

Canonical decompositions of cusped hyperbolic 3-manifolds obtained by Dehn fillings

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1 Introduction

By Epstein and Penner [7], a *cusped* hyperbolic manifold, that is, a non-compact, complete, orientable, hyperbolic manifold of finite volume, is canonically decomposed into a finite collection of hyperbolic ideal polyhedra, which is called the *canonical decomposition*. Given a cusped hyperbolic 3-manifold with at least 2 cusps, one can obtain an infinite family of cusped hyperbolic 3-manifolds by using Thurston's hyperbolic Dehn surgery theorem. In this paper, we study the effect of Dehn fillings on the canonical decompositions of cusped hyperbolic 3-manifolds, and see a phenomenon similar to that which appears in the hyperbolic Dehn surgery theory.

Let M be a cusped hyperbolic 3-manifold of finite volume with $k_1 + k_2$ ($k_1, k_2 > 0$) cusps. For a k_2 -tuple, $\mathbf{s} = (s_1, \dots, s_{k_2})$, of pairs of coprime integers, let $M_0(\mathbf{s})$ be the manifold obtained from M by s_j -Dehn filling on the end j for $j \in \{k_1 + 1, \dots, k_1 + k_2\}$. We will consider the canonical decomposition of $M_0(\mathbf{s})$. For almost every \mathbf{s} , the canonical decomposition of $M_0(\mathbf{s})$ is characterized in Theorem 3.14 as the union of the "stable part" and the "unstable part";

1. the stable part is a subdivision of the polyhedra of the canonical decomposition of M , with sufficiently small "weights" on the cusps to be filled, whose "vertices" are not contained in the cusps to be filled;
2. the unstable part is determined from the local property of each filled end.

There are only finitely many such subdivisions for the stable part described in 1. So, it seems reasonable to say the part is stable, with respect to the change of Dehn surgery coefficient \mathbf{s} . On the other hand, if the end under consideration has a "simple combinatorial type", then the unstable part is described more explicitly. In fact, it is essentially determined via the Euclidean algorithm applied to the Dehn surgery coefficient s_j . The proof uses the characterization of the combinatorial types of the Ford domains of cyclic Kleinian groups due to Jorgensen [8].

This paper is organized as follows. In Section 2, we briefly review the definition of canonical decompositions by Epstein-Penner. In Section 3, we study the relation of Dehn surgeries and canonical decompositions in a general setting and define the stable and the unstable parts. In Section 4, we study the unstable part more explicitly under the condition that the cusp has a simple combinatorial type, where we apply the Jorgensen's work for cyclic Kleinian groups to our setting. Finally, in Section 5, we study an example, and determine the canonical decompositions explicitly for most Dehn surgery coefficients.

2 Canonical decomposition

Let M be a hyperbolic 3-manifold of finite volume with k cusps. A *weight* for M is a k -tuple of positive numbers, $W = (w_1, \dots, w_k)$. The *canonical decomposition of M with weight W* is defined by Epstein and Penner [7] as follows.

1. Choose mutually disjoint (small) horospherical neighborhoods, $C_1(W), \dots, C_k(W)$, of the cusps of M , so that the ratio of the volumes is equal to that of w_1, \dots, w_k .
2. Let $\mathcal{H}(W)$ be the set of horoballs in \mathbb{H}^3 which project onto the union of horospherical neighborhoods $\cup_{j=1}^k C_j(W)$ by the universal covering projection.
3. Let $\mathcal{B}(W)$ be the subset of the positive light-cone in the Minkowski space $\mathbb{E}^{1,3}$ corresponding to $\mathcal{H}(W)$, that is, $\mathcal{B}(W)$ is the set of points b in the positive light-cone such that the horoball $\{x \in \mathbb{H}^3 \mid \langle b, x \rangle \geq -1\}$ is contained in $\mathcal{H}(W)$.
4. Let $\mathcal{C}(W)$ be the closed convex hull of $\mathcal{B}(W)$ in $\mathbb{E}^{1,3}$. Then $\mathcal{C}(W)$ is a closed set contained in the inside of the positive light-cone, and its interior is homeomorphic to the open 4-ball. Moreover, every ray in $\mathbb{E}^{1,3}$ from the origin, which lies in the inside of the positive light-cone, intersects $\mathcal{C}(W)$ at a single point in $\partial\mathcal{C}(W)$.
5. Let $\tilde{\Delta}(W)$ be the polyhedral decomposition of \mathbb{H}^3 obtained from the natural cellular structure on $\partial\mathcal{C}(W)$ via the radial projection from the origin. Then it follows that $\tilde{\Delta}(W)$ is Γ -invariant and locally finite.
6. We define the *canonical decomposition, $\Delta(W)$, of M with weight W* to be the ideal polyhedral decomposition of M obtained from $\tilde{\Delta}(W)$ via the universal covering projection.

In particular, the canonical decomposition with weight $(1, \dots, 1)$ is simply called the *canonical decomposition of M* .

Canonical decompositions can be characterized without using the Minkowski space model as above. For the purpose of doing it, we introduce the notion of the signed *distance*, $d(x, H)$, between a point x and a horoball H in \mathbb{H}^3 defined as

$$d(x, H) = \begin{cases} d(x, \partial H), & \text{if } x \notin H, \\ -d(x, \partial H), & \text{otherwise.} \end{cases}$$

Definition 2.1. For a point x and a set of horoballs \mathcal{H} in \mathbb{H}^3 , a horoball H in \mathcal{H} is said to be a *nearest horoball* to x in \mathcal{H} if $d(x, H)$ attains the minimum among $d(x, H')$ for $H' \in \mathcal{H}$. The set of nearest horoballs to x in \mathcal{H} is denoted by $\mathcal{N}(x, \mathcal{H})$.

Proposition 2.2. *Let M be a cusped hyperbolic manifold, and W a weight for M . Then an ideal polyhedron spanned by a set of points V in $\partial\mathbb{H}^n$ projects onto a polyhedron in $\Delta(W)$ if and only if there is a point x such that the set of centers of horoballs in $\mathcal{N}(x, \mathcal{H}(W))$ is equal to V .*

3 Hyperbolic Dehn fillings and canonical decompositions

3.1 Thurston's hyperbolic Dehn surgery theorem

Definition 3.1 (Dehn filling). Let M be a 3-manifold with an end e with a neighborhood homeomorphic to the product $T^2 \times \mathbb{R}$, where T^2 denotes the torus. We fix a system of generators $\{\mu, \lambda\}$ of $\pi_1(e)$, which is regarded as the image of a pair of elements in $\pi_1(T^2)$ of simple closed curves on T^2 by the canonical identification $\pi_1(e) \cong \pi_1(T^2)$. For a pair of coprime integers $s = (p, q)$, an s -Dehn filling on e is the operation which produces a 3-manifold $M(s)$ from M as follows.

1. Let M' be the manifold obtained from M by removing the neighborhood of the end e corresponding to $T^2 \times \mathbb{R}_+$, where \mathbb{R}_+ denotes the set of positive numbers, and denote the boundary component of M' corresponding to $T^2 \times \{0\}$ by T_e^2 .
2. The manifold $M(s)$ is obtained from M' by gluing the solid torus, V , along T_e^2 and ∂V so that the meridian of V is identified with a simple closed curve on T_e^2 representing the element $p\mu + q\lambda$ of $\pi_1(T_e^2) \cong \pi_1(e)$.

Definition 3.2. We denote the set of coprime pairs of integers by P and the union $P \cup \{\infty\}$ by \widehat{P} , which is regarded as a subset of the one-point compactification of the real plane, $\mathbb{R}^2 = \mathbb{R}^2 \cup \{\infty\} \approx S^2$.

Definition 3.3. Let M be a 3-manifold with k specified ends, e_1, \dots, e_k , each with neighborhoods homeomorphic to $T^2 \times \mathbb{R}$. For a k -tuple $\mathbf{s} = (s_1, \dots, s_k) \in (\widehat{P})^k$, $M(\mathbf{s})$ is the manifold obtained from M by performing the following operation simultaneously on the ends e_1, \dots, e_k .

- If $s_j \in P$, then perform the s_j -Dehn filling on e_j .
- If $s_j = \infty$, then leave e_j unchanged.

We say that $M(\mathbf{s})$ is obtained from M by \mathbf{s} -Dehn filling on the ends e_1, \dots, e_k .

The following is the hyperbolic Dehn surgery theorem by Thurston [9].

Theorem 3.4 (Thurston). Let M be a hyperbolic 3-manifold with k cusps. Then there is a neighborhood U of $\infty = (\infty, \dots, \infty)$ in $(\widehat{\mathbb{R}^2})^k$ such that for any $\mathbf{s} \in U \cap (\widehat{P})^k$, the manifold $M(\mathbf{s})$, obtained from M by the \mathbf{s} -Dehn filling on the cusps, also admits a complete hyperbolic structure.

Remark 3.5. Moreover, the following holds. Let $\{\mathbf{s}_n\}$ be a sequence in $(\widehat{P})^k$ which converges to ∞ , and $\rho_n \in \mathcal{R}(\pi_1(M))$ the composition of the holonomy representation of $M(\mathbf{s}_n)$ and the canonical onto homomorphism $\pi_1(M) \rightarrow \pi_1(M(\mathbf{s}_n))$. Then the sequence $\{\rho_n\}$ converges strongly to the holonomy representation of M .

3.2 Stable part

In what follows, we consider the “stable part” of the canonical decompositions of manifolds obtained from a hyperbolic manifold with at least two cusps by Dehn fillings so that at least one cusp remains unfilled. This is a generalization of what have been studied through [4, 3, 5]. In the remaining of this section, we let M be a hyperbolic 3-manifold with $k_1 + k_2$ cusps for some

positive integers k_1 and k_2 . We will consider manifolds $M(\infty, \dots, \infty, \mathbf{s})$ for $\mathbf{s} \in P^{k_2}$, which will be denoted by $M_0(\mathbf{s})$ for simplicity. Let U be a neighborhood of (∞, \dots, ∞) in $(\widehat{\mathbb{R}^2})^{k_2}$ such that for any $\mathbf{s} \in U \cap P^{k_2}$, $M_0(\mathbf{s})$ admits a complete hyperbolic structure.

Here, to simplify the notations, we will introduce the following symbol.

Definition 3.6. We denote by $[a]_j$ the sequence of length j whose terms are equal to the same number a .

Choose horospherical neighborhoods, C_1, \dots, C_{k_1} , of the cusps $1, \dots, k_1$ of M so that they are mutually disjoint and that their volumes are equal to the same number, v_0 . Let \mathcal{H}_s be the set of horoballs which project onto one of C_1, \dots, C_{k_1} by the universal covering projection, $\pi: \mathbb{H}^3 \rightarrow M$. For any $\mathbf{s} \in U \cap P^{k_2}$, we define a set of horoballs $\mathcal{H}_s(\mathbf{s})$ as follows. Notice that there is a canonical embedding $M \hookrightarrow M_0(\mathbf{s})$, which maps the cusps $1, \dots, k_1$ of M into the cusps $1, \dots, k_1$ of $M_0(\mathbf{s})$. The embedding induces an isomorphism from $\text{Stab}(H, \rho(\pi_1(M)))$ onto a parabolic subgroup of $\rho_s(\pi_1(M))$ for any horoball H in \mathcal{H}_s , where ρ_s denotes the composition of the holonomy of $\pi_1(M_0(\mathbf{s}))$ and the canonical onto homomorphism $\pi_1(M) \rightarrow \pi_1(M_0(\mathbf{s}))$. The image of $\text{Stab}(H, \rho(\pi_1(M)))$ in $\rho_s(\pi_1(M))$ also stabilizes a 1-parameter family of horoballs, and there is one, denoted by $h_s(H)$, among them such that the volume of the quotient by the parabolic group is equal to v_0 . We define $\mathcal{H}_s(\mathbf{s})$ to be the collection of $h_s(H)$ for all $H \in \mathcal{H}_s$. It follows that \mathcal{H}_s is the set of horoballs which project onto the horospherical neighborhoods of the cusps of $M(\mathbf{s})$.

For a set of horoballs, \mathcal{H} , we denote by $\mathcal{P}(\mathcal{H})$ the ideal polyhedra in \mathbb{H}^3 whose vertices are contained in the set of centers of horoballs in \mathcal{H} . Then the map h_s from \mathcal{H}_s to $\mathcal{H}_s(\mathbf{s})$ naturally induces a map, denoted by the same symbol h_s , from $\mathcal{P}(\mathcal{H}_s)$ to $\mathcal{P}(\mathcal{H}_s(\mathbf{s}))$. We remark that some ideal polyhedron may be mapped to a singleton in $\partial\mathbb{H}^3$ in general. However, if we only consider a finite collection of finite sided ideal polyhedra in $\mathcal{P}(\mathcal{H}_s)$, then we may assume that the polyhedra are mapped to those with the same combinatorial types by choosing a sufficiently small neighborhood U of $([\infty]_{k_2})$ in $(\widehat{\mathbb{R}^2})^{k_2}$.

Proposition 3.7. For any ideal polyhedron σ in $\mathcal{P}(\mathcal{H}_s)$ such that $\pi|_\sigma$ is injective, there is a neighborhood U of $([\infty]_{k_2})$ in $(\widehat{\mathbb{R}^2})^{k_2}$ which satisfies the following condition. For any $\mathbf{s} \in U \cap P^{k_2}$, $\pi_s \circ h_s(\sigma)$ is ambient isotopic to the image of $\pi(\sigma)$ by the canonical embedding of M into $M_0(\mathbf{s})$, where $\pi_s: \mathbb{H}^3 \rightarrow M_0(\mathbf{s})$ is the universal covering projection.

Definition 3.8. For any $\epsilon > 0$, we will denote by $\Delta_0(\epsilon)$ the subcomplex of the canonical decomposition $\Delta([1]_{k_1}, [\epsilon]_{k_2})$ of M with weight $([1]_{k_1}, [\epsilon]_{k_2})$ consisting of the polyhedra contained in $\pi(\mathcal{P}(\mathcal{H}_s))$.

As a corollary to Theorem 1.1 of [2], the following holds.

Proposition 3.9. There is a positive number ϵ_0 such that for any $0 < \epsilon \leq \epsilon_0$, the hyperbolic ideal polyhedral complex $\Delta_0(\epsilon)$ embedded in M is equal to $\Delta_0(\epsilon_0)$.

Definition 3.10. Let Δ_s be the hyperbolic ideal polyhedral complex $\Delta_0(\epsilon_0)$ embedded in M obtained by Proposition 3.9. We will call Δ_s the *stable part* for the family of cusped hyperbolic manifolds $\{M_0(\mathbf{s}) \mid \mathbf{s} \in P^{k_2}\}$.

We obtain the following proposition by using the strong convergence of the holonomies mentioned in Remark 3.5.

Proposition 3.11. *There is a neighborhood U of $([\infty]_{k_2})$ in $(\widehat{\mathbb{R}^2})^{k_2}$ which satisfies the following condition. For any $\mathbf{s} \in U \cap P^{k_2}$, there is a subdivision $\Delta_s(\mathbf{s})$ of Δ_s such that for any ideal polyhedron $\pi(\sigma)$ in $\Delta_s(\mathbf{s})$, the corresponding ideal polyhedron $\pi_s \circ h_s(\sigma)$ is contained in the canonical decomposition of $M_0(\mathbf{s})$.*

3.3 Unstable part

Let ϵ be a positive number which is strictly less than ϵ_0 obtained by Proposition 3.9, and denote $\mathcal{H}([1]_{k_1}, [\epsilon]_{k_2})$ simply by \mathcal{H} . For each $j \in \{k_1 + 1, \dots, k_1 + k_2\}$, fix a horoball $H_\infty^{(j)}$ in \mathcal{H} which project onto a neighborhood of the cusp j of M . We can take a set of horoballs $\{H_1^{(j)}, \dots, H_{n_j}^{(j)}\}$ in \mathcal{H} such that the geodesic $\tau_l^{(j)}$ ($l \in \{1, \dots, n_j\}$) connecting the centers of $H_\infty^{(j)}$ and $H_l^{(j)}$ is a lift of an edge of $\Delta([1]_{k_1}, [\epsilon]_{k_2})$, and that the set $\{\tau_1^{(j)}, \dots, \tau_{n_j}^{(j)}\}$ is a complete collection of the representatives of the lifts of edges of $\Delta([1]_{k_1}, [\epsilon]_{k_2})$ which is not contained in Δ_s modulo $\rho(\pi_1(M))$. We denote by $\mathcal{H}^{(j)}$ the set of all $\gamma H_l^{(j)}$ for $\gamma \in \text{Stab}(H_\infty^{(j)}, \rho(\pi_1(M)))$ and $l \in \{1, \dots, n_j\}$. It follows that each horoball in $\mathcal{H}^{(j)}$ projects onto a neighborhood of the cusp j' for some $j' \in \{1, \dots, k_1\}$, and hence the set of horoballs $h_s(\mathcal{H}^{(j)})$, each of which projects onto a neighborhood of a cusp of $M_0(\mathbf{s})$, is well defined for any $\mathbf{s} \in P^{k_2}$ contained in a small neighborhood of $([\infty]_{k_2})$ in $(\widehat{\mathbb{R}^2})^{k_2}$. We denote it by $\mathcal{H}^{(j)}(\mathbf{s})$.

Definition 3.12. For any $\mathbf{s} \in P^{k_2}$ contained in a small neighborhood of $([\infty]_{k_2})$ in $(\widehat{\mathbb{R}^2})^{k_2}$, we denote by $\Delta_u^{(j)}(\mathbf{s})$ the set of ideal polyhedra $\pi_s(\sigma)$ such that the set of vertices of the ideal polyhedron σ in \mathbb{H}^3 is equal to the set of centers of the horoballs in $\mathcal{N}(x, \mathcal{H}^{(j)}(\mathbf{s}))$ for some $x \in \mathcal{H}^3$, and let $\Delta_u(\mathbf{s})$ be the union of $\Delta_u^{(j)}(\mathbf{s})$ ($j \in \{k_1 + 1, \dots, k_1 + k_2\}$). We call $\Delta_u(\mathbf{s})$ the *unstable part* of the canonical decomposition of $M_0(\mathbf{s})$.

Then, by an argument similar to that is used in the proof of Proposition 3.11, we can prove the following proposition.

Proposition 3.13. *There is a neighborhood U of $([\infty]_{k_2})$ in $(\widehat{\mathbb{R}^2})^{k_2}$ with the following property. For any $\mathbf{s} \in P^{k_2} \cap U$, an ideal polyhedron $\pi_s(\sigma)$ embedded in $M_0(\mathbf{s})$ is contained in the canonical decomposition of $M_0(\mathbf{s})$ if and only if it is contained in $\Delta_u(\mathbf{s})$.*

As an immediate corollary to Propositions 3.11 and 3.13, we obtain the following theorem.

Theorem 3.14. *There is a neighborhood U of $([\infty]_{k_2})$ in $(\widehat{\mathbb{R}^2})^{k_2}$ such that, for any $\mathbf{s} \in P^{k_2} \cap U$, the canonical decomposition of $M_0(\mathbf{s})$ is the union of the stable part $\Delta_s(\mathbf{s})$ and the unstable part $\Delta_u(\mathbf{s})$.*

The unstable part $\Delta_u(\mathbf{s})$ is described more precisely when it is “simple” in the following sense.

Definition 3.15. We say that the cusp j ($j \in \{k_1 + 1, \dots, k_1 + k_2\}$) has a *simple combinatorial type* if the number of edges in $\Delta(\epsilon)$ which intersect the cusp is equal to 1, where ϵ is a positive number less than ϵ_0 obtained by Proposition 3.9.

4 Cyclic Kleinian groups

4.1 Ford domain as a dual of canonical decomposition

In [8], Jorgensen studied the Ford domains of cyclic Kleinian groups, and gave, without writing down a proof explicitly, a characterization of the combinatorial types of them. (A nice exposition and the proof for the characterization is given by Drumm and Poritz [6].) In this section, we will use the characterization to determine the unstable part arising from a cusp with a simple combinatorial type.

First, we recall some property of Ford domains and see how it can be used to decide the canonical decompositions.

Definition 4.1. The isometric hemisphere, $Ih(\gamma)$, of an element $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $PSL_2(\mathbb{C})$ with $\gamma(\infty) \neq \infty$ is the hemisphere in the upper half space of the Euclidean 3-space, which can be identified with \mathbb{H}^3 , with boundary \mathbb{C} , whose equator is equal to the circle $\{z \in \mathbb{C} \mid |cz + d| = 1\}$ in \mathbb{C} . The closure of the unbounded component of the complement of $Ih(\gamma)$ in the upper half space will be called the *exterior* of $Ih(\gamma)$.

Definition 4.2. For a Kleinian group G which contains no elements stabilizing ∞ , the *Ford domain* of G is the common exterior of all the isometric hemispheres of the elements of G .

The following lemma will be well-known.

Lemma 4.3. Let H be a horoball in \mathbb{H}^3 centered at ∞ , and γ be an element of $PSL_2(\mathbb{C})$ with $\gamma(\infty) \neq \infty$. Then the set of points x satisfying $d(x, H) \leq d(x, \gamma H)$ is equal to the exterior of $Ih(\gamma^{-1})$.

Corollary 4.4. Let G be a Kleinian group which contains no elements stabilizing ∞ . Let H be a horoball in \mathbb{H}^3 centered at ∞ . Then the Ford domain of G is equal to the set of all points x in \mathbb{H}^3 such that $\mathcal{N}(x, G(H))$ contains H , where $G(H)$ denotes the family of all translates of H by G .

Remark 4.5. Notice that the image, in the fundamental group of the Dehn filled manifold, of the peripheral group of each end to be filled is cyclic. Thus, by the corollary, one can determine the unstable part, arising from a cusp with a simple combinatorial type, by using the Jorgensen's result.

4.2 Layered solid torus

In what follows, we will define a triangulation of the solid torus which has only one vertex in the boundary (and has no other vertices). From the construction, we call the solid torus with the triangulation the *layered solid torus*.

To define the triangulation, we recall the Farey triangulation of \mathbb{H}^2 .

Definition 4.6. The *Farey triangulation*, \mathcal{D} , of \mathbb{H}^2 is the ideal triangulation whose triangles are the translates by the action of $PSL_2(\mathbb{Z})$ on \mathbb{H}^2 of the ideal triangle with vertices 0, 1, and ∞ .

The set of vertices of \mathcal{D} is equal to $\mathbb{Q} \cup \{\infty\}$, which is naturally identified with the set of essential simple closed curves modulo isotopy on the torus, T^2 , as follows. The torus is the

quotient of the complex plane \mathbb{C} by the group of parallel translations $\langle z \mapsto z + 1, z \mapsto z + \sqrt{-1} \rangle$. Then the homotopy classes of essential simple closed curves on T^2 have representatives of the form $\{\mathfrak{S}(z) = s\mathfrak{R}(z)\}$ for some $s \in \mathbb{Q} \cup \{\infty\}$. (s is called the *slope* of the isotopy class.) This gives the desired correspondence.

The condition that three points in $\mathbb{Q} \cup \{\infty\}$ span a triangle of \mathcal{D} is equivalent to that the isotopy classes with the slopes contain representatives which intersect at a single point. In fact, \mathcal{D} can be thought as the set of triangulations of T^2 with only one vertex.

Notice that two adjacent triangles of \mathcal{D} have vertices $\{p/q, (p+r)/(q+s), r/s\}$ and $\{p/q, (p-r)/(q-s), r/s\}$ for some integers p, q, r , and s . Then the triangulation of T^2 corresponding to one of them is obtained from that to another by “pasting a tetrahedron” on it. Precisely, the triangulation of T^2 corresponding to $\{p/q, (p+r)/(q+s), r/s\}$ is obtained from that to $\{p/q, (p-r)/(q-s), r/s\}$ as follows. First, consider the decomposition of T^2 by the two curves with slopes p/q and r/s . By lifting the decomposition, one obtains a $\langle z \mapsto z + 1, z \mapsto z + \sqrt{-1} \rangle$ -invariant tessellation of \mathbb{C} by parallelograms. Each parallelogram has two diagonals, whose slopes are $(p+r)/(q+s)$ and $(p-r)/(q-s)$. Now consider the $\langle z \mapsto z + 1, z \mapsto z + \sqrt{-1} \rangle$ -invariant triangulation of \mathbb{C} obtained by adding the lines with slope $(p+r)/(q+s)$ passing through the vertices of the parallelograms. The quotient of this triangulation by the action of $\langle z \mapsto z + 1, z \mapsto z + \sqrt{-1} \rangle$ is the one corresponding to $\{p/q, (p+r)/(q+s), r/s\}$. First, look at any one of the parallelograms. It is the union of two triangles with an edge with slope $(p+r)/(q+s)$ in common. We put a tetrahedron on it so that two of the four faces coincide with the two triangles. Then, looked down from above, the diagonal of the parallelogram with slope $(p+r)/(q+s)$ is switched to the other, with slope $(p-r)/(q-s)$. We put infinite copies of the tetrahedron in the $\langle z \mapsto z + 1, z \mapsto z + \sqrt{-1} \rangle$ -equivariant way. Then, looking down from the above, the triangulation of \mathbb{C} corresponding to $\{p/q, (p+r)/(q+s), r/s\}$ is switched to that to $\{p/q, (p-r)/(q-s), r/s\}$. If we consider the quotient of this process by $\langle z \mapsto z + 1, z \mapsto z + \sqrt{-1} \rangle$, we can observe that the triangulation of T^2 corresponding to $\{p/q, (p+r)/(q+s), r/s\}$ is switched to that to $\{p/q, (p-r)/(q-s), r/s\}$, by pasting a tetrahedron between them.

Given two triangles σ^- and σ^+ of \mathcal{D} , there is a unique shortest sequence of triangles in \mathcal{D} , $\sigma_0 = \sigma^-, \sigma_1, \dots, \sigma_{n-1}, \sigma_n = \sigma^+$, such that each $\sigma_j \cap \sigma_{j+1} \neq \emptyset$ ($j \in \{0, \dots, n-1\}$). Then, by the construction explained in the previous paragraph, we obtain a “layer” of tetrahedra corresponding to the pairs $\{\sigma_j, \sigma_{j+1}\}$ ($j \in \{0, \dots, n-1\}$) stacked upon the torus.

Definition 4.7. We denote by $\Delta(\sigma^-, \sigma^+)$ the space obtained as the layer of tetrahedra in the above way.

Proposition 4.8. *Let σ^- and σ^+ be two triangles of \mathcal{D} . Then the following hold.*

1. *If $\sigma^- = \sigma^+$, then the underlying space of $\Delta(\sigma^-, \sigma^+)$ is homeomorphic to the torus.*
2. *Suppose that $\sigma^- \neq \sigma^+$ and that they have one or two common vertices. Let c be the union of one or two essential simple closed curves on T^2 with slopes corresponding to the common vertices. Then the underlying space of $\Delta(\sigma^-, \sigma^+)$ is homeomorphic to $T^2 \times [0, 1] / \approx$, where $(x, s) \approx (y, t)$ if and only if $(x, s) = (y, t)$ or $x = y \in c$.*
3. *Suppose that σ^- and σ^+ have no common vertex. Then the underlying space of $\Delta(\sigma^-, \sigma^+)$ is homeomorphic to $T^2 \times [0, 1] / \approx$, where $(x, s) \approx (y, t)$ if and only if $(x, s) = (y, t)$ or $x = y = x_0$ for arbitrarily fixed point $x_0 \in T^2$.*

For different triangles σ^- and σ^+ , there is a unique vertex, v , of σ^- which is not contained in the other triangles in the sequence connecting σ^- and σ^+ . Then the triangulation of the bottom boundary component of $\Delta(\sigma^-, \sigma^+)$ contains the edge with slope v . Let $V(\sigma^-, \sigma^+)$ be the quotient space of $\Delta(\sigma^-, \sigma^+)$ by the orientation reversing simplicial isomorphism on the bottom boundary component of $\Delta(\sigma^-, \sigma^+)$ which fixes every point of the edge with slope v .

Proposition 4.9. *The space $V(\sigma^-, \sigma^+)$ is homeomorphic to the solid torus for any pair of different triangles σ^- and σ^+ of \mathcal{D} . Moreover, the triangulation induced from that of $\Delta(\sigma^-, \sigma^+)$ has a single vertex, which is the vertex of the triangulation on the boundary of the solid torus corresponding to σ^+ .*

Definition 4.10. For a pair of different triangles σ^- and σ^+ of \mathcal{D} , we call $V(\sigma^-, \sigma^+)$ the *layered solid torus* determined by the pair. We fix a basis $\{\mu, \lambda\}$ of $\pi_1(\partial T^2)$ as follows. Let v^\pm respectively be the vertices of σ^\pm which are not contained in the other triangles in the sequence connecting σ^- and σ^+ . Then is an oriented geodesic in \mathbb{H}^2 from v^- to v^+ . Let v_l^+ and v_r^+ respectively be the vertices of σ^+ which lies in the left and the right side of the oriented geodesic. Then we define μ and λ to be represented by the simple closed curves on the boundary with slopes v_l^+ and v_r^+ , respectively. We denote by τ the edge of the triangulation of the boundary of $V(\sigma^-, \sigma^+)$ with slope v^+ .

Proposition 4.11. *For any layered solid torus $V(\sigma^-, \sigma^+)$, let $m(\sigma^-, \sigma^+)$ be the pair, (p, q) , of coprime integers such that $p\mu + q\lambda$ is its meridian and that $p > 0$. Then m induces a bijection from the set of simplicial isomorphism classes of layered solid tori to the set of pairs of coprime integers (p, q) with $p > 0$ and $q < 0$.*

By using the idea mentioned at the beginning of this section, we obtain the following.

Proposition 4.12. *Let M and U be as in Theorem 3.14. Suppose that the cusp j ($j \in \{k_1 + 1, \dots, k_1 + k_2\}$) has a simple combinatorial type. Then, for any $s \in P^{k_2} \cap U$, the unstable part $\Delta_u^{(j)}(s)$ is isomorphic to some layered solid torus with the vertex deleted.*

As a direct consequence from Theorem 3.14 and Proposition 4.12, we obtain the following.

Corollary 4.13. *Let M and U be as in Theorem 3.14. Suppose that every cusp j ($j \in \{k_1 + 1, \dots, k_1 + k_2\}$) has a simple combinatorial type. Then, for any $s \in P^{k_2} \cap U$, the canonical decomposition of $M_0(s)$ is the union of the stable part $\Delta_s(s)$ and k_2 layered solid tori.*

5 Example

Let $L = K_1 \cup K_2$ be the link in S^3 illustrated in Figure 1, and M be its complement in S^3 . We name the ends corresponding to K_1 and K_2 , respectively, e_1 and e_2 . For $s \in \hat{P}$, we denote by $M_0(s)$ the manifold obtained from M by s -Dehn filling on the end e_2 . In this section, we will consider the canonical decompositions of the hyperbolic manifolds $M_0(s)$ with a single cusp for $s \in P$ sufficiently close to ∞ . We remark that the manifold $M_0(1, n)$ for an integer $n \in \mathbb{Z}$ is equal to the complement of the pretzel knot of type $(-2, 3, 2n+1)$. This motivates the author to study this family; it is an example of the family which is obtained by using A'Campo's divides and hence is in particular an example of the family of fibered knot complements (see [1] for example). Moreover, the pretzel knot of type $(-2, 3, 7)$, in our family, is regarded as the simplest hyperbolic

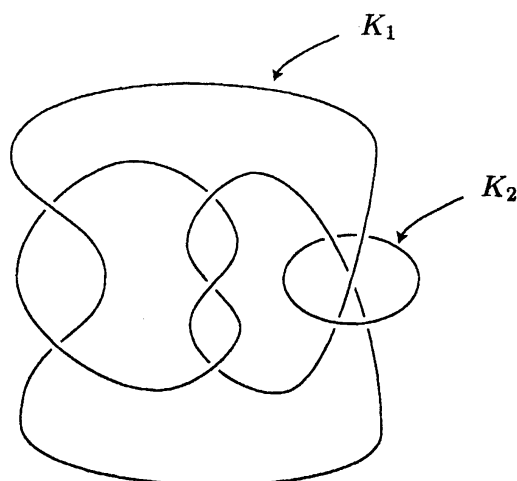


Figure 1: The end corresponding to K_2 will be Dehn filled

knot next to the figure eight knot, and has many special properties in terms of the knot theory. Though our method cannot reach this knot for the present, it seems important to discover what happens when we approach from M to its complement via the hyperbolic Dehn surgery space.

5.1 An ideal polyhedral decomposition of M

First, we must find the stable part Δ_s , which is a subcomplex of the canonical decomposition of M with a sufficiently small weight on the cusp 2. It is achieved as follows.

Step 1. Find the canonical decomposition of M with an arbitrary weight on the cusps. Practically, it is possible by using Weeks' computer program SnapPea [10]; though one may need a "proof by hand" that the decomposition which SnapPea proposed is certainly the desired one.

Step 2. By using the "tilt formula" introduced by Weeks [11], find the canonical decompositions of M with smaller and smaller weights on the cusp 2. It is certainly possible because, after a small change of weight, (i) the canonical decomposition does not change if it is tetrahedral, and (ii) the change will be only the subdivision of the non-tetrahedral polyhedra even if it is not tetrahedral. The process finishes after a finite sequence of modifications from the original decomposition ([2]).

Proposition 5.1. *For any $\epsilon < 16/125$, the canonical decomposition of M with weight $(1, \epsilon)$ is the one illustrated in Figure 2. In the figure, the end e_2 has a neighborhood consisting of a neighborhood of the center vertex of the top polyhedron, and the end e_1 has a neighborhood consisting of neighborhoods of the remaining vertices. In particular, Δ_s consists of the middle and the bottom polyhedra, and the cusp 2 has a simple combinatorial type.*

In fact, the canonical decomposition of M (with the same weight on the cusps) consists of 6 ideal tetrahedra, and after 3 modifications, in each of which a tetrahedral decomposition is modified to another tetrahedral one via a non-tetrahedral one. We will denote the top (resp. middle and bottom) polyhedra by σ_1 (resp. σ_2 and σ_3).

Since σ_3 is an ideal tetrahedron, it suffices to see how σ_2 is subdivided after a Dehn filling in order to find out the part of the canonical decomposition arising from the stable part (Proposition

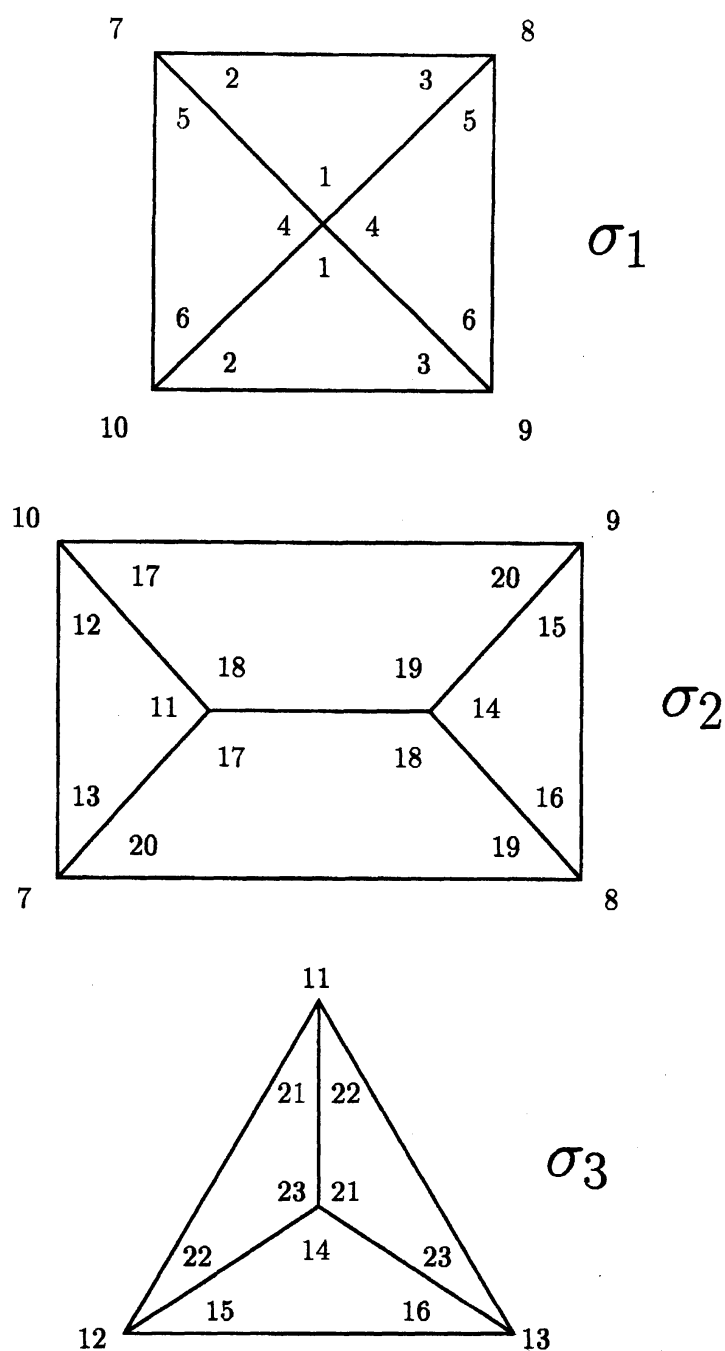


Figure 2: Canonical decomposition of M with a sufficiently small weight on the cusp 2

3.11). By looking the horoball patterns arising from σ_2 , one can prove the following proposition. In the proposition, we will use the canonical meridian-longitude system defined from the link diagram in Figure 1. Moreover, to simplify the notation, we denote the top-left (resp. bottom-left, bottom-right, top-right, middle-left, and middle-right) vertex of σ_2 by v_{tl} (reps. v_{bl} , v_{br} , v_{tr} , v_{ml} , and v_{mr}).

Proposition 5.2. *There is a neighborhood, U , of ∞ in $\widehat{\mathbb{R}^2}$ such that for any $(p, q) \in P \cap U$, the following conditions are satisfied.*

1. *The convex hull of v_{tl} , v_{bl} , v_{br} and v_{tr} is not contained in any ideal polyhedron in the canonical decomposition of $M_0(p, q)$.*
2. *The geodesic connecting v_{tr} and v_{bl} (resp. v_{tl} and v_{br}) is contained in the canonical decomposition of $M_0(p, q)$ if $p(p+q) < 0$ (resp. $p(p+q) > 0$).*

By Propositions 3.11 and 5.2, we obtain the following corollary.

Corollary 5.3. *There is a neighborhood, U , of ∞ in $\widehat{\mathbb{R}^2}$ such that for any $(p, q) \in P \cap U$, the stable part $\Delta_s(p, q)$ is the union of σ_3 and σ_2 subdivided into three ideal tetrahedra as follows.*

1. *If $p(p+q) < 0$, then $\sigma_2 = \langle v_{tl}, v_{bl}, v_{tr}, v_{ml} \rangle \cup \langle v_{bl}, v_{tr}, v_{ml}, v_{mr} \rangle \cup \langle v_{bl}, v_{br}, v_{tr}, v_{mr} \rangle$.*
2. *If $p(p+q) > 0$, then $\sigma_2 = \langle v_{tl}, v_{bl}, v_{br}, v_{ml} \rangle \cup \langle v_{tl}, v_{br}, v_{ml}, v_{mr} \rangle \cup \langle v_{tl}, v_{br}, v_{tr}, v_{mr} \rangle$.*

In the above, the symbol $\langle \cdot \rangle$ denotes the convex hull.

By Corollary 4.13, the canonical decomposition of $M_0(p, q)$ is the union of the stable part $\Delta_s(p, q)$ and a layered solid torus with the vertex deleted. Recall that $M_0(p, q)$ is obtained from M by adding a solid torus to the end e_2 so that the meridian of the solid torus is identified with the loop $p\mu + q\lambda$ in e_2 . We can see that for any $(p, q) \in P \cap U$, there is a unique choice of the pair of (i) a layered solid torus, V , and (ii) the gluing of V to $\Delta_s(p, q)$, because the ideal triangulation of the boundary of V is equal to that of $\Delta_s(p, q)$. Then we finally obtain the following theorem.

Theorem 5.4. *There is a neighborhood, U , of ∞ in $\widehat{\mathbb{R}^2}$ such that for any $(p, q) \in P \cap U$, the canonical decomposition of $M_0(p, q)$ is characterized as follows.*

1. *If $p(2p+q) < 0$, then the canonical decomposition of $M_0(p, q)$ is the union of $\Delta_s(p, q)$ and the layered solid torus, V , with meridian $p\mu + (2p+q)\lambda$, where V is glued to $\Delta_s(p, q)$ so that the edge τ on ∂V is identified with the edge $\langle v_{tl}, v_{bl} \rangle$ in the boundary of $\Delta_s(p, q)$.*
2. *If $(p+q)(2p+q) < 0$, then the canonical decomposition of $M_0(p, q)$ is the union of $\Delta_s(p, q)$ and the layered solid torus, V , with meridian $(2p+q)\mu - p\lambda$, where V is glued to $\Delta_s(p, q)$ so that the edge τ on ∂V is identified with the edge $\langle v_{tl}, v_{tr} \rangle$ in the boundary of $\Delta_s(p, q)$.*
3. *If $(p+q)q < 0$, then the canonical decomposition of $M_0(p, q)$ is the union of $\Delta_s(p, q)$ and the layered solid torus, V , with meridian $(p+q)\mu + q\lambda$, where V is glued to $\Delta_s(p, q)$ so that the edge τ on ∂V is identified with the edge $\langle v_{tl}, v_{tr} \rangle$ in the boundary of $\Delta_s(p, q)$.*
4. *If $pq > 0$, then the canonical decomposition of $M_0(p, q)$ is the union of $\Delta_s(p, q)$ and the layered solid torus, V , with meridian $q\mu - p\lambda$, where V is glued to $\Delta_s(p, q)$ so that the edge τ on ∂V is identified with the edge $\langle v_{tl}, v_{bl} \rangle$ in the boundary of $\Delta_s(p, q)$.*

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