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TOPOLOGICAL TYPES OF GEOMETRIC LIMIT MANIFOLDS OF QUASI-FUCHSIAN GROUPS

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This note is a survey of the author's results given in [7].

Let Σ be a closed orientable surface of genus greater than one. We fix a hyperbolic structure on Σ for convenience, and set $\Pi = \pi_1(\Sigma)$. In [3], Jϕrgensen and Marden gave an example of faithful representations $\rho_n : \mathbb{Z} \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ with $\rho_n(1)$ loxodromic such that the cyclic Kleinian groups $\rho_n(\mathbb{Z})$ converge geometrically to a rank two parabolic group. This is one of typical phenomena which appear in geometric limits. In fact, Kerckhoff and Thurston [4] considered the cyclic action on the Bers slice B_{σ_+} at $\sigma_+ \in \mathrm{Teich}(\Sigma)$ generated by the Dehn twist φ on Σ along a simple closed geodesic l. Then, they showed that any geometric accumulation point of the cyclic orbit $\{(\varphi_*^n(\sigma_-), \sigma_+)\} \subset B_{\sigma_+}$ is a Kleinian group G such that $\mathbb{H}^3/G$ is homeomorphic to $\Sigma \times (0,1) - l \times \{1/2\}$. Then, a tubular neighborhood of $l \times \{1/2\}$ in $\Sigma \times (0,1)$ corresponds to a $\mathbb{Z} \times \mathbb{Z}$-cusp of $\mathbb{H}^3/G$ where Jϕrgensen-Marden phenomenon occurs, see Fig. 1 (a). By using this method iteratively, it is also possible

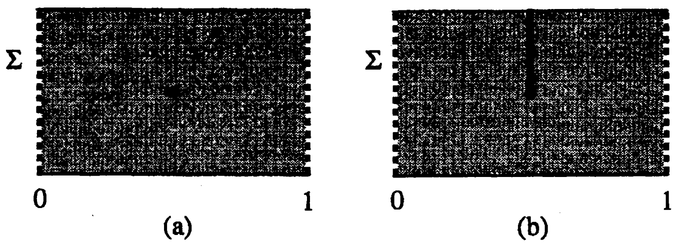

FIGURE 1. The dot in (a) represents $l \times \{1/2\}$ in the model of $\mathbb{H}^3/G$. The vertical bold segment in (b) represents $H \times \{1/2\}$ in the model of $\mathbb{H}^3/G''$.

to construct an example of a geometric limit G' of quasi-Fuchsian groups such that $\mathbb{H}^3/G'$ has infinitely many $\mathbb{Z} \times \mathbb{Z}$-cusps. In particular, G' is infinitely generated. Another important example of geometric limits of quasi-Fuchsian groups is given by Brock [2]. He considered the cyclic action on a Bers slice generated by a homeomorphism $\psi : \Sigma \longrightarrow \Sigma$ such that $\psi|\mathrm{Int}H : \mathrm{Int}H \longrightarrow \mathrm{Int}H$ is pseudo-Anosov for a proper subsurface H of Σ and $\psi|(\Sigma - \mathrm{Int}H)$ is the identity. Then, any geometric accumulation point of the cyclic orbit $\{(\psi_*^n(\sigma_-), \sigma_+)\} \subset B_{\sigma_+}$ is a Kleinian group G'' such that $\mathbb{H}^3/G''$ is homeomorphic to $\Sigma \times (0,1) - H \times \{1/2\}$, see Fig. 1 (b). However, all of these examples are very special ones. In this talk, we will present what kinds of topological types appear generally in geometric limits of quasi-Fuchsian groups.

Let $p : \Sigma \times I \longrightarrow \Sigma$ and $q : \Sigma \times I \longrightarrow I$ be the projections onto the first and second factors, where I is the closed interval $[0,1]$. For any $y \in I$, the preimage

$\Sigma_y = q^{-1}(y)$ is supposed to have the hyperbolic structure so that $p|\Sigma_y : \Sigma_y \longrightarrow \Sigma$ is isometric. A compact connected subsurface F of Σ_y with geodesic boundary is called a *non-pant geodesic subsurface* if F is not homeomorphic to a genus-zero surface with three boundary components, a pair of pants. Note that the interior of a non-pant geodesic subsurface contains a simple closed geodesic.

For a closed subset A of Σ_y, if $\Delta(A)$ is a minimal disjoint union of a geodesic subsurface F and simple closed geodesics $l_1, \dots, l_k$ in Σ_y with $\Delta(A) \supset A$, then the frontier $\mathrm{Fr}(\Delta(A))$ of $\Delta(A)$ in Σ_y is the union $\partial F \cup l_1 \cup \cdots \cup l_k$ of mutually disjoint simple geodesic loops in Σ_y. Let $\lambda(A)$ be the union of all simple closed geodesics l in $\Delta(A)$ such that, for the δ-neighborhood $\mathcal{N}_\delta(l, \Sigma_y)$ with a small $\delta > 0$, at least one component of $\mathcal{N}_\delta(l, \Sigma_y) - l$ is disjoint from A. In particular, $\lambda(A)$ contains $\mathrm{Fr}(\Delta(A))$, see Fig. 2.

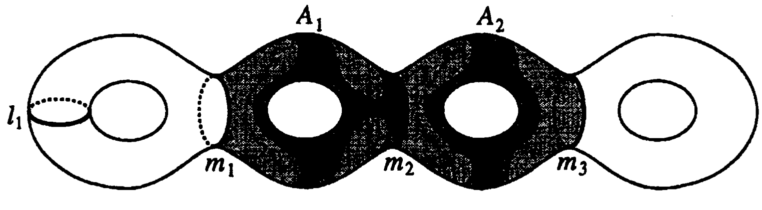

FIGURE 2. The case of $A = l_1 \cup A_1 \cup A_2$. Then, $\Delta(A) = F \cup l_1$, where F is the union of the shaded regions. $\mathrm{Fr}(\Delta(A)) = l_1 \cup m_1 \cup m_3$, and $\lambda(A) = \mathrm{Fr}(\Delta(A)) \cup m_2 = l_1 \cup m_1 \cup m_2 \cup m_3$.

Let $\mathcal{X}$ be the closed subset of $\Sigma \times I$ given below. Then, we set $\mathcal{Y} = q(\mathcal{X})$, $X_y = \Sigma_y \cap \mathcal{X}$ for $y \in \mathcal{Y}$, $\Lambda_y^+ = \Sigma_y \cap \overline{\Sigma \times (y, 1] \cap \mathcal{X}}$ for $y < 1$, $\Lambda_y^- = \Sigma_y \cap \overline{\Sigma \times [0, y) \cap \mathcal{X}}$ for $y > 0$.

Theorem 1. *Let* $\{\rho_n : \Pi \longrightarrow \mathrm{PSL}_2(\mathbb{C})\}_{n=1}^\infty$ *be any algebraically convergent sequence of quasi-Fuchsian representations such that* $\{\rho_n(\Pi)\}_{n=1}^\infty$ *converges geometrically to a Kleinian group* G. *Then, the hyperbolic 3-manifold* $\mathbb{H}^3/G$ *is homeomorphic to* $\Sigma \times I - \mathcal{X}$ *such that* $\mathcal{X}$ *is a closed subset of* $\Sigma \times I$ *satisfying the following conditions* (i)-(iii).

(i) $\Sigma \times I - \mathcal{X}$ *is connected, containing* $\Sigma_{1/2}$, *and disjoint from* $\Sigma_0 \cup \Sigma_1$.
(ii) *For any* $y \in \mathcal{Y}$, X_y *is a disjoint union of a geodesic subsurface and simple geodesic loops in* Σ_y. *For* $\varepsilon = \pm$, *each non-peripheral component of* $X_y - \Delta(\Lambda_y^\varepsilon) \cup \mathrm{Fr}(X_y)$ *is an open non-pant geodesic subsurface of* Σ_y.
(iii) *For any* $y, z \in \mathcal{Y}$ *with* $y < z$, *if a component* l_y *of* $\mathrm{Fr}(X_y) \cup \lambda(\Lambda_y^+)$ *is parallel to a component* l_z *of* $\mathrm{Fr}(X_z) \cup \lambda(\Lambda_z^-)$ *in* $\Sigma \times I - \mathcal{X}$, *then* l_y *and* l_z *are horizontally parallel in* $\mathcal{X}$.

The property $\Sigma_0 \cup \Sigma_1 \subset \mathcal{X}$ in the condition (i) is immediate from that $\mathbb{H}^3/G$ is an open manifold. A subsurface of X_y is *peripheral* if it is horizontally parallel in $\mathcal{X}$ to a subsurface of either Σ_0 or Σ_1. Any component of $X_y - \Delta(\Lambda_y^\varepsilon) \cup \mathrm{Fr}(X_y)$ is called a subsurface of type $\mathbf{B}^\varepsilon$, see Fig 3. Thus, the latter part of the condition (ii) is restated that any non-peripheral subsurface of type $\mathbf{B}^\varepsilon$ is not an open pair of pants. In fact, the ends of $\mathbb{H}^3/G$ corresponding to such subsurfaces are necessarily

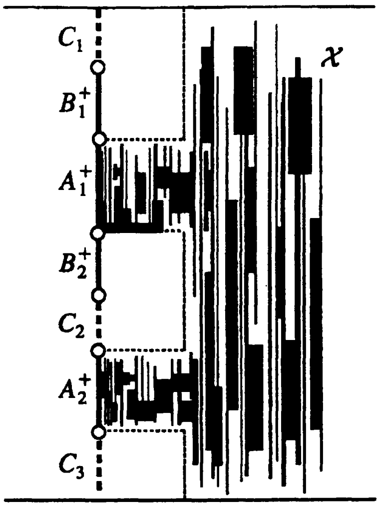

FIGURE 3. A_i^+, B_j^+, C_k represent respectively subsurfaces of Σ_y of types $\mathbf{A}^+$, $\mathbf{B}^+$ and $\mathbf{C}$.

geometrically infinite tame. The condition (iii) is derived from the fact that any two parabolic cusps in a hyperbolic 3-manifold M is not parallel in M.

According to Myers [6], there exists a simple loop l in $\Sigma \times (0,1)$ which is not parallel to a loop in $\Sigma_0 \cup \Sigma_1$ and such that $N = \Sigma \times (0,1) - l$ admits a geometrically finite hyperbolic metric σ. By Hyperbolic Dehn Surgery Theorem in [8], $N(\sigma)$ is a geometric limit of geometrically finite hyperbolic 3-manifolds without parabolic cusps. However, N is not homeomorphic to $\mathbb{H}^3/G$ for any geometric limit G of quasi-Fuchsian groups. This fact is proved by Theorem 1 or directly as an exercise without invoking the theorem.

In general, a closed subset $\mathcal{X}$ in $\Sigma \times I$ satisfying the conditions (i)-(iii) is very complicated. When $\mathcal{Y}$ is a totally disconnected subset of I, $\mathcal{Y}$ is not a perfect set and each connected component of $\mathcal{X}$ is either a geodesic subsurface or a geodesic loop. Even in this rather simple case, there may exist a doubly (or more multiply) accumulation point y in $\mathcal{Y}$. This means that y is an accumulation point of a subset $\{y_n\}$ of $\mathcal{Y}$ such that each y_n is also an accumulation point of $\mathcal{Y}$, see Fig. 4.

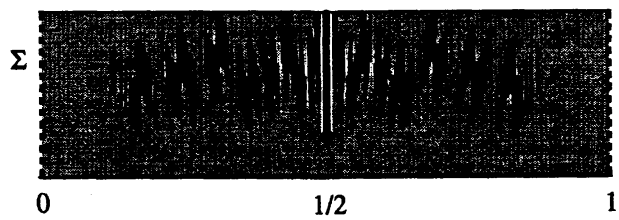

FIGURE 4. '1/2' is a doubly accumulation point of $\mathcal{Y}$.

Remark 2. In particular, Theorem 1 implies that, for any geometric limit G of an algebraically convergent sequence $\{\rho_n\}$ of quasi-Fuchsian representations, $\mathbb{H}^3/G$ is homeomorphic to an open subset of $\Sigma \times (0,1)$. One may suppose that the assertion is obvious since each $\mathbb{H}^3/\rho_n(\Pi)$ is homeomorphic to $\Sigma \times (0,1)$, and since moreover there exists a K_n-quasi-isometry $g_n : \mathcal{N}_{R_n}(x_n, \mathbb{H}^3/\rho_n(\Pi)) \longrightarrow \mathcal{N}_{R_n}(x_\infty, \mathbb{H}^3/G)$ between the R_n-neighborhoods centered at suitable base points x_n and x_∞ with $R_n \nearrow \infty$ and $K_n \searrow 1$. Though the g_n^{-1} and g_{n+1}^{-1}-images of $\mathcal{N}_{R_n}(x_\infty, \mathbb{H}^3/G)$ are mutually homeomorphic, their complements in $\mathbb{H}^3/\rho_n(\Pi)$ and $\mathbb{H}^3/\rho_{n+1}(\Pi)$ do not necessarily have the same topological type. Thus, the maps g_n^{-1} would not offer directly an expanding sequence of embeddings from $\mathcal{N}_{R_n}(x_\infty, \mathbb{H}^3/G)$ $(n = 1, 2, \dots)$ into $\Sigma \times (0, 1)$. We will construct an embedding of $\mathbb{H}^3/G$ into $\Sigma \times (0, 1)$ by using the fact that $\mathbb{H}^3/G$ has the structure of a *block complex.*

Theorem 3. *Let $\mathcal{X}$ be any closed subset of $\Sigma \times I$ satisfying the conditions* (i)-(iii) *in Theorem 1. Then, there exists a geometric limit G of an algebraically convergent sequence of quasi-Fuchsian representations such that $\mathbb{H}^3/G$ is homeomorphic to $\Sigma \times I - \mathcal{X}$.*

A closed subset of $\Sigma \times I$ satisfying the conditions (i)-(iii) in Theorem 1 is called a *crevasse* in $\Sigma \times I$. We need to study crevasses from the topological point of view. This is not only necessary to prove Theorems 1 and 3, but also useful to understand topological properties of geometric limits of quasi-Fuchsian groups. As a special case, these theorems determine the topological types of $\mathbb{H}^3/G$ for geometric limits of any sequence in the Bers slice B_{σ_+}, which is naturally identified with the Teichmüller space Teich(Σ). Then, $\mathbb{H}^3/G$ is homeomorphic to $\Sigma \times I - \mathcal{X}$ for some crevasse $\mathcal{X}$ with $\mathcal{X} \cap \Sigma \times [1/2, 1) = \emptyset$. Though the result does not imply data on the geometric structure on $\mathbb{H}^3/G$, some arguments used in the proofs of our theorems suggest implicitly that the hyperbolic structure on $\mathbb{H}^3/G$ would be controlled by those on the geometrically infinite tame ends $\mathcal{E}$ of $\mathbb{H}^3/G$ corresponding to the subsurfaces in $\mathcal{X}$ of types $\mathbf{B}^{\pm}$. On the other hand, the hyperbolic structures on $\mathcal{E}$ will be determined only by their ending data if Thurston's Ending Lamination Conjecture [9] holds, where the *ending data* means the element of Teich(B) determined by the conformal structure on the front end if $\mathcal{E}$ is geometrically finite and the ending lamination if $\mathcal{E}$ is geometrically infinite. The conjecture is proved by Minsky [5] in the case when the infimum injectivity radius of a hyperbolic 3-manifold is positive, and the project toward the complete solution is making steady progress by some people including himself. Thus, it would not be in distant future when we know all the elements of the *geometric Bers boundary* of Teich(Σ).

Problem 4. Let G_i $(i = 1, 2)$ be geometric limits of algebraically convergent sequences of quasi-Fuchsian groups with homeomorphisms $h_i : \mathbb{H}^3/G_i \longrightarrow \Sigma \times I - \mathcal{X}$ for a given crevasse $\mathcal{X}$. Is $h_2^{-1} \circ h_1 : \mathbb{H}^3/G_1 \longrightarrow \mathbb{H}^3/G_2$ properly homotopic to an isometry if, for any subsurface B in $\mathcal{X}$ of types $\mathbf{B}^{\pm}$, the corresponding ends $\mathcal{E}_i(B)$ in $\mathbb{H}^3/G_i$ have the same ending data?

Outline of the proof of Theorem 1. If an algebraically convergent sequence of quasi-Fuchsian representations $\rho_n : \Pi \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ converges geometrically to a Kleinian group G, then there exists a K_n-quasi-isometry $g_n : \mathcal{N}_{R_n}(x_n, N_n) \longrightarrow \mathcal{N}_{R_n}(x_\infty, M_\infty)$ with $R_n \nearrow \infty$ and $K_n \searrow 1$ for the suitable choice of base points $x_n \in N_n$ and $x_\infty \in M_\infty$, where $N_n = \mathbb{H}^3/\rho_n(\Pi)$ and $M_\infty = \mathbb{H}^3/G$. Since N_n is homeomorphic to $\Sigma \times (0, 1)$, N_n admits a topological fibration $\mathcal{G}_n$ with fiber Σ. Then, the foliation $\widehat{\mathcal{G}}_n$ on $\mathcal{N}_{R_n}(x_\infty, M_\infty)$ is induced from $\mathcal{G}_n|\mathcal{N}_{R_n}(x_n, N_n)$ via g_n.

However, it would be difficult to define a foliation on M_∞ from $\widehat{\mathcal{G}}_n$'s since we do not have geometric data to investigate relations between $\mathcal{G}_n$ and $\mathcal{G}_{n+1}$. In our proof, we will invoke a 'coarse fibration' $\mathcal{S}_n$ on the convex core C_n of N_n 'fibers' of which are pleated surfaces between the two components of ∂C_n. Then, $\mathcal{N}_{R_n}(x_\infty, M_\infty)$ has the coarse foliation $\widehat{\mathcal{S}}_n$ induced from $\mathcal{S}_n|\mathcal{N}_{R_n}(x_n, N_n)$. Let $M_{\infty,\text{p-thin}(\varepsilon)}$ be the union of parabolic cusp components of the ε-thin part $M_{\infty,\text{thin}(\varepsilon)}$ of M_∞ for a sufficiently small $\varepsilon > 0$. For any $x \in M_{\infty,\text{p-thick}(\varepsilon)} = M_\infty - M_{\infty,\text{p-thin}(\varepsilon)}$, there exists a constant $R(x)$ independent of $n \in \mathbb{N}$, such that, for any leaf $F^{(n)}$ of $\widehat{\mathcal{S}}_n$ passing through the 1-neighborhood of x in M_∞, the diameter of the component $F_0^{(n)}$ of $F^{(n)} \cap M_{\infty,\text{p-thick}(\varepsilon)}$ nearest to x is less than $R(x)$. Thus, if necessary passing to a subsequence, we may assume that $\{F_0^{(n)}\}$ converges uniformly to a surface F, and hence in particular $F_0^{(n)}$'s are mutually properly homotopic in $M_{\infty,\text{p-thick}(\varepsilon)}$. This suggests that $\{\widehat{\mathcal{S}}_n\}$ is an expanding sequence of coarse foliations in $M_{\infty,\text{p-thick}(\varepsilon)}$ after slightly modifying $\widehat{\mathcal{S}}_n$ by proper homotopy in $M_{\infty,\text{p-thick}(\varepsilon)}$. Though the limit F may not be an embedded surface, one can replace it by an embedded surface S in the homotopy class of F in $M_{\infty,\text{p-thick}(\varepsilon)}$ by using the least area surface theory. A maximal set $\{Q_b\}$ of these embedded surfaces which are mutually disjoint and not properly homotopic to each other in $M_{\infty,\text{p-thick}(\varepsilon)}$ divides $M_{\infty,\text{p-thick}(\varepsilon)}$ into (in general infinitely many) blocks B_c. Here, B_c being a *block* means that B_c is homeomorphic to $F \times (0,1)$ for a subsurface F of Σ. We note that this block decomposition misses points $x \in M_{\infty,\text{p-thick}(\varepsilon)}$ in a small neighborhood of which does not meet the g_n-images of pleated surfaces. This occurs when $g_n^{-1}(x)$ is $N_n - \text{Int}C_n$ for all sufficiently large $n \in \mathbb{N}$. Each component of $N_n - \text{Int}C_n$ is a geometrically finite end and hence homeomorphic to $\Sigma \times [0,\infty)$. From this fact, we know that x is in a geometrically finite end of M_∞ homeomorphic to $F' \times [0,\infty)$ for some subsurface F' of Σ. Regarding such ends also as blocks, we have a block decomposition of $\text{Int}M_{\infty,\text{p-thick}(\varepsilon)}$. The interior $\mathcal{K}_k$ of the union $\cup_{i=1}^k \overline{B_i}$ of the first k blocks is contained in $\mathcal{N}_{R_n}(x_\infty, M_\infty)$ if n is sufficiently large, and hence it can be embedded in $\Sigma \times I$ via g_n^{-1}. In general, such embeddings are not expanding sequence as was remarked above. So, we will construct an expanding sequence by splitting blocks into sub-blocks if necessary and by embedding them into $\Sigma \times I$ at the sacrifice of a global continuity. Then, the images $\mathcal{K}'_k$ are expanding open submanifolds, but not homeomorphic to the original $\mathcal{K}_k$. We will restore the originals by slit-sliding operations in $\Sigma \times I$. This splitting-to-restoring process is not a meaningless round trip since the process enables us to deal with the embedding problem stepwise. We have an expanding sequence $\{\mathcal{F}_n\}$ of unions of finitely many slits in $\Sigma \times I$ such that the 3-manifold Y_n obtained by sliding along $\mathcal{F}_n$ contains $\mathcal{K}_n$ as an open submanifold. Moreover, $Y_n - \mathcal{K}_n$ contains a submanifold $W_{d(n)}$ such that there exists an embedding $\Phi_n : Y_n - W_{d(n)} \longrightarrow \Sigma \times I$ with $\Phi_n(\mathcal{K}_n) = \Phi_{n+1}(\mathcal{K}_n) \subset \Phi_{n+1}(\mathcal{K}_{n+1})$. Since $\mathcal{K}_\infty = \text{Int}M_{\infty,\text{p-thick}(\varepsilon)}$ is homeomorphic to M_∞, our desired embedding of M_∞ to $\Sigma \times I$ is defined by the expanding sequence $\{\Phi_n(\mathcal{K}_n)\}$.

Once M_∞ is realized as an open subset of $\Sigma \times (0,1)$, the conditions (i)-(iii) are derived from fundamental properties of hyperbolic 3-manifolds, e.g. any two parabolic cusps in a hyperbolic 3-manifold are not mutually parallel. In particular, we will show by invoking some conditions on $\mathcal{X}$ that each component of $\lambda(\Lambda_y^{\pm})$ corresponds to a 'hidden' parabolic cusp of M_∞.

Outline of the proof of Theorem 3. Consider any crevasse $\mathcal{X}$ in $\Sigma \times I$. For any $y \in \mathcal{Y} = q(\mathcal{X})$, the components of $\Delta(\Lambda_y^\varepsilon) - \lambda(\Lambda_y^\varepsilon)$ are called open subsurface of type $\mathbf{A}^\varepsilon$ for $\varepsilon = \pm$, and each component of $\Sigma_y - X_y$ is an open subsurface of type $\mathbf{C}$. Thus, X_y consists of geodesic loops and open subsurfaces of types $\mathbf{A}^\varepsilon$, $\mathbf{B}^\varepsilon$ and $\mathbf{C}$, see Fig. 3 again.

In general, $\Sigma \times I - \mathcal{X}$ is not of finite type, that is, $\pi_1(\Sigma \times I - \mathcal{X})$ is possibly of infinitely generated. So, we will define as follows an expanding sequence $\{W_n\}$ of submanifolds of finite type in $\Sigma \times I - \mathcal{X}$ with $\cup_{n=1}^\infty W_n = \Sigma \times I - \mathcal{X}$. For any $n \in \mathbb{N}$, let $\mathcal{N}_{\mathbf{A},n}$ be the union of small collar neighborhoods in $\Sigma \times I$ of subsurfaces of types $\mathbf{A}^{\pm}$, where the depth of each collar neighborhood in $\mathcal{N}_{\mathbf{A},n}$ is the $1/n$ times of that in $\mathcal{N}_{\mathbf{A},1}$. The number of subsurfaces of types $\mathbf{B}^{\pm}$ and (saturated) geodesic loop components in $\mathcal{X}$ disjoint from $\mathcal{N}_{\mathbf{A},n}$ is finite. Let $\mathcal{B}_n$ (resp. $\mathcal{P}_n$) be the set of such subsurfaces (resp. geodesic loops). Then, we have $\mathcal{B}_n \subset \mathcal{B}_{n+1}$ and $\bigcup \mathcal{P}_n \subset \bigcup \mathcal{P}_{n+1}$. Let M_n^m be a 3-manifold obtained by cutting open $\Sigma \times I - \bigcup \mathcal{P}_n$ along $B_i \in \mathcal{B}_n$ and gluing back the both side of B_i by the m-th iteration φ_i^m of a pseudo-Anosov homeomorphism $\varphi_i : B_i \longrightarrow B_i$. For all sufficiently large $m \in \mathbb{N}$, we may assume that M_n^m is acylindrical if necessary by adding some geodesic loops contained in subsurfaces of type $\mathbf{A}^{\pm}$ to $\mathcal{P}_n$. The extended set is denoted by $\widehat{\mathcal{P}}_n$. By Thurston's Uniformization Theorem, $\mathrm{Int}M_n^m$ admits a hyperbolic structure with two geometrically finite ends corresponding to $\Sigma_0 \cup \Sigma_1$ and $\mathbb{Z}\times\mathbb{Z}$-cusps corresponding to elements of $\widehat{\mathcal{P}}_n$. Note that $\mathrm{Int}M_n^m$ is homeomorphic to $\Sigma \times (0,1)$ minus finitely many geodesic loops in fibers. By Hyperbolic Dehn Surgery Theorem in [8], $\mathrm{Int}M_n^m$ is a geometric limits of hyperbolic 3-manifolds with quasi-Fuchsian holonomies. Let $\zeta_n^m : \pi_1(Q_{n,0}) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ be the restriction of the holonomy of $\mathrm{Int}M_n^m$, where $Q_{n,0}$ is the component of $\Sigma \times I - \bigcup(\mathcal{B}_n \cup \widehat{\mathcal{P}}_n)$ containing $\Sigma_{1/2}$. Then, for each $n \in \mathbb{N}$, one can show that $\{\zeta_n^m\}_{m=1}^\infty$ converges algebraically to a representation ζ_n^∞ such that $N_n^\infty = \mathbb{H}^3/\zeta_n^\infty(\pi_1(Q_{n,0}))$ is homeomorphic to $Q_{n,0}$. In turn, we would like to show that $\{\zeta_n^\infty\}_{n=1}^\infty$ converges 'algebraically' in a reasonable sense, though $Q_{n,0}$ is not a submanifold of $Q_{n+1,0}$. If we use the component W_n of $Q_{n,0} - \overline{\mathcal{N}_{\mathbf{A},n}}$ containing $\Sigma_{1/2}$ instead of $Q_{n,0}$, then W_n is a submanifold of W_{n+1} and $\cup_{n=1}^\infty W_n = \Sigma \times I - \mathcal{X}$. Then, $\{\zeta_n^\infty|\pi_1(W_n)\}$ having a subsequence $\{\xi_a\}$ converging to $\xi_\infty : \pi_1(\Sigma \times I - \mathcal{X}) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ means that $\{\xi_a|\pi_1(W_n)\}$ converges algebraically to $\xi_\infty|\pi_1(W_n)$ for any $n \in \mathbb{N}$. The algebraic convergence of $\{\xi_a|\pi_1(W_n)\}$ is reduced to those of $\{\xi_a|\pi_1(C)\}$ for open subsurfaces C of type $\mathbf{C}$ by Relative Boundedness Theorem [11]. The convergence of $\{\xi_a|\pi_1(C)\}$ is examined by test pleated surfaces $\overline{f}_{C;n} : C(\sigma_n) \longrightarrow N_n^\infty$ given in [10]. Usually, such a convergence theorem is proved by introducing a contradiction under the contrary assumption such that $\{\sigma_n\}$ is unbounded in $\mathrm{Teich}(C)$. Then, we would have an accumulation point $[\nu]$ of $\{\sigma_n\}$ in the Thurston boundary $\mathcal{PL}_0(C)$. Intuitively, the projective lamination $[\nu]$ represents the part of $C(\sigma_n)$ which is split or collapsed most rapidly as $n \to \infty$. However, in our argument, we will need to concern relatively slowly split parts of $C(\sigma_n)$. For example, let l_1 and l_2 be mutually disjoint simple geodesics in C, where C is supposed to have a complete hyperbolic structure of finite area. Suppose that $\sigma_n \in \mathrm{Teich}(C)$ is obtained by the n^2-full Dehn twist along l_1

and the n-full Dehn twist along l_2. Then, $\{\sigma_n\}$ converges to $[l_1] \in \mathcal{PL}_0(C)$ which contains no data about l_2. For any simple geodesic loop $l \in C$ with $l \cap l_2 \neq \emptyset$, we have $\lim_{n\to\infty} \mathrm{length}_{\sigma_n}(l) = \infty$ even if $l \cap l_1 = \emptyset$. Such a divergence of the lengths of certain geodesic loops or measured laminations in $C(\sigma_n)$ is crucial to obtain our desired contradiction.

Once the algebraic convergence of $\{\xi_a\}$ to ξ_∞ is proved, one can show that $H_\infty = \mathbb{H}^3/G$ is homeomorphic to $\Sigma \times I - \mathcal{X}$ for $G = \xi_\infty(\Sigma \times I - \mathcal{X})$ by using the fact that $\mathbb{H}^3/G$ has an expanding sequence $\{H_n\}$ of submanifolds homeomorphic to W_n. This fact can be also proved by using results in Anderson-Canary-McCullogugh [1]. For any $t \in \mathbb{N}$, $\xi_\infty|\pi_1(W_t)$ is algebraically approximated by $\xi_a|\pi_1(W_t) = \zeta^\infty_{n(a)}|\pi_1(W_t)$, and the latter by $\zeta^{m(a)}_{n(a)}|\pi_1(W_t)$. For all sufficiently large $n(a)$, any parabolic elements of $\xi_\infty(\pi_1(W_t))$ corresponds to those of $\zeta^{m(a)}_{n(a)}|\pi_1(W_t)$. It follows that $\xi_\infty(\pi_1(W_t))$ is a geometric limit of $\{\zeta^{m(a)}_{n(a)}(\pi_1(W_t))\}$. Note that the holonomy group G_a of $\mathrm{Int}M^{m(a)}_{n(a)}$ contains $\zeta^{m(a)}_{n(a)}(\pi_1(W_t))$. From this fact, we will show that G is a geometric limit of $\{G_a\}$. In turn, each G_a is the geometric limit of quasi-Fuchsian groups as we have remarked above.

References

[1] J. Anderson, R. Canary and D. McCullogugh, The topology of deformation spaces of Kleinian groups, Ann. of Math. **152** (2000), 693-741.

[2] F. Brock, Iteration of mapping classes and limits of hyperbolic 3-manifolds, Invent. Math. **143** (2001), 523-570.

[3] T. Jφrgensen and A. Marden, Geometric and algebraic convergence of Kleinian groups, Math. Scand. **66** (1990), 47-72.

[4] S. Kerckhoff and W. Thurston, Non-continuity of the action of the modular group at Bers' boundary of Teichmüller space, Invent. Math. **100** (1990), 25-47.

[5] Y. Minsky, On rigidity, limit sets and end invariants of hyperbolic 3-manifolds, J. Amer. Math. Soc. **7** (1994), 539-588.

[6] R. Myers, Simple knots in compact, orientable 3-manifolds, Trans. Amer. Math. Soc. **273** (1982), 75-91.

[7] T. Soma, Geometric limits of quasi-Fuchsian groups, preprint: on line at `http://www.r.dendai.ac.jp/~soma/geomlim.ps.GZ`.

[8] W. Thurston, The geometry and topology of 3-manifolds, Lecture Notes, Princeton Univ., Princeton (1978): on line at `http://www.msri.org/publications/books/gt3m/`.

[9] W. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. **6** (1982), 357-381.

[10] W. Thurston, Hyperbolic structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle, E-print: `math.GT/9801045`.

[11] W. Thurston, Hyperbolic structures on 3-manifolds, III: Deformations of 3-manifolds with incompressible boundary, E-print: `math.GT/9801058`.

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