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BOCHNER FLAT STRUCTURES ON COMPLEX KÄLHER MANIFOLDS

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ABSTRACT. We study the deformation of complete Bochner flat Kähler structures on complex (closed) aspherical Kähler manifolds. More precisely, we shall examine how many distinct complete Bochner flat Kähler metrics keeping the complex structure fixed on the complex hyperbolic Kähler space and the complex euclidean space

INTRODUCTION

When we consider the conformally flat structure on the (compact) real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}/\Gamma$, we know that there is a nontrivial deformation, for example, by Thurston bending ($n \geq 2$). This implies that there exists a non-equivalent family developing pair

$$(\rho, \mathrm{dev}) : (\Gamma, \mathbb{H}^n_{\mathbb{R}}) \longrightarrow (\mathrm{PO}(n+1,1), S^n)$$

starting at the standard developing map dev_0 which maps $\mathbb{H}^n_{\mathbb{R}}$ onto the upper-hemisphere S^n_+ of S^n. (Equivalently there is a nonconjugate family of holonomy representations $\rho : \Gamma \to \mathrm{PO}(n+1,1)$ other than the inclusion $\rho_0 : \Gamma \subset \mathrm{PO}(n,1) \subset \mathrm{PO}(n+1,1)$.) Similarly when we take a closed aspherical manifold $\mathcal{S}/\Gamma$ with virtually solvable fundamental group Γ like infrasolv-manifolds, it is known that if $\mathcal{S}/\Gamma$ admits a conformllay flat structure, $\mathcal{S}/\Gamma$ is necessarily conformal to the euclidean space form $\mathbb{R}^n/\Gamma$ and the developing pair

$$(\rho, \mathrm{dev}) : (\Gamma, \mathbb{R}^n) \longrightarrow (\mathrm{PO}(n+1,1), S^n)$$

is unique up to by an element of $\mathrm{PO}(n+1,1)$ to the standard developing map dev_0 which maps $\mathcal{S}$ onto the sphere with one point removed $S^n - \{\infty\}$. This is so called the topological rigidity of the developing pair. On the other hand, it is well known that the fundamental invariant on the conformal structure of the metrics on a smooth manifold is the Weyl curvature tensor whose vanishing implies the conformal flatness of the Riemannian manifold ($n \geq 4$). In 1949, Bochner introduced a curvature tensor on a Kähler manifold which is thought of as an analogue of the Weyl curvature

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tensor ([1]). When the curvature tensor (Bochner curvature tensor) vanishes, a Kähler structure (respectively, (complete) Kähler metric) is called Bochner flat structure (respectively, (complete) Bochner flatc metric) and a Kähler manifold with this structure is said to be a Bochner flat Kähler manifold. In this note we shall consider the corresponding problem to (complete) Bochner flat structure on a Kähler manifold. As a Kähler manifold we take a complex hyperbolic manifold $\mathbb{H}_{\mathbb{C}}^n/\Gamma$ and a closed aspherical complex Kähler manifold $\mathcal{S}/\Gamma$ with virtually solvable group Γ (for example, a complex euclidesan space form $\mathbb{C}^n/\Gamma$). It is noted that a Kähler manifold with constant holomorphic sectional curvature is a Bochner flat Kähler manifold as well as a fact that a Riemannian manifold of constant sectional curvature is a conformally flat manifold.

Theorem I. *A complete Bochner flat structure on the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$ is unique up to a constant multiple of a hyperbolic Kähler metric. The deformation space $\mathbf{R}(\mathbb{R}, \mathcal{Z}\mathrm{U}(n,1))$ consists of a single representation $\{\rho\}$;*

$$\rho(\theta) = (e^{i\theta}, \cdots, e^{i\theta}).$$

where $\mathcal{Z}\mathrm{U}(n,1)$ is the center S^1 of $\mathrm{U}(n,1)$.

Theorem II.

(1) *If a closed aspherical complex Kähler manifold $\mathcal{S}/\Gamma$ with virtually solvable group Γ admits a Bochner flat structure, then it is holomorphically isometric (up to a constant multiple of the metric) to the complex euclidean space form $\mathbb{C}^n/\Gamma$ with standard euclidean metric.*

(2) *The deformation space $\mathbf{R}(\mathbb{R}, \mathcal{R}\times T^n)$ of all distinct complete Bochner flat structures on the complex euclidean space $\mathbb{C}^n$ modulo the homothety is a convex space $\{(a_1, \cdots, a_n) \in \mathbb{R}^n \mid 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n\}$. For $\rho \in \mathbf{R}(\mathbb{R}, \mathcal{R}\times T^n)$,*

$$\rho(t) = (t, e^{ia_1 t}, \cdots, e^{ia_n t}).$$

This result (2) has been obtained first by R. Bryant [2]. Contrary to that the sphere S^n is the model space in conformal geometry, it is emphasized that the model (complete) Kähler space into which the developing map maps is not unique in Bochner flat geometry.

2. Preliminaries

Let (M, J, g) be a simply connected Kähler manifold of real dimension $2n$ with exact Kähler form Ω. (For example, M is contractible.) There is a 1-form θ such that $d\theta = \Omega$. Consider the product $\mathbb{R}\times M$ for which $p : \mathbb{R}\times M \rightarrow M$ is the projection. We construct the contact form ω and the complex structure J on the contact subbundle $\mathrm{Null}\,\omega = \{V \in T(\mathbb{R}\times M) \mid \omega(V) = 0\}$.

Let t be the coordinate of $\mathbb{R}$. Put

$$\begin{aligned}(2.1)\qquad &\omega = dt + p^*\theta,\\ &\tilde{J}(V) = p_*^{-1}\circ J\circ p_*(V)\quad (\forall\, V\in(\mathrm{Null}\,\omega)_{(t,x)}).\end{aligned}$$

It is easy to see that ω is a contact form of $\mathbb{R}\times M$ on which $\mathbb{R}=\{T_s,\ s\in\mathbb{R}\}$ acts as contact transformations:

$$T_s(t,x) = (t+s,x).$$

Let $\mathrm{Null}\,\omega\otimes\mathbb{C} = T^{1,0}\oplus T^{0,1}$ be the canonical splitting of eigenvalues of $\tilde{J}$. As $d\omega$ is $\tilde{J}$-invariant; $d\omega(\tilde{J}X,\tilde{J}Y) = \Omega(p_*\tilde{J}X,p_*\tilde{J}Y) = \Omega(Jp_*X,Jp_*Y) = \Omega(p_*X,p_*Y) = d\omega(X,Y)$, it implies that $[T^{1,0},T^{1,0}]\subset T^{1,0}$, i.e. $\tilde{J}$ is integrable. By definition, $\tilde{J}$ is a complex structure on $\mathrm{Null}\,\omega$. In addition, $d\omega(\tilde{J}\cdot,\cdot) = g(p_*\cdot,p_*\cdot)$ is a positive definite bilinear form on $\mathrm{Null}\,\omega$.

Definition 1. *The pair* $(\mathrm{Null}\ \omega,\tilde{J})$ *is a strictly pseudoconvex CR-structure on* $\mathbb{R}\times M$.

Proposition 2. (i) *The action $\mathbb{R}$ commutes with the complex structure $\tilde{J}$, i.e. the group $\mathbb{R}$ acts as CR-transformations of $(\omega,\tilde{J})$.*

(ii) *The vector field $\dfrac{d}{dt}$ induced by the $\mathbb{R}$-action is the characteristic vector field (Reeb field) for ω, i.e. $\omega(\dfrac{d}{dt}) = 1$, $d\omega(\dfrac{d}{dt},V) = 0$ $(\forall\, V\in T(\mathbb{R}\times M))$.*

(iii) $d\omega = p^*\Omega$.

Making use of the structure equations modelled on the real hypersurface in $\mathbb{C}^{n+1}$, Chern and Moser have found a CR-invariant tensor which is the fourth-order curvature tensor $S=(S_{\alpha\rho\bar\beta\bar\sigma})$ on a CR-manifold N^{2n+1}. When we persist in the Weyl's conformal geometry to the CR-manifolds, the CR-invariant tensor is a conformal invariant in the following sense: if two contact forms ω,ω' represnt the same CR structure (keeping the complex structure J fixed on the CR-bundle), then $\omega' = u\cdot\omega$ for some positive function u for which the Chern-Moser curvature tensor coincides $S(\omega,J) = S(\omega',J)$. The sphere S^{2n+1} is a CR-manifold viewed as a hyperquadric in $\mathbb{C}^{n+1}$, whose curvature tensor S vanishes identically. The standard contact form ω_0 is obtained from the connection form of the Hopf bundle : $S^1\to S^{2n+1}\longrightarrow\mathbb{CP}^n$. The complex analogue of conformal geometry states that if the Chern-Moser curvature tensor S of a CR-manifold N vanishes, then N is locally CR-equivalent to S^{2n+1} $(n>1)$. In this case, N is said to be a *spherical CR-manifold.*

Note that the formula of S is given by

$$\begin{aligned}(2.2)\qquad S_{\alpha\rho\bar\beta\bar\sigma} &= R_{\alpha\bar\beta\rho\bar\sigma} - \frac{1}{n+2}(R_{\alpha\bar\beta}g_{\rho\bar\sigma} + R_{\rho\bar\beta}g_{\alpha\bar\sigma} + g_{\alpha\bar\beta}R_{\rho\bar\sigma} + g_{\rho\bar\beta}R_{\alpha\bar\sigma})\\ &\quad + \frac{R}{2(n+1)(n+2)}(g_{\alpha\bar\beta}g_{\rho\bar\sigma} + g_{\rho\bar\beta}g_{\alpha\bar\sigma}).\end{aligned}$$

Here $R_{\alpha\bar{\beta}\rho\bar{\sigma}}$ is the Tanaka - Webster curvature tensor. On the other hand, the Bochner curvature tensor B on a Kähler manifold (M, g, J) has the same formula as S. In fact, we have the following coincidence observed by Webster.

Proposition 3. *Let* $\mathbb{R} \to \mathbb{R} \times M \xrightarrow{p} M$ *be the contactization of a Kähler manifold* (M, Ω, J). *When* $(\omega, \tilde{J})$ *is the pseudo-hermitian pair on* $\mathbb{R} \times M$ *such that* $d\omega = p^*\Omega$ *and* $p_*\tilde{J} = Jp_*$, *the Chern-Moser curvature tensor* S *of the* CR*-manifold* $\mathbb{R} \times M$ *coincides with the Bochner curvature tensor* B *of* M:

$$S(\omega, \tilde{J}) = p^*B(\Omega, J).$$

Suppose that (M, g, J) is a Bochner flat Kähler manifold, i.e. $B(\Omega, J) = 0$. Then the associated CR-manifold $(M, \{\mathrm{Null}\,\omega, \tilde{J}\})$ is spherical, i.e. M is uniformizable over S^{2n+1} with respect to the CR-transformation group $\mathrm{Aut}_{CR}(S^{2n+1}) = \mathrm{PU}(n+1, 1)$. Here $\mathrm{PU}(n+1, 1)$ is the unitary Lorentz group. It is also the isometry group of complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^{n+1}$. Denote by (ω_0, J_0) the pseudo-Hermitian structure on the sphere S^{2n+1} which represents the standard CR-structure. Then by the monodromy argument, the universal covering $\mathbb{R} \times M$ (because M is simply connected) can be developed into the sphere;

$$(\rho, \mathrm{dev}) : (\mathbb{R}, \mathbb{R} \times M) \longrightarrow (\mathrm{PU}(n+1, 1), S^{2n+1}), \tag{2.3}$$

where ρ is the holonomy homomorphism of $\mathbb{R}$ into $\mathrm{PU}(n+1, 1)$. By definition, the developing map dev is a CR-immersion satisfying that

$$\begin{aligned} &\mathrm{dev}^*\omega_0 = u \cdot \omega \text{ for some positive function } u \text{ on } \mathbb{R} \times M. \\ &\mathrm{dev}_* \circ \tilde{J} = J_0 \circ \mathrm{dev}_* \text{ on } \mathrm{Null}\ \omega. \end{aligned} \tag{2.4}$$

The closure G of the holonomy group $\rho(\mathbb{R})$ in $\mathrm{PU}(n+1, 1)$ is a connected abelian Lie subgroup acting on S^{2n+1} (acting also on the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^{n+1}$). The standard hyperbolic group theory shows that if G is noncompact, then it has the fixed point subset which is either one point $\{\infty\}$ or exactly two points $\{0, \infty\}$ in S^{2n+1} unique up to conjugate by an element of $\mathrm{PU}(n+1, 1)$. If G is compact, the fixed point subset of S^{2n+1} is either $\{\emptyset\}$ or the subsphere S^{2m-1} $(m = 1, \cdots, n)$ unique up to conjugacy. In the former case, G has the unique fixed point inside the hyperbolic space $\mathbb{H}_{\mathbb{C}}^{n+1}$. According to whether G is noncompact or compact, G belongs to either the similarity group $\mathrm{Aut}(\mathcal{N}) = \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$ or the maximal torus T^{n+1} of $\mathrm{PU}(n+1, 1)$ up to conjugation. Here $\mathcal{N}$ is the Heisenberg nilpotent Lie group identified with $S^{2n+1} - \{0\}$. (See §5.)

Since $\mathbb{R}$ acts freely on $\mathbb{R} \times M$ and dev is a ρ-equivariant immersion, $\rho(\mathbb{R})$ has no fixed point on the image $\mathrm{dev}(\mathbb{R} \times M)$, it follows that (1) $\mathrm{dev}(\mathbb{R} \times M) \subset \mathcal{N} = S^{2n+1} - \{\infty\}$, (2) $\mathrm{dev}(\mathbb{R} \times M) \subset S^{2n+1} - \{0, \infty\}$, (3) $\mathrm{dev}(\mathbb{R} \times M) \subset S^{2n+1} - S^{2m-1}$ $(m = 0, 1, \cdots, n)$. If we denote X one of the domain of

S^{2n+1} as in (1) – (3), then our equivariant CR-immersion reduces :

$$(\rho, \mathrm{dev}) : (\mathbb{R}, \mathbb{R} \times M) \longrightarrow (\mathrm{Aut}_{CR}(X), X). \quad (2.5)$$
$$\mathrm{dev}^* \omega_X = u \cdot \omega \quad (u > 0).$$

Here ω_X is a contact form which represnts the restricted CR-structure on X. Let ξ be the vector field induced by the 1-parameter subgroup $\rho(\mathbb{R})$ on X. As the developing map is equivariant $\mathrm{dev}(T_t(s,x)) = \rho(t)\mathrm{dev}((s,x))$, it follows that $\xi = \mathrm{dev}(\frac{d}{dt})$. Since $\omega(\frac{d}{dt}) = 1$ and (2.5) with $u > 0$, we obtain a resriction $\omega_X(\xi) > 0$ on the developing image $\mathrm{dev}(\mathbb{R} \times M)$. Let $S = \{p \in X \mid \omega_X(\xi_p) = 0\}$ be the singular subset of X. If $\mathcal{W}$ is the connected component $(X - S)^0$ of $X - S$ containing $\mathrm{dev}(\mathbb{R} \times M)$, then (2.5) reduces to the following:

$$(\rho, \mathrm{dev}) : (\mathbb{R}, \mathbb{R} \times M) \longrightarrow (\mathrm{Aut}_{CR}(\mathcal{W}), \mathcal{W}). \quad (2.6)$$

When G is compact, remark that there is a further restriction that $\mathrm{dev}(\mathbb{R} \times M) \subset \mathcal{W} - E$ where E the set of exceptional orbits of G.

Looking at the connected subgroups of $\mathrm{PU}(n+1, 1)$ for (1) – (3), it follows that

Proposition 4. *One of the following cases occur (up to conjugacy) :*

1. *If G is noncompact and fixes $\{\infty\}$ in S^{2n+1}, then $\rho(\mathbb{R})$ is a closed subgroup of the pseudo-hermitian transformation group $\mathrm{Psh}(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(n)$.*
2. *If G is noncompact and fixes $\{0, \infty\}$, then $\rho(\mathbb{R})$ is a closed subgroup lying in $\mathrm{U}(n) \times \mathbb{R}^+$.*
3. *If G is compact, then the fixed point set of G is the subsphere S^{2m-1} $(m = 0, 1, \cdots, n)$. Moreover,*

$$\begin{aligned} G \subset T^{n-m+1} &= P(\mathcal{Z}U(m,1) \times T^{n-m+1}) \\ &\subset P(\mathrm{U}(m,1) \times \mathrm{U}(n-m+1)) = \mathrm{Aut}(S^{2n+1} - S^{2m-1}). \end{aligned}$$

Here $\mathcal{Z}\mathrm{U}(m,1)$ is the center S^1 of $\mathrm{U}(m,1)$.

Corollary 5. *$\rho(\mathbb{R})$ is closed except for the case that G has the fixed point set S^{2m-1} $(m = 0, 1, \cdots, n-1)$. In particular, if $\rho(\mathbb{R})$ is closed, i.e. S^1 or $\mathbb{R}$, then $\rho(\mathbb{R})$ acts properly on $\mathcal{W}$.*

3. Existence of Bochner flat Kähler metric

Suppose that the holonomy group $\rho(\mathbb{R})$ is closed. By Corollary 5, we have an orbifold $\mathcal{W}/\rho(\mathbb{R})$. (If $\rho(\mathbb{R}) \approx \mathbb{R}$, $\mathcal{W}/\rho(\mathbb{R})$ is a smooth manifold.) Let $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))$ be the normalizer of $\rho(\mathbb{R})$ in $\mathrm{Aut}_{CR}(\mathcal{W})$.

Definition 6. *The quotient group is defined as*

$$\mathcal{H} = \mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))/\rho(\mathbb{R}).$$

Then the group $\mathcal{H}$ acts on $\mathcal{W}/\rho(\mathbb{R})$. Thus we get a geometry $(\mathcal{H}, \mathcal{W}/\rho(\mathbb{R}))$. (Note that $\mathcal{H}$ does not necessarily act transitively on $\mathcal{W}/\rho(\mathbb{R})$. This phenomenon occurs in Bochner Kähler geometry.) There exists an equivariant principal bundle:

$$\rho(\mathbb{R}) \rightarrow (\mathrm{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R})), \mathcal{W}) \xrightarrow{\nu} (\mathcal{H}, \mathcal{W}/\rho(\mathbb{R})). \tag{3.1}$$

As we know that $\omega_X(\xi) > 0$ on $\mathcal{W}$ (cf. (2.6)), define a 1-form η on $\mathcal{W}$ to be:

$$\eta(Z) = \frac{1}{\omega_X(\xi)} \cdot \omega_X(Z) \quad (\forall\, Z \in T\mathcal{W}). \tag{3.2}$$

As $\eta(\xi) = 1$ on $\mathcal{W}$, $d\iota_\xi \eta = 0$. Since Null η = Null ω_X, η is a contact form on $\mathcal{W}$.

Lemma 7. *ξ is a characteristic vector field for η on $\mathcal{W}$.*

Proof. Since ξ generates $\rho(\mathbb{R})$, $\rho(t)_*\xi = \xi$ and $\rho(t)^*\omega_X = u_t \cdot \omega_X$ for some $u_t > 0$. (In fact we can show that $\rho(t)^*\omega_X = \omega_X$ for $\mathcal{N}, S^{2n+1} - S^{2m-1}$, $\rho(t)^*\omega_X = e^{2t}\omega_X$ for $X = \mathcal{N} - \{0\} = S^{2n} \times \mathbb{R}^+$.) Hence,

$$\begin{aligned}(\rho(t)^*\eta)_x(Z_x) &= \frac{\omega_X(\rho(t)_*Z_x)}{\omega_X(\xi_{\rho(t)x})} = \frac{\omega_X(\rho(t)_*Z_x)}{\omega_X(\rho(t)_*\xi_x)} \\ &= \frac{\omega_X(Z_x)}{\omega_X(\xi_x)} = \eta_x(Z_x).\end{aligned} \tag{3.3}$$

Hence $0 = \mathcal{L}_\xi \eta = \iota_\xi d\eta + d\iota_\xi \eta = \iota_\xi d\eta$. □

Proposition 8. *There exists a Bochner flat Kähler metric $(\hat{g}, \hat{J})$ on $\mathcal{W}/\rho(\mathbb{R})$. The group $\mathcal{H}$ acts as holomorphic homothetic (not necessarily isometric) transformations.*

Proof. As ν_* : Null $\eta \rightarrow T(\mathcal{W}/\rho(\mathbb{R}))$ is isomorphic at each point of $\mathcal{W}$, the complex structure $\hat{J}$ is defined on $\mathcal{W}/\rho(\mathbb{R})$ by making the diagram below commutative:

$$\begin{array}{ccc} \text{Null } \eta & \xrightarrow{\nu_*} & T(\mathcal{W}/\rho(\mathbb{R})) \\ \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle \hat{J}} \\ \text{Null } \eta & \xrightarrow{\nu_*} & T(\mathcal{W}/\rho(\mathbb{R})). \end{array} \tag{3.4}$$

(If we note that η is also $\tilde{J}$-invariant, then it follows that $\nu_*([X,Y]) = [\nu_*X, \nu_*Y]$ for $X, Y \in (\text{Null } \eta \otimes \mathbb{C})^{1,0}$. As $\tilde{J}$ is integrable on Null η, $\hat{J}$ is a complex structure on $\mathcal{W}/\rho(\mathbb{R})$.) Since $d\eta$ is positive definite (strictly pseudo-convex) and $\tilde{J}$-invariant (i.e. $d\eta(\tilde{J}\cdot, \tilde{J}\cdot) = d\eta(\cdot,\cdot)$ on Null η), we may define a Hermitian metric on $(\mathcal{W}/\rho(\mathbb{R}), \hat{J})$ by setting

$$\hat{g}(\hat{X}, \hat{Y}) = d\eta(\tilde{J}X, Y), \tag{3.5}$$

where $X, Y \in$ Null η such that $\nu_*(X) = \hat{X}$, $\nu_*(Y) = \hat{Y}$. Let $\hat{\Omega}(\hat{X}, \hat{Y}) = \hat{g}(\hat{X}, \hat{J}\hat{Y})$ be the fundamental two form on $\mathcal{W}/\rho(\mathbb{R})$. Recall that $T\mathcal{W} =$

$\{\xi\} \oplus \mathrm{Null}\,\eta$.

$$\begin{aligned}\nu^*\hat{\Omega}(X,Y) &= \hat{\Omega}(\hat{X},\hat{Y}) = \hat{g}(\hat{X},\hat{J}\hat{Y}) \\ &= d\eta(\tilde{J}X,\tilde{J}Y) = d\eta(X,Y) \ (X,Y \in \mathrm{Null}\,\eta).\end{aligned} \tag{3.6}$$

As ξ is characteristic for η by Lemma 7, we have that $d\eta(\xi, X) = 0 = \nu^*\hat{\Omega}(\xi, X)$. Therefore,

$$\nu^*\hat{\Omega} = d\eta \quad \text{on } \mathcal{W}. \tag{3.7}$$

Hence $d\hat{\Omega} = 0$ on $\mathcal{W}/\rho(\mathbb{R})$ so that $\hat{\Omega}$ is a Kähler form on $\mathcal{W}/\rho(\mathbb{R})$. Thus we obtain a Kähler structure $(\hat{g}, \hat{\Omega}, \hat{J})$ on $\mathcal{W}/\rho(\mathbb{R})$. In particular, as $(\eta, \tilde{J})$ represents the spherical CR-structure $(\mathrm{Null}\,\omega, \tilde{J})$ on $\mathcal{W}$, $(\hat{g}, \hat{J})$ is a Bochner flat structure on $\mathcal{W}/\rho(\mathbb{R})$. We examine how the group $\mathcal{H}$ acts on $\mathcal{W}/\rho(\mathbb{R})$. If $h \in \mathrm{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))$, then the projection ν induces an element $\hat{h} \in \mathcal{H}$ such that $\nu(hx) = \hat{h}\nu(x)$. By the definition,

$$(h^*\eta)(Z_x) = h^*\Big(\frac{1}{\omega_X(\xi_x)} \cdot \omega_X(Z_x)\Big) = \frac{1}{\omega_X(\xi_{hx})} \cdot (h^*\omega_X)(Z_x).$$

Let $h^*\omega_X = u \cdot \omega_X$ for some positive function u on $\mathcal{W}$ as before. As Null $\eta =$ Null ω_X, h preserves Null η. On the other hand, there are the following possibilities: **(1)** h satisfies $h \cdot \rho(t) \cdot h^{-1} = \rho(t)$, i.e. $h_*\xi = \xi$; otherwise there exists a constant c such that **(2)** $h \cdot \rho(t) \cdot h^{-1} = \rho(c \cdot t)$, $h_*\xi = c \cdot \xi$. According to (1) or (2), we obtain that $h^*\eta = \eta$, $h^*\eta = c \cdot \eta$. Noting that c is constant and $h_* \circ \tilde{J} = \tilde{J} \circ h_*$ on Null ω_X, by (3.4),

$$\begin{aligned}\hat{g}(\hat{h}_*\hat{X}, \hat{h}_*\hat{Y}) &= d\eta(\tilde{J}h_*X, h_*Y) = dh^*\eta(\tilde{J}X, Y) \\ &= c^j \cdot \hat{g}(\hat{X},\hat{Y}) \quad (j = 0, 1). \\ \hat{h}_* \circ \hat{J} &= \hat{J} \circ \hat{h}_* \quad \text{on Null } \eta.\end{aligned} \tag{3.8}$$

Therefore the group $\mathcal{H} = \mathrm{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))/\rho(\mathbb{R})$ acts as Kähler isometries $(j = 0)$ or homotheties $(j = 1)$ of $\mathcal{W}/\rho(\mathbb{R})$ with respect to $(\hat{g}, \hat{\Omega}, \hat{J})$.

□

Notice that the developing map dev induces an immersion Dev with the commutative diagram:

$$\begin{array}{ccc} \mathbb{R} \times M & \xrightarrow{\mathrm{dev}} & \mathcal{W} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\mathrm{Dev}} & \mathcal{W}/\rho(\mathbb{R}). \end{array} \tag{3.9}$$

Theorem 9 (Geometric uniformization). *Let (M, J, g) be a (real) $2n(\geq 4)$-dimensional simply connected Bochner flat Kähler manifold with exact*

Kähler form. Suppose that the holonomy group $\rho(\mathbb{R})$ is closed. Then there exists a Kähler immersion $\mathrm{Dev} : M \longrightarrow \mathcal{W}/\rho(\mathbb{R})$, *i.e.*

$$\begin{aligned} \mathrm{Dev}^*\hat{\Omega} &= \Omega \quad (\mathrm{Dev}^*\hat{g} = g). \\ \mathrm{Dev}_* \circ J &= \hat{J} \circ \mathrm{Dev}_*. \end{aligned} \tag{3.10}$$

Proof. Since $\rho(t)\,\mathrm{dev}(x) = \mathrm{dev}(tx)$, note that $\xi = \mathrm{dev}(\frac{d}{dt})$. As $\mathrm{dev}^*\,\omega_X = u \cdot \omega$ for some u on $\mathbb{R} \times M$, we obtain that

$$\begin{aligned} u(x) &= u(x) \cdot \omega(\frac{d}{dt}) \\ &= \omega_X(\mathrm{dev}_*(\frac{d}{dt})) = \omega_X(\xi). \end{aligned} \tag{3.11}$$

Then,

$$\begin{aligned} \mathrm{dev}^*\eta &= \mathrm{dev}^*(\frac{1}{\omega_X(\xi)} \cdot \omega_X) \\ &= \frac{1}{\omega_X(\xi)}\mathrm{dev}^*\omega_X = \frac{u}{\omega_X(\xi)} \cdot \omega = \omega \end{aligned} \tag{3.12}$$

Then

$$\begin{aligned} p^*\mathrm{Dev}^*\hat{\Omega} &= \mathrm{dev}^*\nu^*\hat{\Omega} = \mathrm{dev}^* d\eta \\ &= d\mathrm{dev}^*\eta = d\omega = p^*\Omega. \end{aligned} \tag{3.13}$$

Thus, $\mathrm{Dev}^*\hat{\Omega} = \Omega$. Also,

$$\begin{aligned} \mathrm{Dev}_* J p_* &= \mathrm{Dev}_* p_* \tilde{J} = \nu_* \mathrm{dev}_* \tilde{J} \\ &= \nu_* J \mathrm{dev}_* = \hat{J}\nu_* \mathrm{dev}_* = \hat{J}\mathrm{Dev}_* p_*. \end{aligned} \tag{3.14}$$

Thus, $\mathrm{Dev}_* J = \hat{J}\mathrm{Dev}_*$.

□

Remark 10. *In gnereral, when $\rho(\mathbb{R})$ is not closed, we choose a local one-parameter subgroup Δ from $\rho(\mathbb{R})$ for which Δ acts properly on a maximal domain $\mathcal{W}$. Then argue as above. However, the domain $\mathcal{W}$ is quite vague.*

4. Outline of proof

When G is compact, it belongs to the $(n-m+1)$-dimensional torus $T^{n-m+1} \subset P(\mathcal{Z}\mathrm{U}(m,1) \times \mathrm{U}(n-m+1))$ up to conjugacy where $m = 0, 1, \cdots, n$. (Here $\mathcal{Z}\mathrm{U}(0,1) = \mathrm{U}(0,1) = S^1$.) The element of $\rho(\mathbb{R})$ has the form

$$\rho(t) = 1 \times \begin{pmatrix} e^{it\cdot a_1} & & \\ & \ddots & \\ & & e^{it\cdot a_{n-m+1}} \end{pmatrix}$$

for some $a_1, \cdots, a_{n-m+1} \in \mathbb{R}^*$. When $m = n$, $\rho(\mathbb{R})$ is necessarily closed so that

$$\rho(\mathbb{R}) = G = P(\mathcal{Z}\mathrm{U}(n,1) \times \mathrm{U}(1)) = \mathcal{Z}\mathrm{U}(n,1) = S^1.$$
$$\mathbf{N}_{\mathrm{Aut}_{CR}(S^{2n+1}-S^{2n-1})}(\rho(\mathbb{R})) = \mathbf{Z}_{\mathrm{Aut}_{CR}(S^{2n+1}-S^{2n-1})}(\rho(\mathbb{R})) = \mathrm{U}(n,1).$$

Recall that V^{2n+1}_{-1} is the $(2n+1)$-dimensional Lorentz standard space form of constant sectional curvature -1 with transitive unitary Lorentz group $\mathrm{U}(n,1)$. $S^{2n+1} - S^{2n-1}$ is identifed with V^{2n+1}_{-1} as a CR-structure. The center $\mathcal{Z}\mathrm{U}(n,1)$ of $\mathrm{U}(n,1)$ is S^1. Then V^{2n+1}_{-1} is the total space of the principal S^1-bundle over the complex hyperbolic space:

$$\mathcal{Z}\mathrm{U}(n,1) \to V^{2n+1}_{-1} \xrightarrow{P} \mathbb{H}^n_{\mathbb{C}}. \tag{4.1}$$

Denote by $\omega_{\mathbb{H}}$ the connection form of the above principal bundle. Then it is a contact form on V^{2n+1}_{-1}. In particular, $S^1 = \mathcal{Z}\mathrm{U}(n,1)$ induces a characteristic vector field ξ such that $\omega_{\mathbb{H}}(\xi) = 1$. Let $\Omega_{\mathbb{H}}$ be the Kähler form on $\mathbb{H}^n_{\mathbb{C}}$ such that $P^*\Omega_{\mathbb{H}} = d\omega_{\mathbb{H}}$. Let $g_{\mathbb{H}}$ be the Kähler hyperbolic metric of $\mathbb{H}^n_{\mathbb{C}}$. We have that

$$(\mathcal{H}, S^{2n+1} - S^{2n-1}/S^1, \hat{g}, \hat{J}) = (\mathrm{PU}(n,1), \mathbb{H}^n_{\mathbb{C}}, g_{\mathbb{H}}, J_{\mathbb{H}}).$$

We have proved the following.

Proposition 11. *Let (M, J, g) be a simply connected Bochner flat Kähler manifold with exact Kähler form* ($\dim M = 2n \geq 4$). *Suppose that G is compact.*

(i) *If $m = n$, then $\rho(\mathbb{R}) = S^1$, i.e. closed. If g is complete, then the developing map* dev *is an isometry of M onto $\mathbb{H}^n_{\mathbb{C}}$.*
(ii) *Suppose that $\rho(\mathbb{R})$ is closed, i.e. $(= S^1)$ for $m = 0, 1, \cdots, n-1$. If g is complete, then the developing map is an isometry onto $\mathbb{H}^m_{\mathbb{C}} \times \mathbb{CP}^{n-m}$.*
(iii) *Suppose that $\rho(\mathbb{R})$ is not closed. Then g cannot be complete.*

Proposition 12. *Let (M, J, g) be a simply connected Bochner flat Kähler manifold with exact Kähler form* ($\dim M = 2n \geq 4$). *Suppose that G is noncompact. $(G = \rho(\mathbb{R})$.)*

(1) *If the developing map* dev *maps $\mathbb{R} \times M$ into Heisenberg space $\mathcal{N}$ and g is complete, then the developing map* Dev *is an isometry of M onto $\mathcal{N}/\rho(\mathbb{R})$. Moreover, $\mathcal{N}/\rho(\mathbb{R})$ is holomorphic to the complex eulidean space $\mathbb{C}^n$. Especially, $\mathcal{N}/\rho(\mathbb{R})$ is a complete Bochner flat manifold.*
(2) *If the developing map* dev *maps $\mathbb{R} \times M$ into $\mathcal{N} - \{0\} = S^{2n} \times \mathbb{R}^+$, then g cannot be complete.*

Proposition 13. *Let (M, J, g) be as in* (1) *of Proposition 12 and g is complete. Then the representation $\rho : \mathbb{R} \to \mathcal{N} \rtimes \mathrm{U}(n)$ reduces to a representation $\rho : \mathbb{R} \to \mathcal{R} \times T^n$ which has the form; $\rho(t) = ((t, 0), A_t)$ where*

$$A_t = \begin{pmatrix} e^{it\cdot a_1} & & \\ & \ddots & \\ & & e^{it\cdot a_n} \end{pmatrix}.$$

Here a_i's are real numbers such that

$$0 \leq a_1 \leq a_2 \cdots \leq a_n.$$

In fact, a calculation shows $\omega_{\mathcal{N}}(\xi) = 1 + (a_1|z_1|^2 + a_2|z_2|^2 + \cdots + a_n|z_n|^2)$ (cf. §5). So $\omega_{\mathcal{N}}(\xi) > 0$ (i.e. $\mathcal{W} = \mathcal{N}$) if and only if $a_1 \geq 0$. Letting $a = (a_1, \cdots, a_n)$, we denote by g_a the Kähler metric $\hat{g}$ on $\mathcal{N}/\rho(\mathbb{R})$. The complex structure $\hat{J}$ in this case coincides with the standard complex structure $J_{\mathbb{C}}$. (See §5.) We obtain that

$$\begin{aligned} \mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))/\rho(\mathbb{R}) &= (\mathbb{C}^{n-k} \rtimes \mathrm{U}(n-k)) \times \mathrm{U}(\ell_1) \times \cdots \times \mathrm{U}(\ell_m), \\ \mathcal{N}/\rho(\mathbb{R}) &= \mathbb{C}^n \quad (\ell_1 + \cdots + \ell_m = k), \\ (\hat{g}, \hat{J}) &= (g_a, J_{\mathbb{C}}). \end{aligned}$$

As a consequence of Proposition 12, g_a is a complete Bochner flat Kähler metric on $\mathbb{C}^n$ and $\mathcal{H} = (\mathbb{C}^{n-k} \rtimes \mathrm{U}(n-k)) \times \mathrm{U}(\ell_1) \times \cdots \times \mathrm{U}(\ell_m)$ is the full group of isometris of g_a. If all a_i are positive and distinct, then $\mathcal{H} = \mathrm{Iso}(\mathbb{C}^n, g_a) = \mathrm{U}(1) \times \cdots \times \mathrm{U}(1) = T^n$.

Theorem 14. *Let M be a simply connected Bochner flat Kähler manifold with exact form* ($\mathrm{dim}M = 2n \geq 4$). *If the Kähler metric is complete, then the develoing map* Dev *is an isometry of (M, g, J) onto $(\mathbb{H}^m_{\mathbb{C}} \times \mathbb{CP}^{n-m}, g_{\mathbb{H}} \times g_{\mathbb{CP}}, J)$ $(m = 0, 1, \cdots, n)$ or $(\mathcal{N}/\rho(\mathbb{R}), g_a, J)$. Here $(\mathcal{N}/\rho(\mathbb{R}), J)$ is the complex euclidean space $(\mathbb{C}^n, J_{\mathbb{C}})$.*

Let M be a complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ $(n \geq 2)$. Given a complete Kähler metric which is Bochner flat on M, Dev is a holomorphic diffeomorphism of M onto the complex space $\mathbb{H}^m_{\mathbb{C}} \times \mathbb{CP}^{n-m}$, or $\mathbb{C}^n$. Hence, the only possible case is that Dev : $M \longrightarrow \mathbb{H}^n_{\mathbb{C}}$. See Remark 16. By 3 $(m = n)$ of Proposition 4, the complete Bochner flat Kähler structure on the hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ determines uniquely the representation $\rho : \mathbb{R} \rightarrow S^1 = P(\mathcal{Z}U(n,1) \times U^1) = \mathcal{Z}\mathrm{U}(n,1) \subset \mathrm{PU}(n+1,1)$ up to normalization:

$$\rho(t) = (e^{it}, \cdots, e^{it}).$$

Remark 15. *As $\mathbb{H}^n_{\mathbb{C}}$ is viewed as a bounded domain (unit ball) of $\mathbb{C}^n$, the standard Bochner flat euclidean metric restricts to a Bochner flat Kähler metric on $\mathbb{H}^n_{\mathbb{C}}$, but it is not complete.*

Similarly, given a complex euclidean space $\mathbb{C}^n$ $(n \geq 2)$ which supports a complete Bochner flat metric, Dev is a holomorphic diffeomorphism of M onto $\mathbb{C}^n = \mathcal{N}/\rho(\mathbb{R})$. Hence, by Proposition 13, each developing map dev determines the representation: $\rho : \mathbb{R} \rightarrow \mathcal{R} \times T^n$ defined by

$$\rho(t) = ((t, 0), (e^{it\cdot a_1}, \cdots, e^{it\cdot a_n})).$$

Hence, all the distinct isomorphism classes of complete Bochner flat Kähler metrics on $\mathbb{C}^n$, $\mathbf{R}(\mathbb{R}, \mathcal{R} \times T^n)$ up to homothety is in one-to-one correspondence with the convex set $\{(a_1, \cdots, a_n) \in \mathbb{R}^n \mid 0 \le a_1 \le \cdots \le a_n\}$.

Remark 16 (Transformations of complex manifold). *Let $\mathfrak{hol}(M)$ be the group of holomorphic transformations of a complex manifold. It is well known that $\mathfrak{hol}(\mathbb{C}^n)$ is not a Lie group (infinite dimensional). On the other hand, when M is a bounded domain of $\mathbb{C}^n$ or a Hermitian manifold of negative holomorphic curvature (e.g. hyperbolic manifold), it is known that $\mathfrak{hol}(M)$ is a Lie transformation group. Moreover, for a compact complex manifold M, $\mathfrak{hol}(M)$ is a complex Lie transformation group. (Refer to* [4], [5].*)*

5. CR-structure on Heisenberg space $\mathcal{N}$

The rest of this section is spent to how to construct Bochner flat structures on $\mathbb{C}^n$ from the Heisenberg space $\mathcal{N}$. The Heisenberg nilpotent space $\mathcal{N}$ is a Lie group which is the product $\mathbb{R} \times \mathbb{C}^n$ with group law:

$$(a, z) \cdot (b, w) = (a + b - \mathrm{Im} < z, w >,\ z + w), \tag{5.1}$$

where $\mathrm{Im} < z, w >$ is the imaginary part of the Hermitian inner product on $\mathbb{C}^n$

$$< z, w >= \bar{z}_1 \cdot w_1 + \bar{z}_2 \cdot w_2 + \cdots + \bar{z}_n \cdot w_n.$$

It is easy to see that $\mathcal{N}$ is 1-step nilpotent, i.e. the commutator $[\mathcal{N}, \mathcal{N}] = \mathbb{R}$. Put $\mathcal{R} = (\mathbb{R}, 0)$ which is the central subgroup of $\mathcal{N}$. If $\mathrm{Aut}_{CR}(\mathcal{N})$ is the subgroup of CR transformations preserving $\mathcal{N}$, then, $\mathrm{Aut}_{CR}(\mathcal{N}) = \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$. The action of $\mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$ on $\mathcal{N}$ is obtained:

$$((a, z), \lambda \cdot A) \cdot (b, w) = (a + \lambda^2 b - \mathrm{Im} < z, \lambda \cdot Aw >, z + \lambda \cdot Aw). \tag{5.2}$$

The contact form $\omega_{\mathcal{N}}$ on $\mathcal{N}$ is described as follows. Put $\omega = \omega_{\mathcal{N}}$. If $(t, (z_1, \cdots, z_n))$ is the coordinate of $\mathcal{N} = \mathbb{R} \times \mathbb{C}^n$, then

$$\omega = dt + \sum_{j=1}^{n} (x_j dy_j - y_j dx_j) = dt + \mathrm{Im} < z, dz > . \tag{5.3}$$

The subgroup $\mathrm{Psh}(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(n)$ leaves ω invariant. For this, if $\gamma = ((a, w), A) \in \mathcal{N} \rtimes \mathrm{U}(n) = \mathrm{Psh}(\mathcal{N})$, then

$$((a, w), A) \cdot (t, z) = (a + t - \mathrm{Im} < w, Az >, w + Az), \quad \text{and so}$$
$$\gamma^* \omega = dt - d\mathrm{Im} < w, Az > + \mathrm{Im} < w + Az, d(w + Az) > .$$

Since $d\mathrm{Im} < w, Az >= \mathrm{Im} < w, dAz >$, it is easy to see that

$$\gamma^* \omega = dt + \mathrm{Im} < z, dz >= \omega.$$

Recall that J_0 is the CR-structure $(\mathrm{Null}\ \omega_0, J_0)$ on S^{2n+1}. Restricted J_0 to $S^{2n+1} - \{0\} = \mathcal{N}$, we have the CR-structure $(\mathrm{Null}\ \omega, J)$ on $\mathcal{N}$. In general, if $h \in \mathrm{Aut}_{CR}(\mathcal{N})$ is an element, then there exists a positive function u on $\mathcal{N}$ such that

$$h^*\omega_{\mathcal{N}} = u \cdot \omega_{\mathcal{N}}.$$

Moreover, by definition, h is holomorphic (Cauchy-Riemann) on Null ω. Hence, every element h of $\mathrm{Aut}_{CR}(\mathcal{N})$ preserves the CR-structure (Null ω, J).

On the other hand, we have the canonical principal fibration:

$$\mathbb{R} \to (\mathcal{N}, \omega) \xrightarrow{P} (\mathbb{C}^n, \Omega_0) \tag{5.4}$$

where $d\omega = P^*\Omega_0$ such that $\Omega_0 = 2\sum_{j=1}^{2n} dx_j \wedge dy_j$ is the standard Kähler form of $\mathbb{C}^n$ and $g_0 = \Omega_0(J_0\ ,\ )$ is the complex euclidean metric. (In other words, the CR-structure J on Null ω is obtained from the standard complex structure $J_{\mathbb{C}}$ on $\mathbb{C}^n$ by the commutative diagram:

$$\begin{array}{ccc} \text{Null } \omega & \xrightarrow{P_*} & T(\mathbb{C}^n) \\ \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J_{\mathbb{C}}} \\ \text{Null } \omega & \xrightarrow{P_*} & T(\mathbb{C}^n). \end{array} \tag{5.5}$$

Let $\rho : \mathbb{R} \to \mathcal{R} \times T^n$ be the represntation $\rho(t) = ((t, 0), (e^{it \cdot a_1}, \cdots, e^{it \cdot a_n}))$ such that

$$0 \leq a_1 \leq \cdots \leq a_n. \tag{5.6}$$

Note that if all $a_i = 0$, then $\rho(\mathbb{R})$ is the center of $\mathcal{N}$.

Recall that $\rho(\mathbb{R})$ is a closed subgroup of $\mathrm{Psh}(\mathcal{N})$ isomorphic to $\mathbb{R}$. As $\mathrm{Psh}(\mathcal{N})$ acts properly on $\mathcal{N}$, $\rho(\mathcal{N})$ acts properly and freely on $\mathcal{N}$. Let

$$\rho(\mathbb{R}) \to \mathcal{N} \xrightarrow{\nu} \mathcal{N}/\rho(\mathbb{R}) \tag{5.7}$$

be the principal bundle. Note that the orbit space $\mathcal{N}/\rho(\mathbb{R})$ is biholomorphic to $\mathbb{C}^n$. For this, let $f : \mathcal{N} \to \mathbb{C}^n$ be a map defined by

$$f((t, (z_1, \cdots, z_n))) = (e^{-ita_1} \cdot z_1, \cdots, e^{-ita_n} \cdot z_n). \tag{5.8}$$

Since $f_* : (\mathrm{Null}\,\omega)_{(t,z)} \to T_{f(t,z)}\mathbb{C}^n$ is isomorphic, f_* induces a complex structure J' on $\mathbb{C}^n$ such that $f_*J = J'f_*$. As $P_* : (\mathrm{Null}\,\omega_{(0,z)}, J) \to (T_z\mathbb{C}^n, J_{\mathbb{C}})$ is holomorphic and $f(0, z) = z$, $f_* \circ P_*^{-1} : T_z\mathbb{C}^n \to T_z\mathbb{C}^n$ satisfies that $(f_* \circ P_*^{-1}) \circ J_{\mathbb{C}z} = J'_z \circ (f_* \circ P_*^{-1})$. Hence, the complex structure J' is conjugate to the standard complex structure. Since f induces a diffeomorphism $\hat{f} : \mathcal{N}/\rho(\mathbb{R}) \to \mathbb{C}^n$ such that the diagram is commutative:

$$\begin{array}{ccc} & \mathcal{N} & \\ {\scriptstyle \nu}\swarrow & & \searrow{\scriptstyle f} \\ \mathcal{N}/\rho(\mathbb{R}) & \xrightarrow{\hat{f}} & \mathbb{C}^n. \end{array}$$

Noting that $\nu_* : (\text{Null } \omega, J) \to (T\mathcal{N}/\rho(\mathbb{R}), \hat{J})$ is holomorphic, $\hat{f}$ is a holomorphic diffeomorphism of $(\mathcal{N}/\rho(\mathbb{R}), \hat{J})$ onto $(\mathbb{C}^n, J')$.

Recall that $\rho(\mathbb{R})$ acts on $\mathcal{N}$ by

$$\rho(t)(s,z) = (s+t, A_t z) \quad ((s,z) \in \mathcal{N}).$$

Let ξ be the vector field on $\mathcal{N}$ induced by $\rho(\mathbb{R})$. Then,

$$\xi = \frac{d}{dt} + \sum_{j=1}^{n} a_j (x_j \frac{d}{dy_j} - y_j \frac{d}{dx_j}) \quad \text{on } \mathcal{N}.$$

Using (5.3), $\omega(\xi) = 1 + (a_1|z_1|^2 + a_2|z_2|^2 + \cdots + a_n|z_n|^2)$. By the hypothesis (5.6), $\omega(\xi) > 0$ everywhere on $\mathcal{N}$. We have the contact form as in (3.2):

(5.9) $$\eta(Z) = \frac{1}{\omega(\xi)} \cdot \omega(Z) \quad (\forall\, Z \in T\mathcal{N}).$$

By Lemma 7, it follows that

(5.10) $$\eta(\xi) = 1, d\eta(\xi, X) = 0 \quad (\forall X \in T\mathcal{N}).$$

As in (3.5), we have a Hermitain metric on $(\mathcal{N}/\rho(\mathbb{R}), \hat{J}) = (\mathbb{C}^n, J_{\mathbb{C}})$:

$$\hat{g}(\hat{X}, \hat{Y}) = d\eta(JX, Y)$$

where $X, Y \in$ Null η such that $\nu_*(X) = \hat{X}$, $\nu_*(Y) = \hat{Y}$. Let $\hat{\Omega}(\hat{X}, \hat{Y}) = \hat{g}(\hat{X}, \hat{J}\hat{Y})$ be the fundamental two form on $\mathcal{N}/\rho(\mathbb{R}) = \mathbb{C}^n$. Using (5.10), it follows that $\nu^*\hat{\Omega} = d\eta$, i.e. $d\hat{\Omega} = 0$. Therefore, $\hat{\Omega}$ is a Kähler form on $\mathbb{C}^n$. Thus we obtain a Bochner flat Kähler structure $(\hat{g}, \hat{\Omega}, J_{\mathbb{C}})$ on $\mathbb{C}^n$. For $a = (a_1, \cdots, a_n)$, we put $\hat{g} = g_a$. Since $\nu^*\hat{\Omega} = d\eta$ and $(\text{Null}\, \eta, J)$ is a spherical CR-structure (i.e. $S(\eta, J) = 0$), Proposition 3 implies that the Bochner curvature $B(g_a) = 0$. Hence, $(\mathbb{C}^n, g_a)$ is a Bochner flat Kähler manifold. We omit the Kähler metric g_a is complete whenever $0 \leq a_1 \leq \cdots \leq a_n$.

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