

## Local Zero Estimate

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Consider a set of  $m$  analytic functions  $\Phi := (\Phi_1, \dots, \Phi_m)$  defined in a neighbourhood of a point. Let  $F$  be a polynomial of degree  $k$ . Changing  $F$ , we take the supremum  $\theta_\Phi(k)$  of the vanishing orders of  $F(\Phi)$  at the point that do not vanish identically. If  $\Phi_1, \dots, \Phi_m$  are polynomials,  $\theta_\Phi(k)$  is majorized by a linear function of  $k$ . In general  $\theta_\Phi(k)$  depends upon stiffness of the functions  $\Phi_1, \dots, \Phi_m$ . This estimate is treated by researchers of transcendental number theory and called *zero-estimate*.

Here, we are concerned with zero-estimate at a point. There is a similar problem, *multiplicity estimate*, which treats multiplicity of intersections of functions. These problems coincide in one variable case. Hence these term are often used in the same meaning.

First we introduce that  $\theta_\Phi(k)$  and  $\alpha(\Phi)$  are closely related to transcendence, applying theory of local rings. In particular, we obtain zero-estimate of exponentials in Nash functions. Next we introduce nice zero-estimate obtained by Gabrielov for Noetherian functions along a trajectory of a Noetherian vector field. (Gabrielov and Khovanskii also obtained a multiplicity estimate for an isolated complete intersection of Noetherian functions.) We remark that Gabrielov's result immediately yields zero-estimate for Noetherian functions on higher dimensional integral manifolds.

First we introduce some notations. In particular,  $\theta_\Phi(k)$  and  $\alpha(\Phi)$  are key invariants for our observations.

$K$ : a field of characteristic 0

$(R, \mathfrak{m})$ : local  $K$  algebra, i.e.  $R$  is a commutative ring with the unique maximal ideal  $\mathfrak{m}$  such that  $R/\mathfrak{m}$  is canonically isomorphic to  $K$

$\nu(f) := \sup\{p : f \in \mathfrak{m}^p\}$ : order of  $f$

$\Phi := (\Phi_1, \dots, \Phi_m) \subset R^m$ : chain of order  $m$

$K[\Phi]^k := \{f \in K[\Phi] : \deg(f) \leq k\}$ : set of polynomials in the elements of

$\Phi$  of degree not greater than  $k$

$\theta_{\Phi}(k) := \sup\{\nu(f) : f \in K[\Phi]^k \setminus \{0\}\}$ : growth function of  $K[\Phi]$

$\alpha(\Phi) := \limsup_{k \rightarrow \infty} \log_k \theta_{\Phi}(k)$ : growth order of  $K[\Phi]$

$K(\Phi)$  (for a subset  $\Phi$  of an integral local  $K$  algebra  $R$ ): field of quotients of integral domain  $K[\Phi]$

$\mathcal{O}_n := K\{\mathbf{x}\}$  ( $K = \mathbb{C}, \mathbb{R}$ ): local  $K$  algebra of convergent power series in  $\mathbf{x} := (x_1, \dots, x_n)$

## 1. Algebraic theory

Here we remember purely algebraic properties of invariants  $\theta_{\Phi}(k)$  and  $\alpha_{\Phi}$  associated with a chain  $\Phi$  in a local  $K$  algebra of characteristic 0. The following is obtained by the pigeon box principle applied to Hilbert function of  $K[\Phi]$  and Samuel function of  $R$ .

**Theorem 1** ([I<sub>4</sub>], lower estimate). *Let  $(R, \mathfrak{m})$  be a  $d$  dimensional local  $K$  algebra over field  $K$  of characteristic 0. If  $\Phi \subset R$  and  $e := \text{trdeg}_K K(\Phi)$ , there exists  $c > 0$  such that  $\theta_{\Phi}(k) \geq c \cdot k^{e/d}$ ,  $\alpha(\Phi) \geq e/d$ .*

The following is analogue of Liouville numbers, examples of transcendental numbers obtained as *gap series*.

**Example 2** ([I<sub>3</sub>], Ueda function). *If we put  $\Phi := \{x, \sum_{i=1}^{\infty} \frac{1}{(2^i)!} x^{2^i}\} \subset \mathcal{O}_2$ , we have  $\alpha(\Phi) = \infty$ .*

**Theorem 3** ([I<sub>4</sub>], upper estimate). *Let  $(R, \mathfrak{m})$  be a local  $K$  algebra over field  $K$  of characteristic 0. We suppose that  $R$  is an integral domain. If  $\Phi \subset \Psi \subset R$  and  $\Psi$  is algebraic over  $K(\Phi)$ , then there exists  $a > 0$  and we have  $\theta_{\Phi}(k) \leq \theta_{\Psi}(k) \leq \theta_{\Phi}(ak)$  and  $\alpha(\Phi) = \alpha(\Psi)$ .*

**Theorem 4** ([I<sub>3</sub>], [I<sub>4</sub>], algebraicity criterion). *Let  $(R, \mathfrak{m})$  be a local  $K$  algebra over field  $K$  of characteristic 0. Suppose that  $\Phi \subset R$  generates a  $\mathfrak{m}$  primary ideal of  $R$ , then  $\text{trdeg}_K K(\Phi) \geq \dim R$ . Moreover, the following conditions are equivalent.*

1.  $\text{trdeg}_K K(\Phi) = \dim R$ .
2.  $\alpha(\Phi) = 1$ .
3. *There exists  $a, b \in \mathbb{R}$  such that  $\theta_\Phi(k) \leq ak + b$ .*

Suppose that  $R := \mathcal{O}_n/\mathfrak{a}$  be an analytic local  $K$  algebra for an analytic singularity embedded in  $K^n$ . Let  $\Phi := \{x_1 \bmod \mathfrak{a}, \dots, x_n \bmod \mathfrak{a}\}$  be the restrictions of coordinates of ambient space. Then the theorem above implies that this singularity is algebraic if and only if  $\alpha(\Phi) = 1$ .

**Example 5.** *Let  $f \in \mathcal{O}_2$  be a function germ. Then chain  $\Phi := \{x_1^p, x_2^q, f\}$  ( $p, q \in \mathbb{N}$ ) generates a primary ideal of  $\mathcal{O}_2$ . Hence  $f$  is Nash (i.e algebraic over  $K(x_1, x_2)$ ) if and only if  $\alpha_\Phi = 1$  and if and only if  $\theta_\Phi(k)$  is majorized by a linear function of  $k$ .*

## 2. Exponential functions

The following is a basic fact in the field of transcendental number theory.

**Theorem 6** (exponentials in linear forms). *Let us put*

$$\begin{aligned} \mathbf{x} &:= (x_1, \dots, x_p), \quad \mathbf{y} := (y_1, \dots, y_q), \\ \Psi &:= \{\lambda_{11}x_1, \dots, \lambda_{1h_1}x_1, \dots, \lambda_{p1}x_p, \dots, \lambda_{ph_p}x_p, \\ &\quad \mu_{11}y_1, \dots, \mu_{1k_1}y_1, \dots, \mu_{q1}y_q, \dots, \mu_{qk_q}y_q\} \subset \mathcal{O}_{p+q}, \\ \Phi &:= \{\mathbf{x}, \exp \Psi\} \subset \mathcal{O}_{p+q}. \end{aligned}$$

*If we put*

$$s_i := \dim_{\mathbb{Q}} \sum_{j=1}^{h_p} \mathbb{Q}\lambda_{ij}, \quad t_i := \dim_{\mathbb{Q}} \sum_{j=1}^{k_q} \mathbb{Q}\mu_{ij},$$

*then we have  $\alpha(\Phi) = \max\{s_1 + 1, \dots, s_p + 1, t_1, \dots, t_q\}$ .*

In transcendence theory more complicated cases are teated. For example we know the following.

**Example 7.**

$$\alpha(e^x, e^y, e^{\sqrt{2}x + \sqrt{3}y}) = 1.5$$

*in  $\mathcal{O}_2$ .*

Applying “complementary inequality for homomorphisms of local algebras” [I<sub>1</sub>], [I<sub>2</sub>], we can generalize Theorem 6 as follows.

**Theorem 8** (exponentials in Nash functions [I<sub>4</sub>]). *Suppose that*

$$\mathbf{x} := (x_1, \dots, x_p),$$

$\Theta := \{\Theta_1, \dots, \Theta_p\} \subset \mathcal{O}_n$  *is algebraic over*  $K(\mathbf{x})$ ,

$\Phi := \{\lambda_{11}\Phi_1, \dots, \lambda_{h_1}\Phi_1, \dots, \lambda_{q1}\Phi_q, \dots, \lambda_{qh_q}\Phi_q\}$  *is algebraic over*  $K(\Theta)$ ,

$\Psi := \{\mu_{11}\Psi_1, \dots, \mu_{1k_1}\Psi_1, \dots, \mu_{r1}\Psi_r, \dots, \mu_{rk_r}\Psi_r\}$  *is algebraic over*  $K(\mathbf{x})$   
*and algebraically independent over*  $K(\Theta)$ ,

*the rank of Jacobian matrix (with value in*  $\mathcal{O}_n$ ) *of*  $\Phi$  *is*  $q$  *(full rank).*

*If we put*  $s_i := \dim_{\mathbb{Q}} \sum_{j=1}^{h_q} \mathbb{Q}\lambda_{ij}$ ,  $t_i := \dim_{\mathbb{Q}} \sum_{j=1}^{k_r} \mathbb{Q}\mu_{ij}$ ,

*then we have*

$$\alpha(\Theta, \exp \Phi, \exp \Psi) = \max\{s_1 + 1, \dots, s_p + 1, t_1, \dots, t_q\}.$$

We can deduce, for example, the following.

**Example 9.**  $\alpha(x, e^{x^2}) = 2$ ,  $\alpha(x, e^{\sqrt{x+1}}, e^{\sqrt{2(x+1)}}) = 3$  *in*  $\mathcal{O}_1$ .

### 3. Noetherian functions

Hearing the author's talk on Theorem 8 in 1997, Khovanskii remarked that similar estimates are possible for the big family of Noetherian functions. Perhaps the corollaries below are such estimates.

First, let us recall the definition of Noetherian functions introduced by Khovanskii and Tougeron. Suppose that  $\Phi := (\Phi_1, \dots, \Phi_m) \subset \mathcal{O}_n^m$  is a finite set such that there exists polynomials  $P_{ij}$  of  $m+n$  variables of degree  $d$  such that  $\frac{\partial \Phi_i}{\partial x_j} = P_{ij}(\mathbf{x}, \Phi)$ . Then we call  $\Phi$  a *Noetherian chain* of order  $m$  and degree not exceeding  $d$ . An polynomial  $f$  in  $\mathbf{x}$  and  $\Phi$  of degree not exceeding  $s$  is called a *Noetherian function* of degree  $s$  with respect to Noetherian chain  $\Phi$ .

Gabrielov proved the following using topological integration theory.

**Theorem 10** ([G<sub>1</sub>]: Noetherian functions on a trajectory). *Suppose that*  $\Phi \subset \mathcal{O}_n$  *is a Noetherian chain of order*  $m$  *and degree*  $d$ ;

$\xi$ : *a germ of a vector field at*  $0 \in K^n$  *whose coefficients are Noetherian functions of degree*  $\leq s$  *such that*  $\xi(0) \neq 0$ ;

$M$ : *the germ of integral manifold of*  $\xi$  *through*  $0$ .

*Then we have*

$$\theta_{(\mathbf{x} \cup \Phi)|M}(k) \leq 2^{2(n+m)-1} \sum_{i=1}^{n+m} [k + (i-1)(s+d-1)]^{2(n+m)},$$

$$\alpha((\mathbf{x} \cup \Phi)|M) \leq 2(n + m).$$

It should be noted that similar kind of estimates are obtained by Nesterenko in the eighties (e.g. [N]).

Gabrielov and Khovanskii [GK] generalized this to higher dimensional case, a *multiplicity* estimate of an isolated complete intersection defined by Noetherian functions.

It took long time for the present author to notice the simple fact that Gabrielov's theorem above (1-dimensional case) implies *zero* estimate of Noetherian functions on higher dimensional integral manifold as follows.

**Corollary 11** (Noetherian functions on an integral manifold). *Suppose that  $\Phi \subset \mathcal{O}_n$  is a Noetherian chain of order  $m$  and degree  $d$ ,  $\xi_1, \dots, \xi_r$  are germs of vector fields which are independent at  $0 \in K^n$  and whose coefficients are Noetherian functions of degree  $\leq s$  and that there exists a germ of integral manifold  $M$  of dimension  $r$  through  $0$  for  $\xi_1, \dots, \xi_r$ . Then we have*

$$\theta_{(\mathbf{x} \cup \Phi)|M}(k) \leq 2^{2(n+m)-1} \sum_{i=1}^{n+m} [k + (i-1)(s+d-1)]^{2(n+m)},$$

$$\alpha((\mathbf{x} \cup \Phi)|M) \leq 2(n + m).$$

*Proof* Let  $f$  be any Noetherian function of order  $k$ . We can choose a linear combination  $\eta$  of  $\xi_1, \dots, \xi_r$  over  $\mathbb{C}$  such that  $f$  does not vanish identically on the trajectory  $N \subset M$  of  $\eta$ . We can apply Theorem 10 to  $\Phi$ ,  $\eta$  and  $N$  obtaining the following

$$\nu_N(f|N) \leq 2^{2(n+m)-1} \sum_{i=1}^{n+m} [k + (i-1)(s+d-1)]^{2(n+m)}.$$

The estimate for  $f$  follows at once from obvious inequality  $\nu_M(f) \leq \nu_N(f|N)$ .  $\square$

If we put  $\xi_1 = \frac{\partial}{\partial x_1}, \dots, \xi_r = \frac{\partial}{\partial x_r}$  in the above, then  $s = 0$  and we have the following.

**Corollary 12** (Noetherian functions on Euclidean domain). *Suppose that  $\Phi \subset \mathcal{O}_n$  is a Noetherian chain of order  $m$  and degree  $d$ , then we have*

$$\theta_{\mathbf{x} \cup \Phi}(k) \leq 2^{2(n+m)-1} \sum_{i=1}^{n+m} [k + (i-1)(d-1)]^{2(n+m)},$$

$$\alpha(\mathbf{x} \cup \Phi) \leq 2(n + m).$$

## 4. Problems

**Problem I.** Can we prove the finiteness of  $\alpha$  for exponentials in Nash functions without the full rank condition for Jacobian (see Theorem 6)?

**Problem II.** Geometers may have tendency to seek for conditions for tameness of functions and hence like upper estimates of order,  $\theta$  and  $\alpha$ . On the other hand, number theorists seek for transcendental elements and hence will like lower estimates.

Does there exist a good lower estimate for Noetherian functions?

**Problem III.** Apparently, the family of all Noetherian functions (taking all chains) is far bigger than that of Nash functions. But the author does not know whether all Nash functions are Noetherian or not.

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