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AN INTRODUCTION TO THE SPECIAL MCKAY CORRESPONDENCE

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1. Introduction

This note is based on the paper "Special McKay correspondence " [9] by the author and we will show you more concrete exmaple to see the structures.

Let us consider an example of 2-dimensional quotient singularity which is obtained from the action of a cyclic group

First, we recall the toric resolution of cyclic quotient singularities because the quotient space $\mathbb{C}^2/G$ is a toric variety.

Let $\mathbb{R}^2$ be the 2-dimensional real vector space, $\{e^i|i=1,2\}$ its standard base, L the lattice generated by e^1 and e^2, $N := L+\sum \mathbb{Z}v$, where the summation runs over all the elements $v = 1/r(1,a) \in G = C_{r,a}$, and

$$\sigma := \left\{ \sum_{i=1}^{2} x_i e^i \in \mathbb{R}^2, \quad x_i \geq 0, \forall i, 1 \leq i \leq 2 \right\}$$

the naturally defined rational convex polyhedral cone in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. The corresponding affine torus embedding Y_σ is defined as $\mathrm{Spec}(\mathbb{C}[\check{\sigma} \cap M])$, where M is the dual lattice of N and $\check{\sigma}$ the dual cone of σ in $M_{\mathbb{R}}$ defined as $\check{\sigma} := \{\xi \in M_{\mathbb{R}} | \xi(x) \geq 0, \forall x \in \sigma\}$.

Then $X = \mathbb{C}^2/G$ corresponds to the toric variety which is induced by the cone σ within the lattice N.

Fact 1 We can construct a simplicial decomposition S with the verteces on the Newton Boundary, that is, the convex hull of the lattice points in σ except origin.

Fact 2 If $\widetilde{X} := X_S$ is the corresponding torus embedding, then X_S is non-singular. Thus, we obtain the minimal resolution $\pi = \pi_S : \widetilde{X} = X_S \longrightarrow \mathbb{C}^2/G = Y$. Moreover, each lattice point of the Newton boundary corresponds to an exceptional divisor.

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Example Let us look at the example of the cyclic quotient singularity of type $C_{7,3}$ which is generated by the matrix $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^3 \end{pmatrix}$ where $\epsilon^7 = 1$. The toric resolution of this quotient singularity is given by the triangulation of a lattice $N\colon = \mathbf{Z}^2 + \frac{1}{7}(1,3)\mathbf{Z}$ with the lattice points: See Figure 1.1.

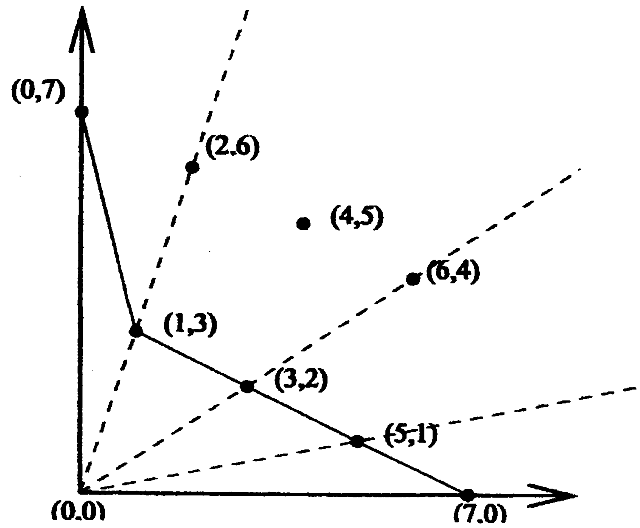

FIGURE 1.1. toric resolution of $\mathbf{C}^2/G$

From this Newton polytope, we can see that there are 3 exceptional divisors and the dual graph gives the configuration of the exceptional components with a deformed coordinate from the original coordinate (x, y) on $\mathbf{C}^2$ as in Figure 1.2.

Therefore we have 4 affine pieces in this example and we have 4 coordinate systems corresponding to each affine piece.

In any example of the original McKay correspondence, all of non-trivial irreducible representations appear as the corresponding representations to the exceptional divisors. However, in this case, there are only three exceptional curves though there are six non-irreducible representations. Thereofore we must find the three "special" representations among the six representations, if there exists the generalized McKay correspondence.

At the end of this note, we can find the corresponding special irreducible representations, and you will see two ways to find them. The first original definition will be written in the section 3, and you will be able to see more easy way finally.

This paper is organized as follows: In the next section, we will give a brief history of the McKay correspondence and we will discuss the special representations and the generalized McKay correspondence in

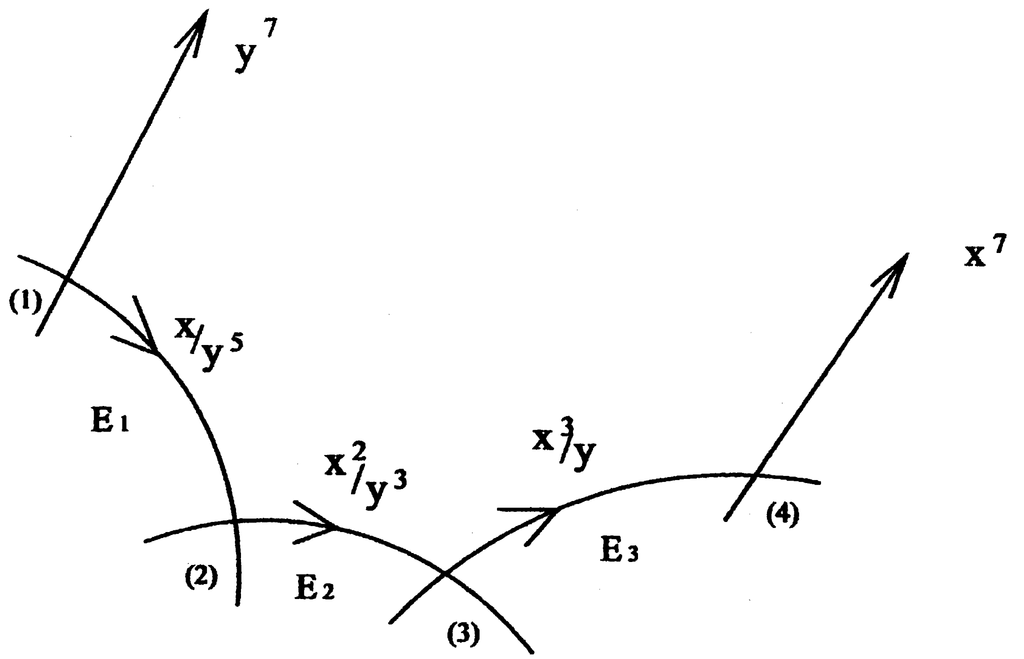

FIGURE 1.2. configuration of $\tilde{X}$

the following section. In section four, we treat G-Hilbert schemes as a resolution of singularities, consider the relation with the toric resolution in the cyclic case, and show how to find the special representations by combinatorics. In the final section, we will compare the difiiculties of the caliculations of the special representations between the original way and this new combinatorial way with an example.

2. McKay correspondence

We will review a history of the McKay correspondece in this section.

The McKay correspondence is originally a correspondence between the topology of the minimal resolution of a 2-dimensional rational double point, which is a quotient singularity by a finite group G of $SL(2,\mathbb{C})$, and the representation theory (irreducible representations or conjugacy classes) of the group G. We can see the correspondence via Dynkin diagrams, which came from McKay's observation in 1979 [16].

Let G be a finite subgroup of $SL(2,\mathbb{C})$, then the quotient space $X := \mathbb{C}^2/G$ has a rational double point at the origin. As there exists the minimal resolution $\widetilde{X}$ of the singularity, we have the exceptional divisors E_i. The dual graph of the configuration of the exceptional divisors is just the Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8.

On the other hand, we have the set of the irreducible representations ρ_i of the group G up to isomorphism and let ρ be the natural representation in $SL(2,\mathbb{C})$. The tensor product of these representations

$$\rho_i \otimes \rho = \Sigma_{j=0}^{r} a_{ij}\rho_j,$$

where r is the number of the non-trivial irreducible representations, gives a set of integers a_{ij} and it determines the Cartan matrix which defines the Dynkin diagram. [1]

Then we have a one-to-one numerical correspondence between non-trivial irreducible representations $\{\rho_i\}$ and irreducible exceptional curves $\{E_i\}$, that is, the intersention matrix of the exceptional divisors can be written as $(-1)\times$ Cartan matrix.

This phenomenon was explained geometrically in terms of vector bundles on the minimal resolution by Gonzalez-Sprinberg and Verdier ([7]) [2] by case-by-case computations in 1983. In 1985, Artin and Verdier [1] proved this more generally with relexive modules and this theory was developed by Esnault and Knörrer ([5], [6]) for more general quotient surface singularities. After Wunram [21] constructed a nice generalized McKay correspondence for any quotient surface singularities in 1986 in his dissertation, Riemenschneider intoruduced the notion of "special representation etc." and made his propaganda for the more generalized McKay correspondence [18]. [3]

In dimension three, we have several "McKay correspondences" but they are just bijections between two sets: Let X be the quotient singularity $\mathbb{C}^3/G$ where G is a finite subgroup of $SL(3,\mathbb{C})$. Then X has a Gorenstein canonical singularity of index 1 but not a terminal singularity. It is known that there exist crepant resolutions $\widetilde{X}$ of this singularity. The crepant resolution is a minimal resolution and preserves the triviality of the canonical bundle in this case.

As the McKay correspondence, following bijections are known:

(1) (Ito-Reid [12]) cohomology group $H^{2i}(\widetilde{X},\mathbb{Q}) \leftrightarrow$ {the conjugacy classes of "age" i in G}.

(2) (Ito-Nakajima [10])Grothendieck group $K(\widetilde{X}) \leftrightarrow$ {the irreducible representations of G}, where G is a finite abelian group.

(3) (Bridgeland-King-Reid [3]) Derived category $D(\widetilde{X}) \leftrightarrow$ {the irreducible representations of G} for any finite subgroups.

[1] More precisely, the Cartan matrix is defined as the matrix $2E - A$, where E is the $(r-1)\times(r-1)$ identity matrix and $A = \{a_{ij}\}$ $(i,j \neq 0)$.

[2] They gave the name *McKay correspondence* (in French, *la correspondance de McKay*) in this paper!

[3] Similar generalization for $G \subset GL(2,\mathbb{C})$ was obtained by Gonzalez-Sprinberg.

Remark 2.1. In (1), the age of $g \in G$ is defined as follows: After diagonalization, if $g^r = 1$, we obtain $g' = \mathrm{diag}(\epsilon^a, \epsilon^b, \epsilon^c)$ where ϵ is a primitive r-th root of unity. Then $\mathrm{age}(g) := (a+b+c)/r$. For the identity element id, we define $\mathrm{age}(id) = 0$ and all ages are integers if $G \subset SL(3, \mathbb{C})$.

The correspondence (2) can be included in (3), but note that the 2-dimensional numerical McKay correspondence can be explained very clearly as a corollary of the result in [10].

As a generalization of the first McKay correspondence (1), we have precise correspondence for each $2i$-th cohomology with conjugacy classes of age i for any $i = 1, \cdots, n-1$ in dimension n which was given by Batyrev and Kontsevich by "motivic integration" under the assumption of the existence of a crepant resolution, and this idea was developed to "string theoretic cohomology" for all quotient singularities (cf.[2]).

And we can see that the string theoretic Euler number of the resolution is the same as the order of the acting group G in case $G \subset GL(n, \mathbb{C})$, but it is different from the usual topological Euler number of the minimal resolution. Of course, it is very interesting to consider the geometrical meaning of these new invariants.

By the way, in (2) we don't have such a difference among representations like *age*. But the author is interested in the relation between the group theory and the classical topological invariants. Then we would like to remind the reader of the notion of special representations which give some difference between irreducible representations. The special representations were defined by Riemenschneider and Wunram [18], which correspond to an exceptional divisor of the minimal resolution of a 2-dimensional quotient singularity.

In particular, we would like to discuss special representations and the minimal resolution for quotient surface singularities from now on. Around 1996, Nakamura and the author showed another way to the McKay correspondence with the help of the G-Hilbert scheme, which is a 2-dimensional G-fixed set of the usual Hilbert scheme of $|G|$-points on $\mathbb{C}^2$ and isomorphic to the minimal resolution. Kidoh [14] proved that the G-Hilbert scheme for general cyclic surface singularities is the minimal resolution. Then Riemenschneider checked the cyclic case and conjectured that the representations which are given by the Ito-Nakamura type McKay correspondence via G-Hilbert scheme are just special representations in 1999 ([19]) and this conjecture was proved by A. Ishii recently ([8]). In this paper, we will give another characterization of the special representations by combinatorics for the cyclic quotient case using results on the G-Hilbert schemes.

As a colorful introduction to the McKay correspondence, the author would like to recommend a paper presented at the Bourbaki seminar by Reid [17] and also on the Web page (http://www.maths.warwick.ac.uk/~miles/McKay), one can find some recent papers related to the McKay correspondence.

3. Special representations

In this section, we will discuss the special representations. Let G be a finite small subgroup of $GL(2, \mathbb{C})$, that is, the action of the group G is free outside the origin, and ρ be a representation of G on V. G acts on $\mathbb{C}^2 \times V$ and the quotient is a vector bundle on $(\mathbb{C}^2 \setminus \{0\})/G$ which can be extended to a reflexive sheaf $\mathcal{F}$ on $X: = \mathbb{C}^2/G$.

For any reflexive sheaf $\mathcal{F}$ on a rational surface singularity X and the minimal resolution $\pi: \tilde{X} \to X$. We define a sheaf $\widetilde{\mathcal{F}}: = \pi^*\mathcal{F}/\text{torsion}$.

Definition 3.1. ([5]) The sheaf $\widetilde{\mathcal{F}}$ is called a *full sheaf* on $\tilde{X}$.

Theorem 3.2. ([5]) *A sheaf $\widetilde{\mathcal{F}}$ on $\tilde{X}$ is a full sheaf if the following conditions are fulfilled:*

1. *$\widetilde{\mathcal{F}}$ is locally free,*
2. *$\widetilde{\mathcal{F}}$ is generated by global sections,*
3. *$H^1(\tilde{X}, \widetilde{\mathcal{F}}^\vee \otimes \omega_{\tilde{X}}) = 0$, where $\vee$ means the dual.*

Note that a sheaf $\widetilde{\mathcal{F}}$ is indecomposable if and only if the corresponding representation ρ is irreducible. Therefore we obtain an indecomposable full sheaf $\widetilde{\mathcal{F}}_i$ on $\tilde{X}$ for each irreducible representation ρ_i, but in general, the number of the irreducible representations is larger than that of irreducible exceptional components. Therefore Wunram and Riemenschneider inroduced the notion of a speciality for full sheaves:

Definition 3.3. ([18]) A full sheaf is called *special* if and only if

$$H^1(\tilde{X}, \widetilde{\mathcal{F}}^\vee) = 0.$$

A reflexive sheaf $\mathcal{F}$ on X is *special* if $\widetilde{\mathcal{F}}$ is so.

A representation ρ is *special* if the associated reflexive sheaf $\mathcal{F}$ on X is special.

With these definitions, following equivalent conditions for the speciality hold:

Theorem 3.4. ([18], [21])

1. *$\widetilde{\mathcal{F}}$ is special $\Longleftrightarrow$ $\widetilde{\mathcal{F}} \otimes \omega_{\tilde{X}} \to [(\mathcal{F} \otimes \omega_{\tilde{X}})^{\vee\vee}]^\sim$ is an isomorphism,*
2. *$\mathcal{F}$ is special $\Longleftrightarrow$ $\mathcal{F} \otimes \omega_{\tilde{X}}/\text{torsion})$ is reflexive,*
3. *ρ is a special representation $\Longleftrightarrow$ $(\Omega^2_{\mathbb{C}^2})^G \otimes (\mathcal{O}_{\mathbb{C}^2} \otimes V)^G \to (\Omega^2_{\mathbb{C}^2} \otimes V)^G$ is surjective.*

Then we have following nice generalized McKay correspondence for quotient surface singularities:

Theorem 3.5. ([21]) *There is a bijection between the set of special non-trivial indecomposable reflexive modules $\mathcal{F}_i$ and the set of irreducible components E_i via $c_1(\widetilde{\mathcal{F}}_i)E_j = \delta_{ij}$ where c_1 is the first Chern class, and also a one-to-one correspondence with the set of special non-trivial irreducible representations.*

As a corollary of this theorem, we get the original McKay correspondence for finite subgroups in $SL(2, \mathbb{C})$ back because in thsi case all irreducible representations are special.

4. G-Hilbert schemes and combinatorics

In this section, we will discuss G-Hilbert schemes and a new way to find the special representations for cyclic quotient singularities by combinatorics.

Hilbert scheme of n-points on $\mathbb{C}^2$ can be described as a set of ideals:

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{I \subset \mathbb{C}[x, y] \mid I : \text{ideal}, \dim \mathbb{C}[x, y]/I = n\}.$$

It is a $2n$-dimensional smooth projective variety. The G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{C}^2)$ was introduced in the paper by Nakamura and the author ([11]) as follows:

$$\mathrm{Hilb}^G(\mathbb{C}^2) = \{I \subset \mathbb{C}[x, y] \mid I : G\text{-invariant ideal}, \ \mathbb{C}[x, y]/I \cong \mathbb{C}[G]\},$$

where $|G| = n$. This is a union of components of fixed points of G-action on $\mathrm{Hilb}^n(\mathbb{C}^2)$ and in fact it is just the minimal resolution of the quotient singularity $\mathbb{C}^2/G$. It was proved for $G \in SL(2, \mathbb{C})$ in [11] first by the properties of $\mathrm{Hilb}^n(\mathbb{C}^2)$ and finite group action of G and they state a McKay correspondence in terms of ideals of G-Hilbert schemes.

Later Kidoh ([14]) proved that the G-Hilbert scheme for any small cyclic subgroup in $GL(2, \mathbb{C})$ is also the minimal resolution of the corresponding cyclic quotient singularities and Riemenschneider conjectured that the G-Hilbert scheme for any $G \subset GL(2, \mathbb{C})$ is the minimal resolution of the quotient singularity $\mathbb{C}^2/G$ and it was based on his result. That is, he checked the irreducible representation which are given by the ideals of G-Hilbert scheme, so-called Ito-Nakamura type McKay correspondence, are just the same as the special representations defined by himself [19], see also [18] Recently A. Ishii ([8]) proved more generally that the G-Hilbert scheme for any small $G \subset GL(2, \mathbb{C})$ is always isomorphic to the minimal resolution of the singularity $\mathbb{C}^2/G$ and the conjecture is true:

Theorem 4.1. ([8]) *Let G be a finite small subgroup of $GL(2,\mathbb{C})$.*

(i) G-Hilbert scheme $Hilb^G(\mathbb{C}^2)$ is the minimal resolution of $\mathbb{C}^2/G$.

(ii) For $y \in Hilb^G(\mathbb{C}^2)$, denote by I_y the ideal corresponding to y and let m be the maximal ideal of $\mathcal{O}_{\mathbb{C}^2}$ corresponding to the origin 0. If y is in the exceptional locus, then, as representations of G, we have

$$I_y/mI_y \cong \begin{cases} \rho_i \oplus \rho_0 & \text{if } y \in E_i \text{ and } y \notin E_j \text{ for } j \neq i, \\ \rho_i \oplus \rho_j \oplus \rho_0 & \text{if } y \in E_i \cap E_j, \end{cases} \tag{4.2}$$

where ρ_i is the special representation associated with the ireducible exceptional curve E_i.

Remark 4.3. In dimension two, we can say that G-Hilbert scheme is the same as a 2-dimensional irreducible component of the G-fixed set of $\text{Hilb}^n(\mathbb{C}^2)$. A similar statement holds for $G \subset SL(3,\mathbb{C})$ in dimension three, that is, the G-Hilbert scheme is a 3-dimensional irreducible component of the G-fixed set of $\text{Hilb}^n(\mathbb{C}^3)$ and a crepant resolution of the quotient singularity $\mathbb{C}^3/G$. In this case note that $\text{Hilb}^n(\mathbb{C}^3)$ is not smooth.

Moreover, Haiman proved that S_n-Hilbert scheme $\text{Hilb}^{S_n}(\mathbb{C}^{2n})$ is a crepant resolution of $\mathbb{C}^{2n}/S_n = n$-th symmetric product of $\mathbb{C}^2$, i.e.,

$$\text{Hilb}^{S_n}(\mathbb{C}^{2n}) \cong \text{Hilb}^n(\mathbb{C}^2)$$

in process of the proof of $n!$ conjecture. (cf. [13])

From now on, we restrict our considerations to $G \subset GL(2,\mathbb{C})$ cyclic. Wunram constructed the generalized McKay correspondence for cyclic surface singularities in the paper [20] and we have to consider the corresponding geometrical informations (the minimal resolution, reflexive sheaves and so on) to obtain the special representations. Here we would like to give a new characterization of the special representations in terms of combinatorics. It is much easier to find the special representation because we don't need any geometrical objects, but based on the result of G-Hilbert schemes.

Let us discuss the new characterization of the special representations in terms of combinatorics. Let G be a cyclic group $C_{r,a}$ which is generated by a matrix $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^a \end{pmatrix}$ where $\epsilon^r = 1$ and $\gcd(r,a) = 1$ and consider a character map $\mathbb{C}[x,y] \longrightarrow \mathbb{C}[t]/t^r$ as $x \mapsto t$ and $y \mapsto t^a$, then we have a corresponding characters for each monomials in $\mathbb{C}[x,y]$.

Let I_p be the ideal of the G-fixed point p in the G-Hilbert scheme, then we can define the following sets.

Consider a G-invariant subscheme $Z_p \subset \mathbb{C}^2$ for which $H^0(Z_p, \mathcal{O}_{Z_p}) = \mathcal{O}_{\mathbb{C}^2}/I_p$ is the regular representation of G. Then the G-Hilbert scheme can be regarded as a moduli space of such Z_p.

Definition 4.4. The set of monomials in $\mathbb{C}[x, y]$ $Y(Z_p)$ is called *G-cluster* if all monomials on $Y(Z_p)$ are not in I_p and it can be drawn as a Young diagram of $|G|$ boxes.

Definition 4.5. For any small cyclic group G, let $B(G)$ be the set of monomials which are not divisible, by any G-invariant monomial and call it G-basis.

Definition 4.6. If $|G| = r$, then let $L(G)$ be $\{1, x, \cdots, x^{r-1}, y, \cdots, y^{r-1}\}$, i.e., the set of monomials which cannot be devided by x^r, y^r or xy. We call it L-space for G because the shape of this diagram looks as the chapital "L."

Definition 4.7. The monomial $x^m y^n$ is of *weight* k if $m + an = k$.

Let us describe the method to find the special representations of G with these diagrams:

Theorem 4.8. *For a small finite cyclic subgroup of $GL(2, \mathbb{C})$, the irreducible representation ρ_i is special if and only if the corresponding monomial in $B(G)$ are not contained in the set of monomials $B(G) \setminus L(G)$.*

Proof. In Theorem 3.4 (3), we have the definition of the special representation, and it is not easy to compute all special representations. However look at the behavior of the monomials in $\mathbb{C}[x, y]$ under the map $\Phi_i\ (\Omega^2_{\mathbb{C}^2})^G \otimes (\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G \to (\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ for each representation ρ_i:

First, let us consider the monomial bases of each set. Let $V_i = \mathbb{C}e_i$ and $\rho(g)e_i = \epsilon^{-i}$. An element $f(x, y)dx \wedge dy \otimes \rho_i$ is in $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ if and only if

$$g^* f(x, y)dx \wedge dy \cdot \epsilon^{1+a} \otimes \epsilon^{-i} = f(x, y)dx \wedge dy,$$

that is,

$$g^*(f(x, y)dx \wedge dy) = \epsilon^{i-(a+1)}(f(x, y)dx \wedge dy).$$

Therefore the monomial base for $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ is a set of monomials $f(x, y)$ such that

$$g : f(x, y) \mapsto \epsilon^{i-(a+1)} f(x, y)$$

under the action of G, that is, monomials of weight $i - (a + 1)$.

Similarly, we have the monomial bases for $(\Omega^2_{\mathbb{C}^2})^G$ as the set of monomials $f(x.y)$ of weight $r - (a + 1)$.

The monomial bases for $(\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G$ is given as a set of monomials $f(x,y)$ of weight i.

Let us check the surjectivity of the map Φ_i. If Φ_i is surjective, then all the monomial bases in $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ can be obtained as a product of the monomial basis of two other sets. Therefore the degree of the monomials in $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ must be higher than the degree of the monomials in $(\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G$.

Now look at the map Φ_{a+1}. The vector space $(\mathcal{O}_{\mathbb{C}^2} \otimes V_{a+1})^G$ is generated by the monomials of weight $a+1$, i.e., $x^{a+1}, xy, \cdots, y^b$ where $ab = a+1 \mod r$. On the other hand, $(\Omega^2_{\mathbb{C}^2} \otimes V_{a+1})^G$ is generated by the degree 0 monomial 1. Then the map Φ_{a+1} is not surjective.

By this, if a monomial of type $x^m y^n$, where $mn \neq 0$, is a base of $(\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G$, then there exists a monomial $x^{m-1}y^{n-1}$ in $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ and the degree become smaller under tha map Φ_i. This means Φ_i is not surjective.

Moreover, if the bases of $(\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G$ is generated only by x^i and y^j where $aj = i \mod r$, then the degrees of the monomials in $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ is bigger and Φ_i is surjective. Thus we have the assertion. □

Remark 4.9. From this theorem, we can also say that a representation ρ_i is special if and only if the number of the generators of the space $(\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G$ is 2.

Theorem 4.10. *Let p be a fixed point by G-action, then we can define an ideal I_p by the G-cluster and the configuration of the exceptional locus can be described by these data.*

Proof. The defining equation of the ideal I_p is given by

$$\begin{cases} x^a = \alpha y^c, \\ y^b = \beta x^d, \\ x^{a-d}y^{b-c} = \alpha\beta, \end{cases}$$

where α and β are complex numbers and both x^a and y^c (resp. y^b and x^d) correspond the same representation (or character).

The pair (α, β) is a local affine coorinate near the fixed point p and it is also obtained from the calculation with toric geometry. Moreover each axis of the affine chart is just a exceptional curve or the original axis of $\mathbb{C}^2$. The exceptional curve is isomorphic to a $\mathbb{P}^1$ and the points on it is written by the ratio like $[x^a : y^b]$ (resp. $[x^d : y^c]$) which is corresponding to a special representation ρ_a (resp. ρ_d). The fixed point p is the intersection point of 2 exceptional curves E_a and E_d.

Thus we can get the whole space of exceptional locus by deformation of the point p and patching the affine pieces. □

We will see a concrete example in the following section. Here we would like to make one remark as a corollary:

Corollary 4.11. *For A_n-type simple singularities, all $n+1$ affine charts can be described by $n+1$ Young diagrams of type* $(1, \cdots, 1, k)$.

Proof. In A_n case, xy is always G-invariant, hence $B(G) = L(G)$. Therefore we have $n+1$ G-clusters and each of them corresponds to the monomial ideal (x^k, y^{n-k+2}, xy). □

5. Example

Now let us go back to our example in the first section: the cyclic quotient singularity of type $C_{7,3}$ which is generated by the matrix $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^3 \end{pmatrix}$ where $\epsilon^7 = 1$.

By the Theorem 3.4, we have a way to find the three special representations corresponding to the exceptional curves, i.e., let us consider when the map $(\Omega^2_{\mathbb{C}^2})^G \otimes (\mathcal{O}_{\mathbb{C}^2} \otimes V)^G \to (\Omega^2_{\mathbb{C}^2} \otimes V)^G$ is surjective.

As a basis of the $(\mathcal{O}_{\mathbb{C}^2} \otimes V)^G$, we have the following:

$k=1$ $\{x^4, xy\}$
$k=2$ $\{x^5, x^2y\}$
$k=3$ $\{x^6, x^3y, y^2\}$
$k=4$ $\{1\}$
$k=5$ $\{x\}$
$k=6$ $\{x^2\}$
$k=7$ $\{x^3, y\}$

The basis of the space $(\Omega^2_{\mathbb{C}^2})^G$ is $\{x^3, y\}$.

Moreover, we have the following basis of the space $(\Omega^2_{\mathbb{C}^2} \otimes V)^G$:

$k=1$ $\{x\}$
$k=2$ $\{x^2\}$
$k=3$ $\{x^3, y\}$
$k=4$ $\{x^4, xy\}$
$k=5$ $\{x^5, x^2y\}$
$k=6$ $\{x^6, x^3y, y^2\}$
$k=7$ $\{x^7, x^4y, xy^2\}$

Thus the map will be surjective when $k = 1, 2$ or 3, and we have three special representations ρ_1, ρ_2 and ρ_3.

Now we have a chance to use our new way to find the special representations! Let us draw the diagram which corresponds to the G-basis and L-space. First we have the following G-basis $B(G)$ and the corresponding characters in a same diagram. In Figure 5.1 we draw the L-space as shaded part in $B(G)$.

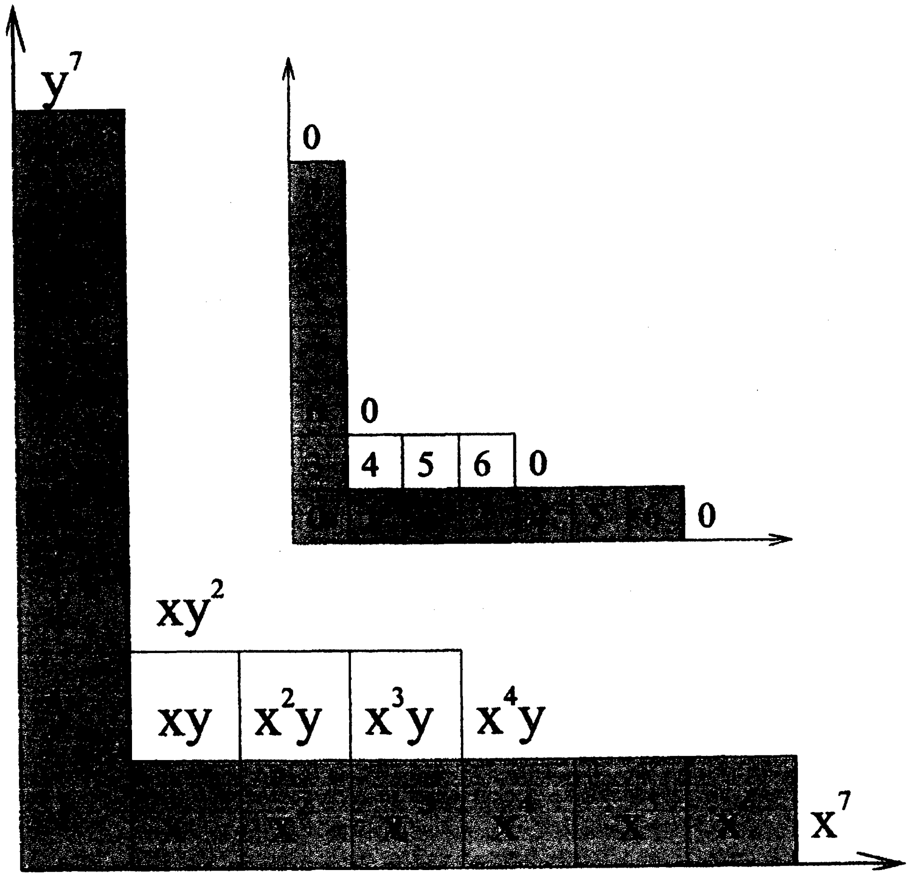

FIGURE 5.1. G-basis $B(G)$ and the characters

Now we have three monomials xy, x^2y and x^3y in $B(G) \setminus L(G)$ and they correspond to the characters (resp. representations) 4, 5 and 6 (resp. ρ_4, ρ_5 and ρ_6). Therefore we can find a set of special representations, that is, $\{\rho_1, \rho_2, \rho_3\}$, and find the corresponding G-clusters, representing the origin of the affine charts of the resolution, can be drawn as 4 young diagrams and get the corresponding special representations in this case. See Figure 5.2.

Let us see the meanings of the corresponding G-clusters in this case. From $Y(Z_p)$ for (2), we obtain an ideal $I_2 = (y^5, x^2, xy^2)$ for the origin of the affine chart (2) in Figure 1.2, and the corresponding representations are ρ_1, ρ_2 and ρ_0. If we take the maximal ideal m of $\mathcal{O}_{\mathbb{C}^2}$ corresponding to the origin 0, then we have

$$I_2/mI_2 \cong \rho_1 \oplus \rho_2 \oplus \rho_0.$$

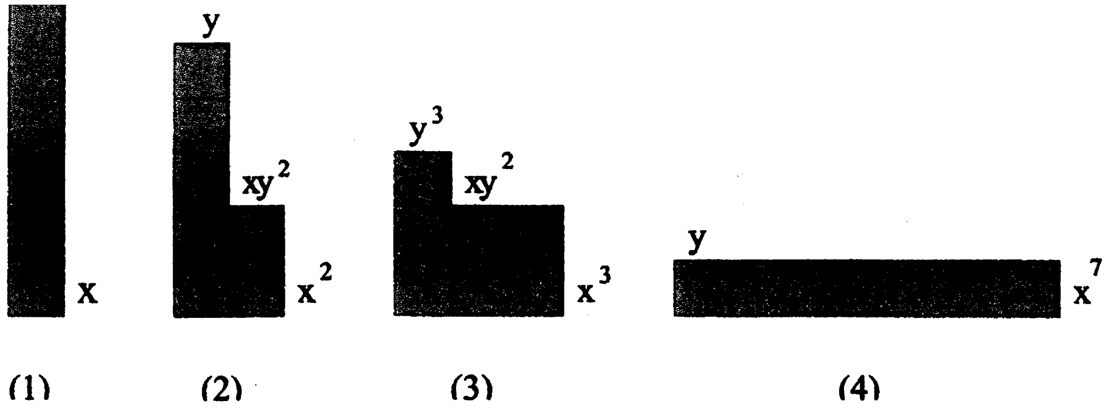

FIGURE 5.2. G-cluster $Y(Z_p)$

Similarly we have the ideal $I_3 = (y^3, x^3, xy^2)$ and

$$I_3/mI_3 \cong \rho_2 \oplus \rho_3 \oplus \rho_0.$$

These descriptions coincide with the results of Theorem 4.1 for an intersecting point at $E_1 \cap E_2$.

For any other points p on the exceptional component E_i, we must have

$$I_p/mI_p \cong \rho_i \oplus \rho_0. \qquad (*)$$

In fact, we can see that on the exceptional divisor E_2 in this example was determined by the ratio $x^2 : y^3$, that is, the corresponing ideal of a point on E_2 can be described as $I_p = (\alpha x^2 - \beta y^3, xy^2 - \gamma)$. Therefore the ratio $(\alpha : \beta)$ gives the coordinate of the exceptional curve ($\cong \mathbb{P}^1$) and we also have $(*)$.

We discussed special McKay correspondence in 2-dimensional case in this paper. In dimension three, it is convenient to consider crepant resolutions as a minimal resolution and we have much more complicated situation. Even in the case $G \subset SL(3, \mathbb{C})$, we have $H^4(\tilde{X}, \mathbb{Q}) \neq 0$ in general. Of course we can use the same definition for the special representations in the higher dimensional case, but all non-trivial irreducible representations of $G \subset SL(3, \mathbb{C})$ are special. On the other hand, the number of the exceptional divisors is less than that of the non-trivial irreducible representations. Therefore, it looks very difficult to generalize this special McKay correspondence. That is, we should make a difference, say a kind of the grading, in the set of the special (or non-trivial) representations like "age" of the conjugacy classes.

However, there are good news: In 2000, Craw [4] constructed a cohomological McKay correspondence for the G-Hilbert schemes where G is an abelian group, and in this correspondence we can see the 2-dimensional special McKay correspondence. And recently, the author found a way to obtain a polytope which corresponds to the 3-dimensional G-Hilbert schemes for abelian subgroups in $SL(3, \mathbb{C})$ by combinatorics. There are many crepant resolutions in general in higher dimension, but G-Hilbert scheme for $G \subset SL(3, \mathbb{C})$ is a unique crepant resolution, and the configuration of the exceptional locus of the special crepant resolution, G-Hilbert scheme, can be determined in terms of a Gröbner basis. (Let us call this the Gröbner method.) Moreover, we can get another characterization of special representations for cyclic quotient surface singularities by this Gröbner method. So the author is dreaming of having a more simple and beautiful formulation of the McKay correspondence in the future.

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