_q) \subset d\pi_q(\mathcal{D}^2_q) = \xi_l$, that is, a point of $\mathbb{P}(\xi_l)$. In this way, we obtain a mapping from l to $\mathbb{P}(\xi_l)$. As the domain U with a nice foliation is regarded as a union of leaves $\cup_{l\cap U\neq\emptyset}(l \cap U)$, we obtain a map from a nice domain U to a prolongation $\mathbb{P}(\xi)$. We call this mapping $\Phi_{\mathcal{D}}\colon U \to \mathbb{P}(\xi)$ the *development mapping* associate to the Engel structure $\mathcal{D}$. The twisting condition (1.3), and the argument following it, ensure that this development mapping is diffeomorphic locally. From the construction, the development mapping $\Phi_{\mathcal{D}}$ preserves the characteristic line field and another line field in the Engel distribution twisting in the sense of the twisting condition, with respect to the given $\mathcal{D}$ on U and the prolongation $\mathcal{D}(\xi)$ on $\mathbb{P}(\xi)$. Therefore the development mapping is an Engel diffeomorphic locally (see [M2]). We consider the lift $\tilde{\Phi}_{\mathcal{D}}\colon U \to \tilde{\mathbb{P}}(\xi)$ of the development mapping. We also call this the development mapping.

2 Engel structures on $M \times S^1$, $M \times I$. — Statements of results.

First of all we define the invariant, twisting number, for an Engel structure. In this paper, Engel structures on $M \times S^1$ and $M \times I$ with trivial characteristic foliations are investigated. We say a characteristic foliation L is *trivial* if it is isotopic to a foliation which consists of leaves $\{pt\} \times S^1 \subset M \times S^1$ or $\{pt\} \times I \subset M \times I$. In the following, we suppose that these isotopies have been applied. Namely, we assume in the following that a trivial characteristic foliation consists of leaves $\{pt\} \times S^1$ or $\{pt\} \times I$. In this case, we can define invariants for Engel structures. Let $\mathcal{D}$ be an Engel structure on $M \times S^1$ with a trivial characteristic foliation. An Engel structure, as a 2-plane field, is spanned by the characteristic line field and a line field twisting along leaves of the characteristic foliation (see Section 1.3). Now each leaf of characteristic foliation is a fibre $\{pt\} \times S^1 \subset M \times S^1$. Then we can define the twisting number for Engel structures on $M \times S^1$ and $M \times I$. We begin with $M \times S^1$. Let $\mathcal{D}$ be an Engel structure on $M \times S^1$ with a trivial characteristic foliation, and $\xi = \xi(\mathcal{D})$ its deprolongation. Let $l \in L(\mathcal{D}^2)$ be a leaf of the characteristic foliation corresponding to a point $p \in M = (M \times I)/L(\mathcal{D}^2)$. We note that l is diffeomorphic to S^1. Then we obtain a mapping from $l \cong S^1$ to $\mathbb{P}(\xi_p) \cong \mathbb{R}P^1 \cong S^1$, defined as $\theta \mapsto \mathcal{D}_{(p,\theta)} \cap T_pM$, which can be regarded as a mapping $S^1 \to S^1$. The *twisting number* $\mathrm{tw}(\mathcal{D})$ of $\mathcal{D}$ is defined as the degree of this mapping. We suppose that the orientation of a fibre l is defined by the characteristic vector field X_0. Considering the basis (v_0, v_1) of ξ such that $[X_0, v_0] = v_1$, we obtain an orientation of ξ, and then that of $S^1 \cong \mathbb{P}(\xi_p)$. We consider the above degree with respect to those orientations. We note that it is independent of the choice of points $p \in M$. In other words, the Engel structure $\mathcal{D}$ with the twisting number $\mathrm{tw}(\mathcal{D}) = n$ is Engel diffeomorphic to $(\mathbb{P}_n(\xi), \mathcal{D}_n(\xi))$ by the development mapping, where $\xi = \xi(\mathcal{D})$. Next, we define the minimal twisting number for Engel structures on $M \times I$ with trivial characteristic foliations. Let $\bar{\mathcal{D}}$

be an Engel structure on $M \times I$ with the trivial characteristic foliation, and $\bar{\xi} = \xi(\bar{\mathcal{D}})$ its deprolongation. We take a pair (V_0, V_1) of non-vanishing vector fields on M, which is a positively oriented basis of ξ as above, such that V_0 defines the induced Legendrian foliation $\mathcal{F}_0 = \mathcal{F}(M \times \{0\}, \bar{\mathcal{D}})$. Then we obtain a family $g_t \colon M \to \mathbb{R}$ of functions, which satisfies that $g_0 \equiv 0$ and that the vector fields $(V_0 \cdot \cos(g_t\pi) + V_1 \cdot \sin(g_t\pi))$ define the line fields $\bar{\mathcal{D}} \cap TM_t$, by identifying M_t with M. We call the non-negative integer $n \in \mathbb{Z}_{\geq 0}$ such that $n \leq \min_{p \in M} g_1 < n+1$ the *minimal twisting number* of $\bar{\mathcal{D}}$. Let $\mathrm{tw}_{-}(\bar{\mathcal{D}})$ denote it. We note that it is independent of the choice of V_1 and the orientation of V_0.

Next, we consider the induced Legendrian foliations. Let $\mathcal{D}$ be an Engel structure on $M \times S^1$ or $M \times I$ with a trivial characteristic foliation $L(\mathcal{D}^2)$. Let us identify cross sections of $M \times S^1$ and $M \times I$ with M itself by the standard projection. This projection is the projection along the characteristic foliation $L(\mathcal{D}^2)$ too. We note that this M is transverse to the characteristic foliation $L(\mathcal{D}^2)$. It is known that the even-contact structure $\mathcal{D}^2$ induce a contact structure $\xi(\mathcal{D}^2)$ on M (see [M2] and [Ge]). It does not depend on the choice of the cross section (see Section 1.2). Similarly the Engel distribution $\mathcal{D}$ induce an 1-dimensional foliation $\mathcal{F}(M, \mathcal{D})$ on M by the integral of the line field defined by the intersection of $\mathcal{D}$ and M. We note that the leaves of this foliation $\mathcal{F}(M, \mathcal{D})$ are tangent to the induced contact structure $\xi(\mathcal{D}^2)$ everywhere. Curves in contact 3-manifolds, which are tangent to the contact structures everywhere, are called *Legendrian curves.* We call this foliation $\mathcal{F}(M, \mathcal{D})$ the *induced Legendrian foliation.*

Now, we are ready to state the results of this paper. First, we consider Engel structures on $M \times S^1$ with trivial characteristic foliations.

Theorem 1. (1) *Let ξ, ζ be parallelizable contact structures, that is, they have global framings. If the prolonged Engel structures $\bar{\mathcal{D}}(\xi)$ and $\bar{\mathcal{D}}(\zeta)$ are isotopic preserving the characteristic foliation, then the contact structures ξ and ζ are isotopic.*
(2) *For any Engel structure $\mathcal{D}$ on a 4-dimensional manifold $M \times S^1$ with a*

trivial characteristic foliation, there exist a contact structure ξ *on* M *and a natural number* $n \in \mathbb{N}$, *for which the* n*-fold prolonged Engel structure* $\bar{\mathcal{D}}_n(\xi)$ *of* ξ *on* $M \times S^1$ *is isotopic to* $\mathcal{D}$.

In other words, this theorem implies the following.

Corollary 1. *Engel structures on* $M \times S^1$ *with trivial characteristic foliations are characterized, up to isotopies, by isotopy classes of contact structures on* M *and the twisting number* $\mathrm{tw}(\mathcal{D}) \in \mathbb{N}$.

We note that we can construct an Engel structure $\bar{\mathcal{D}}_n(\xi)$ on $M \times S^1$ for any isotopy class $[\xi]$ of of contact structures on M which are trivial as plane fields, and a natural number $n \in \mathbb{N}$.

Next, we show that Engel structures on $M \times I$ with the trivial characteristic foliation are determined by the induced contact structures, the minimal twisting numbers, and the induced Legendrian foliations on both ends $M \times \partial I$.

Theorem 2. (1) *Let* $\mathcal{D}$ *and* $\tilde{\mathcal{D}}$ *be Engel structures on* $M \times I$ *with the trivial characteristic foliations. If they have the same induced contact structure, induced Legendrian foliations on both ends* $M \times \partial I$, *and minimal twisting number, then they are isotopic relative to the ends.*
(2) *Let* ξ *be a parallelizable contact structure on a* 3*-manifold* M, $(\mathcal{F}_0, \mathcal{F}_1)$ *a pair of Legendrian foliations on* (M, ξ), *and* $n \in \mathbb{Z}_{\geq 0}$ *a non-negative integer. Then there exists an Engel structure* $\mathcal{D} = \mathcal{D}(\xi, \mathcal{F}_0, \mathcal{F}_1, n)$ *on* $M \times I$, *which has the induced contact structure* $\xi(\mathcal{D}) = \xi$, *the induced Legendrian foliations* $\mathcal{F}(M \times \{i\}, \mathcal{D}) = \mathcal{F}_i, i = 0, 1$, *and the minimal twisting number* $\mathrm{tw}_-(\mathcal{D}) = n$.

This theorem implies the following.

Corollary 2. *Engel structures on* $M \times I$ *with the trivial characteristic foliation are determined by the induced contact structures and the induced Legendrian foliations on both ends* $M \times \partial I$ *and the minimal twisting num-*

This Theorem 1 implies that Engel structures on $M \times S^1$ with the trivial characteristic foliation are classified, up to isotopy, if contact structures on the 3-manifold M are classified. The classification of contact structure has been an important problem for a long time. There are some results on this subject (for example, [El1], [El2], [K], [Et], [Gi], [H]). Here we take S^3 for example.

Example. Let us consider Engel structures on $S^3 \times S^1$ with trivial characteristic foliations. Contact structures on S^3 are classified, up to isotopy, to the following structures (see [El1], [El2]): a tight structure ζ, overtwisted structures ξ_m, $m \in \mathbb{Z}$. Then, any Engel structure on $S^3 \times S^1$ with a trivial characteristic foliation is isotopic to one of the followings:

$$\bar{\mathcal{D}}_n(\zeta), \quad \bar{\mathcal{D}}_n(\xi_m), \quad n \in \mathbb{N},\ m \in \mathbb{Z}.$$

These Engel structures are not isotopic each other.

References

[A] J. Adachi, *Engel structures with trivial characteristic foliations*, Algebr. Geom. Topol. **2** (2002), 239–255.

[BCG3] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, P. A. Griffiths, *Exterior differential systems*, Mathematical Sciences Research Institute Publications, **18**. Springer-Verlag, New York, 1991.

[C] E. Cartan, *Sur quelques quadratures dont l'element differentiel contient des fonctions arbitraires*, Bull. Soc. Math. France **29** (1901), 118–130.

[El1] Ya. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math. **98** (1989), 623–637.

[El2] Ya. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 1-2, 165–192.

[Et] J. B. Etnyre, *Tight contact structures on lens space*, Commun. Contemp. Math. **2** (2000), no. 4, 559–577.

[Ge] V. Gershkovich, *Exotic Engel structures on* $\mathbb{R}^4$, Russian J. Math. Phys. **3** (1995), no. 2, 207–226.

[Go] A. Golubev, *On the global stability of maximally nonholonomic two-plane fields in four dimensions*, Internat. Math. Res. Notices 1997, no.**11**, 523–529.

[Gi] E. Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. **141** (2000), no. 3, 615–689.

[H] K. Honda, *On the classification of tight contact structures I*, Geom. Topol. **4** (2000), 309–368.

[K] Y. Kanda, *The classifications of contact structures on 3-torus*, Comm. Anal. Geom. **5** (1997), 413–438.

[M1] R. Montgomery, *Generic distributions and Lie algebras of vector fields*, J. Differential Equations **103** (1993), no. 2, 387–393.

[M2] R. Montgomery, *Engel deformations and contact structures*, Northern California Symplectic Geometry Seminar, Amer. Math. Soc. Transl. Ser. 2, **196**, 103–117, Amer. Math. Soc., Providence, RI, 1999.

[VG] A. Vershik and V. Gershkovich, *Nonholonomic dynamical systems. Geometry of distributions and variational problems*, Encyclopaedia Math. Sci. **16** (1994), 1–81.

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