数理解析研究所講究録 1329 巻 2003 年 133-138

SUM OF THE EDGE LENGTHS OF A GEODESIC GRAPH

奈良女子大学・理学部　市原 一裕 (Kazuhiro Ichihara)
Department of Information and Computer Sciences,
Nara Women's University

Abstract

Consider an embedding of a complete graph of order four into the 2-sphere such that each edge becomes a shortest geodesic connecting its end-points. Then we show that the sum of the edge lengths is at most 4π, and is bigger than 3π if the graph is not contained in any hemisphere.

はじめに

2002 年 12 月に京都大学数理解析研究所で行われた研究集会「双曲空間に関連する研究とその展望」では、「Area of a cellular complex in a hyperbolic manifold」という題目で、双曲多様体に測地的に埋め込まれた 2 次元胞体複体の面積に関して発表しました。本稿では、その主定理の証明において鍵となった、Guddum の定理 [1] の拡張を与えます。尚、研究集会で発表した結果に関しては、プレプリント [3] を参照下さい。

1. Results

In this article, we consider a finite graph geodesically embedded into a surface with constant curvature metric, and estimate the sum of the edge lengths. As usual, we regard a finite graph as a 1-dimensional cellular complex by setting a vertex as a 0-cell and an edge as a closed 1-cell. Given an embedding f of a finite graph into a surface, its image G is obviously identified with the original graph. Thus we say that the image of a vertex

2000 *Mathematics Subject Classification.* 53C22, 05C10, 57M15, 51M10.
Key words and phrases. geodesic graph, spherical geometry.
The author is partially supported by JSPS Research Fellowships for Young Scientists.

and an edge under f a *vertex* and an *edge* of G. By S^2, we mean the two dimensional sphere endowed with the Riemannian metric of constant curvature +1. Then our result is the following.

Theorem 1. *Let G be the image of an embedding of the complete graph K_4 of order 4 into S^2. Suppose that*

(1) *each edge of G is a shortest geodesic arc on S^2 connecting its endpoints, and*

(2) *G is not contained in any hemisphere of S^2.*

Let E be the sum of the length of the edges of G. Then $3\pi < E \leq 4\pi$ holds.

This almost follows from the result of Guddum [1]. In fact, his theorem in [1] implies the inequality $3\pi \leq E \leq 4\pi$. In the next section, we will prove the theorem above using purely elementary spherical geometry.

A generalization of this estimate to the case of any graph embedded in n-sphere S^n will appear in [4]. Our estimate depends upon the combinatorics of the graph only.

Here we append an easy observation for more general cases. Let F_g be a closed, orientable surface of genus $g \geq 2$ with a fixed Riemannian metric of constant curvature −1. For convenience, let F_0 denote S^2.

Proposition 1. *Let G be the image of an embedding of a graph into F_g where $g \neq 1$. Suppose that*

(1) *each edge e of G is a shortest geodesic arc on F_g connecting its endpoints, and*

(2) *the closure of each component of $F_g - G$ is a convex polygon on F_g.*

Then the sum of the length of the edges of G is greater than $\pi|2-2g|$.

Proof. Let σ be a complementary face of G, i.e., the closure of a component of $F_g - G$. By *Area*(σ) and *Length*($\partial\sigma$) , we denote the area of σ and the length of the boundary $\partial\sigma$ of σ respectively. We consider the ratio $Area(\sigma)/Length(\partial\sigma)$. This ratio is strictly less than the corresponding ratio for the disk on F_g which has equilong boundary as σ. See [2] for a survey. By elementary calculations, the ratio for such a disk is shown to be less than 1 for any $g \neq 1$. This implies $Length(\partial\sigma) > Area(\sigma)$ holds. By summing the inequalities up over all complementary faces, we have $\sum Length(\partial\sigma) > \sum Area(\sigma) = 2\pi|2-2g|$, where the last equality follows from the Gauss-Bonne's theorem. Then the sum of the length of the edges of G, which is equal to the half of $\sum Length(\partial\sigma)$, is greater than $\pi|2-2g|$. □

2. Proof

Let us start with recalling fundamentals of spherical geometry. Let u_1, u_2, u_3 be points on S^2 such that no two of them are antipodal and no great circle includes all the three points. Let Λ_i be the closed hemisphere whose boundary contains the other two points than u_i and whose interior contains u_i for $i = 1, 2, 3$. The *spherical triangle* Δ with the vertices u_1, u_2, u_3 is defined as the intersection $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3$. Then we have the following:

- Δ is convex, i.e., any pair of points in Δ is connected by a geodesic arc in Δ. Moreover the arc is shortest among the arcs connecting the points, and the length is equal to the spherical distance between the points which is strictly less than π.
- The length of an edge of Δ is less than the sum of the length of the other two edges (*the triangle inequality*).

Proof of Theorem 1. Let v_1, v_2, v_3, v_4 be the vertices of G. Let e_{ij} denote the edge of G connecting v_i and v_j for $1 \leq i, j \leq 4$. Note that the assumption (1) implies that the length of e_{ij} is equal to the spherical distance d_{ij} on S^2 between v_i and v_j for $1 \leq i, j \leq 4$. Thus it suffice to show that

$$3\pi < \sum_{1 \leq i < j \leq 4} d_{ij} \leq 4\pi .$$

In the following, the antipodal point of v_i is denoted by v_{i+4} for $1 \leq i \leq 4$. Also d_{ij} denotes the spherical distance between v_i and v_j for $1 \leq i, j \leq 8$.

First we consider the case that a couple of the vertices, say v_1 and v_2, are antipodal, equivalently, $d_{12} = \pi$. This implies that $d_{13} + d_{32} = d_{14} + d_{42} = \pi$ holds. Together with $0 < d_{34} \leq \pi$, we have $3\pi < \sum_{1 \leq i < j \leq 4} d_{ij} \leq 4\pi$.

Next consider the case that all the four vertices are contained in a great circle. Suppose for example that v_1, v_2, v_3, v_4 lies in a great circle Γ in this order. Since G is the image of an embedding, the edge e_{13} is not contained in Γ. This implies that e_{13} is a half of a great circle and $d_{13} = \pi$. Also we see that $d_{24} = \pi$ and so we obtain $\sum_{1 \le i < j \le 4} d_{ij} = 4\pi$.

Thus, in the following, we assume that $d_{ij} \neq \pi$ for $1 \le i, j \le 4$ and at most three vertices of G lie on a great circle.

Next consider the case that three vertices are contained in a great circle. Suppose for example that v_1, v_2 and v_3 lie on a great circle. Then, by the triangle inequality, we have $d_{41} + d_{42} > d_{12}$, $d_{42} + d_{43} > d_{23}$ and $d_{43} + d_{41} > d_{31}$. These are added to obtain

$$2(d_{41} + d_{42} + d_{43}) > d_{12} + d_{23} + d_{31} = 2\pi \ .$$

Thus

$$\sum_{1 \le i < j \le 4} d_{ij} = (d_{41} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} > \pi + 2\pi = 3\pi \ .$$

In the same way as above, we have $d_{45} + d_{46} + d_{47} > \pi$. Since $d_{4j} = \pi - d_{4(j+4)}$ for $j = 1, 2, 3$,

$$\begin{aligned} \sum_{1 \le i < j \le 4} d_{ij} &= (d_{41} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} \\ &= 3\pi - (d_{45} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} \\ &< 3\pi - \pi + 2\pi = 4\pi \end{aligned}$$

holds.

Finally we consider the case that the four vertices are in a general position: We assume that $d_{ij} \neq \pi$ for $1 \le i, j \le 4$ and at most two vertices of G lie on a great circle. This means that for any three of the points there is a triangular face which includes the three points as vertices.

Then, by the triangle inequality, we have $d_{53} + d_{63} > d_{56}$, $d_{54} + d_{64} > d_{56}$, $d_{53} + d_{54} > d_{34}$ and $d_{63} + d_{64} > d_{34}$. Add these to obtain

$$d_{53} + d_{63} + d_{54} + d_{64} > d_{34} + d_{56} \ .$$

Here note that $d_{ij} = \pi - d_{(i-4)j}$ for $i = 5, 6$, $j = 1, 2, 3$, and $d_{56} = d_{12}$. These imply that

$$4\pi - (d_{13} + d_{23} + d_{14} + d_{24}) > d_{34} + d_{12} \ .$$

Consequently we have

$$4\pi > \sum_{1\le i<j\le 4} d_{ij} \; .$$

In the following, let Δ be the spherical triangle bounded by e_{12}, e_{23} and e_{31}.

Claim 1. *The antipodal point v_8 of v_4 is included in the interior of Δ.*

Proof. Let Γ_i be the great circle including an edge of Δ but not including v_i for $i = 1,2,3$. By the assumption above, v_4 and hence v_8 never lie on $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Note that $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ decomposes S^2 into eight spherical triangles.

Assume for a contradiction that v_8 is not included in the interior of Δ. Then v_4 is included in the interior of one of the seven spherical triangles other than the antipodal image of Δ. This implies all the four points v_1, v_2, v_3 and v_4 are included in the closed hemisphere bounded by one of Γ_1, Γ_2 or Γ_3. Since the four vertices are assumed in a general position, there is a hemisphere which contains whole G. This contradicts the assumption (2) of the theorem. □

Claim 2. *The inequality $d_{12} + d_{13} > d_{82} + d_{83}$ holds.*

Proof. Since the length of each edge is less than π, the edge e_{13} intersects the great circle including v_2 and v_8 at just one point v_9. Let d_{i9} or d_{9i} denote the distance between v_i and v_9 for $1 \le i \le 9$. The distance d_{19} is realized by a geodesic arc included in e_{13} and also is d_{93}. Thus $d_{13} = d_{19} + d_{93}$ holds.

The distance d_{29} is realized by a geodesic arc e_{29} in Δ since Δ is convex. In particular, the arc e_{29} contains v_8 and so $d_{29} = d_{28} + d_{89}$ holds.

Together with the triangle inequality $d_{12} + d_{19} > d_{29}$ and $d_{98} + d_{93} > d_{83}$, we conclude

$$d_{12} + d_{13} = d_{12} + d_{19} + d_{93} > d_{29} + d_{93} = d_{28} + d_{89} + d_{93} > d_{28} + d_{83} \; .$$

□

In the same way, we have $d_{21} + d_{23} > d_{81} + d_{83}$ and $d_{31} + d_{32} > d_{81} + d_{82}$. By adding these inequalities, we obtain

$$d_{12} + d_{23} + d_{31} > d_{81} + d_{82} + d_{83} \; .$$

Together with the equations $d_{8j} = \pi - d_{4j}$ for $i = 1, 2, 3$, we conclude that

$$\sum_{1 \le i < j \le 3} d_{ij} > 3\pi - \sum_{1 \le k \le 3} d_{k4} \,.$$

This completes the proof. □

ACKNOWLEDGMENTS

The author would like to thank Kouki Taniyama for letting him know the paper [1].

REFERENCES

1. J. W. Gaddum. Distance sums on a sphere and angle sums in a simplex. Amer. Math. Monthly 63 (1956), 91–96.
2. H. Howards, M. Hutchings and F. Morgan. The isoperimetric problem on surfaces. Amer. Math. Monthly 106 (1999), no. 5, 430–439.
3. K. Ichihara. Area of a geodesic cellular complex in a hyperbolic 3-manifold. Preprint.
4. K. Ichihara and K. Taniyama. On the sum of external angles of a convex polyhedron. In preparation.

630–8506 奈良県奈良市北魚屋西町　奈良女子大学理学部情報科学科 (DEPARTMENT OF INFORMATION AND COMPUTER SCIENCES, FACULTY OF SCIENCE, NARA WOMEN'S UNIVERSITY, KITA-UOYA NISHIMACHI, NARA 630–8506).

E-mail address: `ichihara@vivaldi.ics.nara-wu.ac.jp`