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Moduli of punctured Riemann surfaces and the Takhtajan-Zograf metric

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ABSTRACT. We show a convergence theorem of Eisenstein series for degenerating Riemann surfaces, which is an improved version of the former one of the author . We will apply it to investigate L_2-cohomology of the Takhtajan-Zograf metric.

§1. PRELIMINARIES

1.1 Eisenstein series.

Let S be a punctured hyperbolic surface of type (g, n) $(n > 0)$. It can be represented as a quotient H/Γ of the upper half plane $H = \{z \in \mathbb{C} | \mathrm{Im} z > 0\}$ by the action of a torsion free finitely generated Fuchsian group $\Gamma \in \mathrm{PSL}_2(\mathbb{R})$. The group is generated by $2g$ hyperbolic transformations $A_1, B_1, \ldots, A_g, B_g$ and parabolic transformations $P_1, \ldots P_n$ satisfying the relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A_g^{-1} B_g^{-1} P_1 \ldots P_n = 1.$$

The fixed points of the parabolic elements $P_1, \ldots P_n$ will be denoted by $z_1, z_2, \ldots, z_n \in \mathbb{R} \cup \{\infty\}$ respectively and called inequivalent cusps. The projection of the cusps $z_1, z_2, \ldots, z_n$ are the punctures $p_1, p_2, \ldots, p_n$ of S. For each $i = 1, \ldots, n$, denote by Γ_i the stabilizer of z_i in Γ that is the cyclic subgroup of Γ generated by P_i. Pick $\sigma_i \in \mathrm{PSL}_2(\mathbb{R})$ such that $\sigma_i \infty = z_i$ and $\langle \sigma_i^{-1} P_i \sigma_i \rangle = \langle z \mapsto z + 1 \rangle$. Then, for $a > 1$, the $a-$cusp region $C_i(a)$ associated to p_i is represented as a quotient $\langle \sigma_i^{-1} P_i \sigma_i \rangle \diagdown \{z \in H | \mathrm{Im} z > a\} \simeq \Gamma \diagdown \{z \in H | \mathrm{Im} z > a\}$,

$$C_i(a) \simeq [a, \infty) \times S^1, \text{ equipped with the metric } ds^2 = (dy^2 + dx^2)/y^2.$$

Let $\Delta : C^\infty(S) \to C^\infty(S)$ be the negative hyperbolic Laplacian of S. Regarded as an operator in $L^2(S)$ with domain $C_0^\infty(S)$, Δ is essentially self-adjoint. Denote by $\overline{\Delta}$ the unique self-adjoint extension (that is, Friedrichs extension). Then the continuous spectrum of $\overline{\Delta}$ can be described in terms of Eisenstein series ([He]Chap.Seven, [K]Chap.V, [V]§3.2).

The Eisenstein series attached to z_i is defined by

$$E_i(z, s) = \sum_{\gamma \in \langle P_i \rangle \diagdown \Gamma} \mathrm{Im}(\sigma_i^{-1} \gamma z)^s, \qquad \mathrm{Re}\ s > 1.$$

The series is absolutely convergent in the half-plane Re $s > 1$ and in the upper half-plane, it satisfies

$$\Delta E_i(z,s) = s(s-1)E_i(z,s). \tag{1.1}$$

A. Selberg originally showed that the series admits meromorphic continuation to the whole complex s-plane, holomorphic on $\{\mathrm{Re}\ s = \frac{1}{2}\}$and satisfies a system of functional equations ([Sl]§7). Several mathematicians also verified it by the various methods ([dV], [He] Th.11.6, [K]$pp.23-46$, [Mu]). $E_i(z,s)$ has Fourier expansions at punctures p_j, ([He]Prop.8.6 , [K]§2.2, [L-P]§8, [V]§3.1)

$$E_i(\sigma_j z, s) = \delta_{ij} y^s + \phi_{ij}(s) y^{1-s} + \sum_{m\neq 0} c_m(s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|m|y) e^{2\pi\sqrt{-1}mx}, \tag{1.2}$$

$K_{s-\frac{1}{2}}$ the MacDonald-Bessel function ([Wa], p.78), that has the following asymptotics ([Wa], p.202)

$$y^{\frac{1}{2}} K_{s-\frac{1}{2}}(y) \sim \sqrt{\frac{\pi}{2}} e^{-y}, \quad \text{as } y \nearrow \infty, \qquad \text{for any complex } s. \tag{1.3}$$

In the proof of Theorem 1, we need a more precise information about the ratio of Both terms in (1.3). We use 3.70(6)(p.78), 7.2(p.197) in [Wa] and can easily see

$$\left| \frac{y^{\frac{1}{2}} K_{s-\frac{1}{2}}(y)}{\sqrt{\frac{\pi}{2}} e^{-y}} - 1 \right| < \frac{B_s}{y}, \quad \text{as } (\mathbb{R} \ni)\ y \nearrow \infty, \tag{1.4}$$

where B_s can be chosen to be a positive number depending only on s.

1.2 Modified infinite-energy harmonic maps.

In this part, we will introduce the modified infinite-energy harmonic functions that are defined by S. Wolpert ([W2]), while the infinite-energy harmonic maps are originally constructed by M. Wolf ([Wf]), for parametrizing degeneration of hyperbolic surfaces. Denote by $(S_l(l>0), \rho_l(w)|dw|^2)$ a degenerating family of hyperbolic surfaces of type (g,n). We assume that several disjoint simple closed geodesics $l_1, l_2, \dots, l_k$ on S_l will be pinched (We denote their hyperbolic lengths by the same notations). Let Δ_l be the negative Laplacian of S_l. To compare functions on the limit surface $(S_0, \rho(z)|dz|^2)$ and $(S_l, \rho_l(w)|dw|^2)$, M. Wolf has constructed infinite-energy harmonic maps $w^l : S_0 \to S_l \setminus \{l_1, l_2, \dots, l_k\}$ ([J], [Wf], [W2]). A node on S_0 is a pair of cusps and distinct nodes involve distinct cusps. we call the cusps of S_0 that arise from the cusps (resp. arise from the pinching geodesics) of S_l the *old cusps* (resp. the *new cusps*). But w^l is not adequate for us to compare the Eisenstein series for S_l and for S_0 on cusp regions around old cusps, because w^l is not the identity map on the cusp regions and the Eisenstein series has a singularity at the associated cusp for Re $s > 1$. Thus we will use Wolpert's infinite-energy harmonic map, denoted by f^l, that is modified from w^l so that the meridians and longitudes of a cusp will be mapped to the meridians and longitudes of the collar or cusp in the image ([W2]).

Now we can arrange that given $b > 1$, for b-cusp regions $C_i^0(b)$ on S_0 and b-cusp regions $C_i^l(b)$ on S_l $(i = 1, 2, \dots, n)$,

$$f^l|_{C_i^0(b)} = id : C_i^0(b) \to C_i^l(b). \tag{1.5}$$

1.3 The Weil-Petersson and the Takhtajan-Zograf metrics.

Denote by $T_{g,n}$ Teichmüller space of hyperbolic surfaces of type (g,n). Now we consider the tangent and cotangent spaces at S of $T_{g,n}$. The cotangent space is $Q(S)$, the integrable holomorphic quadratic differentials on S. Let $B(S)$ be the L^∞-closure of Γ-invariant, bounded, $(-1,1)$-forms, i.e. the Beltrami differentials for S. For $\mu \in B(S), \varphi \in Q(S)$, the integral $(\mu,\varphi) = \int_S \mu\varphi$ defines a paring, let $Q(S)^\perp$ be the annihilator of $Q(S)$. The tangent space at S to $T_{g,n}$ is $B(S)/Q(S)^\perp \simeq HB(S)$, the Serre dual space of $Q(S)$, i.e. the harmonic Beltrami differentials on S. Then for $\mu,\nu \in HB(S)$, the Weil-Petersson and the Takhtajan-Zograf metrics are defined as follows ([T-Z]),

$$\langle \mu,\nu\rangle_{\mathrm{WP}} = \iint_S \mu(z)\overline{\nu(z)}y^{-2}dxdy \tag{1.6}$$

$$\begin{aligned}\langle \mu,\nu\rangle_{(\mathrm{i})} &= \iint_S E_i(z,2)\mu(z)\overline{\nu(z)}y^{-2}dxdy \\ &= \int_0^\infty\int_0^1 \mu(\sigma_i z)\overline{\nu(\sigma_i z)}dxdy\end{aligned} \tag{1.7}$$

$$\langle \mu,\nu\rangle_{\mathrm{TZ}} = \sum_{i=1}^{n} \langle \mu,\nu\rangle_{(i)}\,.$$

In the theory of automorphic functions, those two inner products are called, respectively the Petersson product and the Rankin product, while they are defined for general automorphic forms in the setting (refer to [Hi] §5.4). Both Weil-Petersson and Takhtajan-Zograf metric are Kählerian and incomplete ([O1], [T-Z]).

§2. A refined version of convergence theorem of Eisenstein series

In this section we will show a new convergence theorem of Eisenstein series, which is improved from the former version in [O2]. A little improvement is involved but, is essential for us to investigate the behavior of Takhtajan-Zograf metric near the boundary of moduli space more precisely than in [O2].

2.1 The Harish-Chandra transformation.

Here we prepare several fundamental notations from T. Kubota's book. ([K], Theorem 1.3.2). For $\epsilon > 0$, set a $\mathrm{PSL}_2(\mathbb{R})$-invariant kernel function on $H \times H$

$$k_\epsilon(z,z') = \begin{cases} 1, & \text{if } d(z,z') < \epsilon \\ 0, & \text{otherwise,}\end{cases}$$

where $d(z,z')$ denotes the hyperbolic distance between z and z' in H. Then there exists a constant $\Lambda_\epsilon(s)$ depending only on ϵ and the index s such that for any $\sigma \in \mathrm{PSL}_2(\mathbb{R})$,

$$\Lambda_\epsilon(s)\mathrm{Im}(\sigma z)^s = \iint_H k_\epsilon(z,z')\mathrm{Im}(\sigma z')^s \frac{dx'dy'}{y'^2}, \qquad (z' = x' + y').$$

([K], Theorem 1.3.2). The correpondence $s(s-1) \mapsto \Lambda_\epsilon(s)$ is sometimes called the Harish-Chandra transformation. We set $B(z,\epsilon) = \{w \in H | d(w,z) < \epsilon\}$ for $z \in H, \epsilon > 0$. With the help of Mathematica ([Mt]), we find

$$\Lambda_\epsilon(s) = \iint_{B(i,\epsilon)} y^{s-2}dxdy = \iint_{x^2+(y-\cosh\epsilon)^2\le\sinh^2\epsilon} y^{s-2}dxdy$$
$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\sinh\epsilon} (\cosh\epsilon + r\sin\theta)^{s-2} r dr d\theta$$
(Here we set $x = r\cos\theta, y - \cosh\epsilon = r\sin\theta$)
$$= \pi\Gamma^2(\frac{3-s}{2})(\cosh\epsilon)^s(\tanh\epsilon)^2 {}_2F_1(1-\frac{s}{2},\frac{3-s}{2};2;(\tanh\epsilon)^2). \tag{2.1}$$

Here ${}_2F_1(\alpha,\beta;\gamma;z)$ $(\gamma\neq 0,-1,-2,\dots)$ is the hypergeometric function,

$${}_2F_1(\alpha,\beta;\gamma;z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\cdot\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)\cdot 1\cdot 2\cdots n} z^n$$

and satisfies the differential equation,

$$z(1-z)\frac{d^2u}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{du}{dz} - \alpha\beta u = 0 \quad ([\mathrm{Wa}]).$$

Then

$$\Lambda_\epsilon(s) \sim \pi\Gamma^2(\frac{3-s}{2})\epsilon^2 \text{ as } \epsilon\to 0$$

holds. But we need the next more precise estimate of the ratio of both terms above in our proof of Theorem 1. It can be easily seen from the definitions that, for Re $s > 1$,

$$\Lambda_\epsilon(\mathrm{Re}s)^{-1} \le \frac{c(\mathrm{Re}s)\epsilon^{-2}}{\pi\Gamma^2(\frac{3-\mathrm{Re}s}{2})} \text{ as } \epsilon\to 0 \tag{2.2}$$

holds, where $c(\mathrm{Re}s)$ is a positive constant depending only on Re s.

Now we quote the next lemma ([O2],Lemma 1.).

Lemma 1. *We use the same notations as in § 1. Let the index Re $s > 1$. For any $i = 1,2,\dots,n$ and any $a > 1$,*

$$|E_i(z,s)| \le E_i(z,Res) < M_1(Re\ s,a), \qquad \textit{for } z\in\partial C_i(a). \tag{2.3}$$

Here $M_1(Re\ s,a)$ is a constant depending only on $Re\ s, a$, independent of complex structure and topological type of the surface, precisely represented as follows;

$$M_1(Re\ s,a) = \frac{3\cdot(2a)^{\mathrm{Re}s-1}}{(Re\ s-1)\Lambda_{\epsilon_0(a)}(Re\ s)} \qquad (\textit{we may set } \epsilon_0(a) = \frac{1}{2a}).$$

Since $E_i(z,Res)$ is subharmonic on S, we finally see

$$|E_i(z,s)| < M_1(Re\ s,a), \qquad \textit{on } S - C_i(a). \tag{2.4}$$

We use the setting as in §1., 1.1. Let Γ be the Fuchsian group uniformizing S of type (g,n) $(n>0)$ with $z_1=\infty$ and $P_1(z) = z+1$. The next proposition is a new version of Wolpert's result ([W2] p.260)

Proposition 1. *Let the index of Eisenstein series Re $s > 1$. Then*

$$|E_1(z,s)| < C(Re\ s)\ (Imz)^{-(\mathrm{Re}s+1)}, \qquad for\ Imz < 1. \tag{2.5}$$

Here $C(Re\ s)$ is a constant depending only on Re s, independent of complex structure and topological type of the surface.

Furthermore, the coefficients $\{c_m(s)\}_{m\neq 0}$ appearing in the Fourier expansion of $E_1(z,s)$ around $z_1 = \infty$ (1.2) satisfy

$$\sum_{m\neq 0} |c_m(s)|^2 |m|^{-2(\mathrm{Re}s+1)-1-\delta} < \infty, \quad for\ any\ \delta > 0. \tag{2.6}$$

Remark 1. The order $-(\mathrm{Re}s + 1)$ of y in (1) are different from $-\mathrm{Re}s$ the one in p.260, [W2]. The reason is that our constant $C(\mathrm{Re}s)$ is universal, while the constant C in [W2] depends on complex structure of the surface S.

2.2 the convergence of Eisenetein series.

We will show a convergence theorem of Eisenstein series, which is refined from the old version stated in [O2] Theorem 1., concerning convergence on the cusp regions around the old cusps. We state

Theorem 1. *We set the same notations as in §1. Let the index Re $s > 1$. Let $(f^l)^* E_i^l(z,s)$ be the pull-back of $E_i^l(z,s)$ on S_l by the modified harmonic map $f^l : S_0 \to S_l$, introduced in §1, 1.2.*

(1) Assume that $\{l_1, \dots, l_k\}$ do not separate S_l. Let q_j $(j = 1, \dots, k)$ be the new cusp arising from l_j. Denote by $C_j(b)$ $(b > 1)$ be the cusp region around q_j in S_0, each composed of usual two $b-$cusp regions. Then for any $i = 1, \dots, n$, as $\overrightarrow{l} = (l_1, \dots, l_k) \to \overrightarrow{0}$,

$$(f^l)^* E_i^l(z,s) - E_i^0(z,s) \longrightarrow 0 \tag{2.7}$$

uniformly on $S_0 - \bigcup_{j=1}^{k} C_j(b)$. Here $E_i^0(z,s)$ is the Eisenstein series attached to the old puncture p_i for S_0.

(2) Assume that $\{l_1, \dots l_k\}$ separate S_l. Denote by $S_{0,1}^i$ and $S_{0,2}^i$ respectively the component of S_0 containing p_i and the union of the components of S_0 not containing p_i. Let q_j $(j = 1, \dots, k)$ be the new cusp arising from l_j. Denote by $C_j(b)$ $(b > 1)$ be the cusp region around q_j in S_0, each composed of usual two $b-$cusp regions. Then

(i) For any $i = 1, \dots, n$, as $\overrightarrow{l} \to \overrightarrow{0}$,

$$(f^l)^* E_i^l(z,s) - E_i^0(z,s) \longrightarrow 0 \tag{2.8}$$

uniformly on $S_{0,1}^i - \bigcup_{j=1}^{k} C_j(b)$. Here $E_i^0(z,s)$ is the Eisenstein series attached to p_i for $S_{0,1}^i$.

(ii) For any $i = 1, \dots, n$ and any $b > 1$, as $\overrightarrow{l} \to \overrightarrow{0}$,

$$(f^l)^* E_i^l(z,s) \longrightarrow 0 \tag{2.9}$$

uniformly on $S^i_{0,2} - \bigcup_{j=1}^{k} C_j(b)$.

Furthermore, for $b > 1$ *fixed,*

$$|(f^l)^* E^l_i(z,s)| = O\left(\max_{j=1,\dots,k} l_j^{(2-\delta)\mathrm{Re}s-2}\right), \quad \text{for any small } \delta > 0$$

on $S^i_{0,2} - \bigcup_{j=1}^{k} C_j(b)$.

The new part of Theorem 1 is restated as the following proposition.

Proposition 2. *We set the same notations as in Theorem* 1. *Assume that* $\mathrm{Re}\, s > 1$. *We state our claim just on* $E^l_1(z,s)$ *for notational simplicity. For any* $i = 1,2,\dots,n$, *as* $\overrightarrow{l} \to \overrightarrow{0}$,

$$E^l_1(z,s) - E^0_1(z,s) \quad \to \quad 0$$

uniformly on $C_i(B)$ $(B > b)$, *where* b *is the number taken in* (1.5).

§3. Comparisons of the W-P and T-Z metric along general degenerations

3.1 A review of H. Masur's work.

We review a construction of a basis of quadratic differentials spanning the cotangent spaces of a general degenerating family of punctured Riemann surfaces. In [Ms], he has constructed such a basis for any degenerating family of compact surfaces. But from his result, we can easily obtain the same kinds of quadratic diferentials for a family of surfaces with cusps.

Assume that S_0 has singularities at points q_j $(j = 1,2,\dots,k)$, these have neibourhoods $N_j = \{(z_j,w_j) \in \mathbb{C}^2 \mid |z_j|,|w_j| < 1, z_j \cdot w_j = 0\}$, respectively, $N_j = N_j^1 \bigcup N_j^2$ is a union of disks; $N_j^1 = \{z_j |\ 0 \le |z_j| \le 1\}, N_j^2 = \{w_j |\ 0 \le |w_j| \le 1\}$. The components S_α of S_0 $(\alpha = 1,\dots,r)$ are called parts of S_0. We have to assume that the S_α are hyperbolic, i.e. $2g_\alpha - 2 + n_\alpha + \tilde{n}_\alpha > 0$ where g_α are the genus of S_α and n_α $(\tilde{n}_\alpha)$ are the numbers of the *old* (*new*) cusps of S_α respectively (we regard a node attached to just one component as a pair of two *new* cusps). Let g, n be the genus and the number of *old* cusps of S_0. Then we see the equations $g = g_1 + \dots + g_r + k - (r-1)$, $n = n_1 + \dots + n_r$, $2k = \tilde{n}_1 + \dots + \tilde{n}_r$. These yield

$$3g - 3 + n = \sum_{\alpha=1}^{r} (3g_\alpha - 3 + n_\alpha + \tilde{n}_\alpha) + k.$$

Any part S_α possesses a $(3g_\alpha - 3 + n_\alpha + \tilde{n}_\alpha)$- dimensional universal family, $T_{g_\alpha, n_\alpha+\tilde{n}_\alpha}$ i.e. Teichmüller space of type $(g_\alpha, n_\alpha + \tilde{n}_\alpha)$. We set a basis of Beltrami differentials ν_i^αon S_α with compact supports in $S_\alpha - \bigcup_{j=1}^{k} N_j - \{$cusp neighborhoods around *old* cusps of $S_\alpha\}$ (For example, we may take restrictions to compact support of the duals of a basis of integrable quadratic differentials): let

$$\tau^\alpha = (\tau_1^\alpha, \tau_2^\alpha \dots, \tau^\alpha_{3g_\alpha-3+n_\alpha+\tilde{n}_\alpha})$$

be associated local coordinates for $T_{g_\alpha,n_\alpha+\tilde{n}_\alpha}$ around S_α, where we set $\mu_\alpha(\tau) = \sum_{i=1}^{3g_\alpha-3+n_\alpha+\tilde{n}_\alpha} \tau_i^\alpha \nu_i^\alpha$. If we vary the complex structure of parts S_α, and set $\tau = (\tau^1, \tau^2, \dots, \tau^r)$, we obtain a family $\{S_\tau\}$ and quasiconformal homeomorhisms $f^{\mu_\alpha(\tau)} : S_\alpha \to S_{\alpha,\tau^\alpha}$ which comprise a quasiconformal homeomorphism $f^\tau : S_0 = \bigcup_{\alpha=1}^{r} S_\alpha \to \bigcup_{\alpha=1}^{r} S_{\alpha,\tau^\alpha}$, the last set denoted by S_τ. The map f^τ is conformal on N_j^1, N_j^2 $(j = 1, \cdots, k)$ and thus z_i, w_i serve as local coordinates for $f^\tau(N_j^1), f^\tau(N_j^2)$ respectively. For each $t = (t_1, t_2, \dots, t_k), |t_j| < 1$, the new Riemann surface $S_{t,\tau}$ is constructed from S_τ by removing the disks $\{z_j | \, |z_j| < |t_j|\}$ and $\{w_j | \, |w_j| < |t_j|\}$ and identifying z_j with $w_j = t/z_j$, constructing annuli $N_j^{t_j} = \{z_j | |t_j| < |z_j| < 1\} \simeq \{w_j | |t_j| < |w_j| < 1\}$. If K is a compact subset of $S_0 \setminus \{q_1, \cdots, q_k\}$, for small (t, τ) we will consider K as a compact subset of $S_{t,\tau}$ via f^τ and the natural inclusion of $S_{t,\tau}$ in S_τ (See [Ms],p.625).

Here following Wolpert ([W1] Lemma1.1), we take a modification F^τ of f^τ so that each lift $\tilde{F}_\alpha^\tau : H \to H$ of $F^\tau|_{S_\alpha} : S_\alpha \to S_{\alpha,\tau^\alpha}$ will coincide with an element of $\mathrm{PSL}_2(\mathbb{R})$ on any cusp region corresponding to old cusps of S_α (See also [P-S] Lemma2.2). It should be remarked that F^τ and f^τ have equivalent initial tangents $\partial/\partial\tau_\nu^\alpha$, that is, the corresponding Beltrami differentials will have the same pairing with each integrable quadratic differential on S_α, which we need in the proof of Proposition 4 (See [W1] Remark1.2).

Thus We have gotten a local parameter space $(t, \tau) = (t_1, t_2, \dots, t_k, \tau^1; \tau^2; \dots; \tau^r) \in D \subset \mathbb{C}^{3g-3+n}$, a neighbourhood of the origin. By changing the indicies, we sometimes use a notation $\tau = (\tau_{k+1} \cdot \tau_{k+2}, \dots, \tau_{3g-3+n-k})$. We state the important proposition, essentially due to H. Masur [Ms] (see also [B], [Sc], [Tr]). We arrange his results so that they should fit to our setting, that is, a degenerating family of punctured Riemann surfaces.

Proposition 3. *There is a basis of regular quadratic differentials*

$$\{\phi_j(z,t,\tau)dz^2, \phi_\nu(z,t,\tau)dz^2\}_{j=1,\dots,k;\nu\geq k+1}$$

, dual to $\{\partial/\partial t_j, \partial/\partial\tau_\nu\}_{j=1,\dots,k;\nu\geq k+1}$, *satisfying the next properties:*

i) $\phi_\nu(z,0,0)$ *has support in the component of* S_0 *where the Beltrami differential corresponding to* $\partial/\partial\tau_\nu$ *has support.*

ii) the followings hold, where $(\cdot,\cdot)$ *means the Serre dual pairing;*

$$(\phi_i, \partial/\partial t_j) = \delta_{ij}, \qquad \text{for } i,j \leq k \tag{3.1}$$

$$(\phi_i, \partial/\partial\tau_\nu) = O(|t_i|), \quad \text{for } i \leq k, \nu \geq k+1 \tag{3.2}$$

$$(\phi_\mu, \partial/\partial t_j) = 0, \qquad \text{for } \mu \geq k+1, j \leq k \tag{3.3}$$

$$\lim_{(t,\tau)\to(0,0)} (\phi_\mu, \partial/\partial\tau_\nu) = \delta_{\mu-k,\nu}, \qquad \text{for } \mu,\nu \geq k+1. \tag{3.4}$$

iii) On $z_j \in N_j^1$, *for* $i \leq k$,

$$\phi_i(z_j,t,\tau) = -\frac{t_i}{\pi}\left[\frac{\delta_{ij}}{z_j^2} + a_{-1}(z_j,t,\tau) + \frac{1}{z_j^2}\sum_{r=1}^{\infty}\left(\frac{t_j}{z_j}\right)^r \cdot t_j^{m(r)} \cdot a_r(t,\tau)\right], \tag{3.5}$$

where $m(r) \geq 0$, a_{-1} *has at most a simple pole at* $z_j = 0$, a_r $(r \geq 1)$ *is holomorphic. On* $z_j \in N_j^1$, $\nu \geq k+1$,

$$(3.6)\quad \phi_\nu(z_j,t,\tau) = \phi_\nu(z_j,0,0) + \frac{1}{z_j^2}\sum_{r=1}^{\infty}\left(\frac{t_j}{z_j}\right)^r \cdot t_j^{\tilde{m}(r)} \cdot b_r(t,\tau) + \sum_{r=-1}^{\infty} z_j^r \cdot c_r(t,\tau),$$

where $\tilde{m}(r) \geq 0$, $\phi_\nu(z_j,0,0)$ *has at most a simple pole and* b_r, c_r *is holomorphic and* $c_r(0,0) = 0$. *Similar equations hold on* N_j^2 *with respect to* (w_j,t,τ)*-coordinates.*
iv) the followings hold, where $<\cdot,\cdot>$ *means the natural inner product of quadratic differentials;*

$$(3.7)\qquad <\phi_i,\phi_i> \approx -|t_i|^2(\log|t_i|)^3, \qquad \text{for } i \leq k,$$

$$(3.8)\qquad <\phi_i,\phi_j> = O(|t_i|)O(|t_j|), \qquad \text{for } i,j \leq k, i \neq j$$

$$(3.9)\qquad <\phi_i,\phi_\mu> = O(|t_i|), \qquad \text{for } i \leq k, \mu \geq k+1,$$

$$(3.10)\qquad \lim_{(t,\tau)\to(0,0)} <\phi_\mu(z,t,\tau),\phi_\nu(z,t,\tau)> = <\phi_\mu(z,0,0),\phi_\nu(z,0,0)>, \qquad \text{for } \mu,\nu \geq k+1.$$

Remark 2. It seems difficult that we would apply the method of Masur's original proof to our case because he used compactness of general fibers of the degenerating family in his proof.

3.2 Comparisons along general degenerations.

Before we state the main theorem, we give a preparatory tool for investigating the boundary behaviors of the Takhtajan-Zograf metric. We get the representation of $\{\partial/\partial t_j, \partial/\partial\tau_\nu\}_{j=1,\dots,k;\nu\geq k+1}$ in terms of harmonic Betrami differentials approximately.

Proposition 4. *Let* $\rho(z,t,\tau)|dz|$ *be the Poincaré metric with curvature* -1. *We define harmonic Beltrami differentials* $\eta_j(z,t,\tau) = \rho(z,t,\tau)^{-2}\overline{\phi_j(z,t,\tau)}$, $\eta_\nu(z,t,\tau) = \rho(z,t,\tau)^{-2}\overline{\phi_\nu(z,t,\tau)}$, $(j = 1,\dots,k, \nu = k+1,\dots,3g-3+n)$. *And for* $i \leq k, \mu \geq k+1$, *we put*

$$(3.11)\qquad \partial/\partial t_i = \sum_{j=1}^{k} u_{ij}(t,\tau)\eta_j(z,t,\tau) + \sum_{\nu=k+1}^{3g-3+n} u_{i\nu}(t,\tau)\eta_\nu(z,t,\tau)$$

$$(3.12)\qquad \partial/\partial\tau_\mu = \sum_{j=1}^{k} u_{\mu j}(t,\tau)\eta_j(z,t,\tau) + \sum_{\nu=k+1}^{3g-3+n} u_{\mu\nu}(t,\tau)\eta_\nu(z,t,\tau).$$

Then we obtain the followings:

i) $u_{ii}(t,\tau) \approx -1/|t_i|^2(\log|t_i|)^3$, *for* $i \leq k$

ii) $u_{ij}(t,\tau) = O(1/|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3)$, *for* $i,j \leq k, i \neq j$

iii) $u_{i\nu}(t,\tau) = O(-1/|t_i|(\log|t_i|)^3)$, *for* $i \leq k, \nu \geq k+1$

iv) $u_{\mu j}(t,\tau) = O(-1/|t_j|(\log|t_j|)^3)$, *for* $\mu \geq k+1, j \leq k$

v) $u_{\mu\nu}(t,\tau) = \delta_{\mu\nu} + \sum_{i=1}^{k} O(-1/(\log|t_i|)^3)$, *for* $\mu,\nu \geq k+1$

Finally we combine Theorem 1 and Proposition 1, 3 to get one of our main theorems.

Theorem 2. *We obtain order estimates of the Riemannian tensors $h_{i\bar{j}}(t,\tau)$ ($g_{i\bar{j}}(t,\tau)$) of the Takhtajan-Zograf (the Weil-Petersson) metric near the boundary of Teichmüller space;*

$$\begin{aligned}
&i)\ g_{i\bar{i}}(t,\tau) = -1/|t_i|^2(\log|t_i|)^3 + O(-1/|t_i|^2(\log|t_i|)^6), &&\text{for } i \le k\\
&ii)\ g_{i\bar{j}}(t,\tau) = O(1/|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3), &&\text{for } i,j \le k, i \neq j\\
&iii)\ \lim_{(t,\tau)\to(0,0)} g_{\mu\bar{\nu}}(t,\tau) = g_{\mu\bar{\nu}}(0,0), &&\text{for } \mu,\nu \ge k+1\\
&iv)\ g_{i\bar{\mu}}(t,\tau) = O(-1/|t_i|(\log|t_i|)^3), &&\text{for } i \le k, \mu \ge k+1.
\end{aligned}$$

$$\begin{aligned}
&i)\ h_{i\bar{i}}(t,\tau) = O(-1/|t_i|^2(\log|t_i|)^3), &&\text{for } i \le k\\
&ii)\ h_{i\bar{j}}(t,\tau) = O(1/|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3), &&\text{for } i,j \le k, i \neq j\\
&iii)\ \lim_{(t,\tau)\to(0,0)} h_{\mu\bar{\nu}}(t,\tau) = h_{\mu\bar{\nu}}(0,0), &&\text{for } \mu,\nu \ge k+1\\
&iv)\ h_{i\bar{\mu}}(t,\tau) = O(-1/|t_i|(\log|t_i|)^3), &&\text{for } i \le k, \mu \ge k+1.
\end{aligned}$$

We state a conjecture that is inspired by M.Wolf's asymptotic formula of the hyperbolic metrics for degenerating Riemann surfaces ([Wf], Corollary 5.4).

The second-term conjecture (Obitsu and Wolpert). *Use the notations as in Theorem 2. The next asymptotic formula for the Weil-Petersson metric for T_g holds; for $\mu,\nu \ge k+1$,*

$$\begin{aligned}
&g_{\mu\bar{\nu}}(t,\tau)\\
&= g_{\mu\bar{\nu}}(0,\tau) + \frac{4\pi^4}{3}\sum_{i=1}^{k}(\log|t_i|)^{-2}\Big\langle \eta_\mu, (E_{i,1}(z,2)+E_{i,2}(z,2))\eta_\nu\Big\rangle_{WP}(0,\tau)\\
&\qquad + O\Big(\sum_{i=1}^{k}(\log|t_i|)^{-3}\Big).
\end{aligned}$$

Here, $E_{i,1}(z,2), E_{i,2}(z,2)$ are the Eisenstein series associated with the i-th node and the components of the degenerate Riemann surface. i.e. in the second-term, the associated Takhtajan-Zograf metrics appear.

§4. An application to L_2–cohomology of Moduli space

First of all, we review the result of L.Saper.

Theorem 3 ([Sa]). *Denote by M_g the moduli space of compact Riemann surfaces of genus $g > 1$. Then, We have the isomorphisms*

$$H^*_{(2)}(M_g, \omega_{WP}) \simeq H^*(\overline{M_g}, \mathbb{R}),$$

where the left-hand sides are the L_2-cohomolgy groups with respect to the Weil-Petersson metric, and the right-hand sides are the usual cohomolgy groups of the Deligne-Mumford compactification of the moduli space with coefficients in $\mathbb{R}$.

We can mimic the proof of Theorem 3 with using Theorem 2 to deduce the next generalization.

Theorem 4. *Denote by $M_{g,n}$ the moduli space of punctured Riemann surfaces of genus g with n punctures, $3g-3+n>0$. Then, we have the isomorphisms*

$$H^*_{(2)}(M_{g,n},\omega_{WP}) \simeq H^*_{(2)}(M_{g,n},\omega_{TZ}) \simeq H^*(\overline{M_{g,n}},\mathbb{R}),$$

, where the middle are the L_2-cohomolgy groups with respect to the Takhtajan-Zograf metric, and the left-hand sides and the right-hand sides are respectively the obvious counterparts of them in Thorem 3.

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Addendum.

Very recently, I and S. Wolpert have proved the second-term conjecure! Precise proof and several applications will appear elsewhere.