# Pseudo-free action of cyclic groups

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## 1　Determinant of equivariant elliptic operators

$M = M^{2m}$ : $2m$-dimensional closed connected oriented Riemannian manifold

$G$ : finite group acting on $M$

$G$-action is orientation-preserving, isometric and effective

$D : \Gamma(E) \to \Gamma(F)$ : $G$-equivariant elliptic operator

Equivariant determinant of $D$ is defined by

$$G \ni g \longrightarrow \det(D, g) = \frac{\det(g| \ker D)}{\det(g|\mathrm{coker}\, D)} \in S^1 \subset \mathbf{C}^*$$

$\det_D := \det(D, \cdot) : G \longrightarrow S^1$ is a group homomorphism

$S^1$ : Abelian $\Longrightarrow$ $\det_D([G, G]) = 1$

$I_D : G \longrightarrow \mathbf{R}/\mathbf{Z}$ (additive group homomorphism) is defined by

$$I_D(g) := \frac{1}{2\pi\sqrt{-1}} \log \det(D, g) \quad (\mathrm{mod}\ \mathbf{Z})$$

**Theorem 1** $I_D$ is an additive group homomorphism and hence

(a) $I_D(g) + I_D(h) - I_D(gh) = 0$  $(\forall g,\ h \in G)$

(b) $\det(D, g)^N = 1 \iff N I_D(g) = 0$

**Proposition 1**   $g^p = 1 \ (p \geq 2) \implies$

$$I_D(g) \equiv \frac{p-1}{2p}\text{Ind}(D) - \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\xi_p^{-k}}\text{Ind}(D, g^k) \quad (\text{mod } \mathbf{Z})$$

where $\xi_p = e^{2\pi\sqrt{-1}/p}$ : primitive $p$-th root of unity,

$$\text{Ind}(D, g^k) = \text{Tr}(g^k|\ker D) - \text{Tr}(g^k|\text{coker } D) \in \mathbf{C},$$

$$\text{Ind}(D) = \text{Ind}(D, 1) = \dim \ker D - \dim \text{coker } D \in \mathbf{Z}$$

$\text{Ind}(D)$, $\text{Ind}(D, h)$ $(h \in G)$  and hence $I_D(h)$ is calculated by using the Atiyah-Singer index theorem.

$\text{Ind}(D)$ is a characteristic number defined by using

the principal symbol of $D$

$$\begin{cases} M \supset \text{Fix}(g) = \{q_1, q_2, \cdots, q_n\} \text{ consists of points} \\ T_{q_i}M = \oplus_{j=1}^m N(\tau_{ij}) \quad \left(h|N(\tau) = \begin{pmatrix} \cos\frac{2\pi\tau}{p} & -\sin\frac{2\pi\tau}{p} \\ \sin\frac{2\pi\tau}{p} & \cos\frac{2\pi\tau}{p} \end{pmatrix}\right) \end{cases}$$

$$\implies \text{Ind}(D, h) = \sum_{i=1}^n \frac{\text{Tr}(h|E_{q_i}) - \text{Tr}(h|F_{q_i})}{\prod_{j=1}^m(1-\xi_p^{\tau_{ij}})(1-\xi_p^{-\tau_{ij}})}$$

**Proposition 2**   Let $S$ be the signature operator and assume that $p$ is an odd prime number. Then we have

$$\text{Ind}(S) = \text{Sign}(M), \quad \text{Ind}(S, h) = \sum_{i=1}^n \prod_{j=1}^m \left(-\sqrt{-1}\cot\frac{\pi\tau_{ij}}{p}\right)$$

where $\text{Sign}(M)$ is the signature of $M$.

$\text{Spin}(2m)$ : universal covering group of $SO(2m)$ $(m \geq 2)$

$\text{Spin}^c(2m) := \text{Spin}(2m) \times S^1/\sim \quad ((s, z) \sim (-s, -z))$

$\exists P$ : $\text{Spin}^c$-structure of $M \iff w_2(TM) = \exists u \in H^2(M; \mathbf{Z}) \quad (\text{mod } 2)$

$\exists \text{Spin}$ or $\exists$ almost complex structure $\implies \exists \text{Spin}^c$-structure

$\eta = P \times_{\text{Spin}^c(2m)} \mathbf{C}$ : associated complex line bundle over $M$

($\text{Spin}^c$-structure is determined by $c_1(\eta)$ if $H^1(M; \mathbf{Z}_2) = 0$)

$G$-action on a $\text{Spin}^c$-manifold $M$ is called a $\underline{\text{Spin}^c\text{-action}}$

if the action lifts to an action on the Spin$^c$-structure of $M$

**Remark 1**   Let $G$ be a cyclic group and assume that $m \geq 2$. Then

if $H^1(M; \mathbf{Z}) = 0 (\Longrightarrow H^1(SO(M); \mathbf{Z}) = 0)$ and $c_1(\eta)$ is $G$-invariant

$\Longrightarrow$ any $G$-action lifts to the Spin$^c$-structure $P(\xrightarrow{S^1} SO(M))$.

(A.Hattori and T.Yoshida, Lifting compact group actions in fiber bundles, Japan. J. Math. **2**, 13-25 (1976). )

$L$ : (arbitrary) complex $G$-line bundle over $M$

$$D_L : \Gamma(S_+ \otimes L) \xrightarrow{\nabla} \Gamma(T^*M \otimes S_+ \otimes L) \xrightarrow{cm} \Gamma(S_- \otimes L) : L\text{-valued Dirac operator}$$

(where $S_\pm$ are half spinor bundles and $cm$ is the Clifford multiplication)

**Proposition 3**

$h$ acts on the fibers $\eta|q_i$, $L|q_i$ via multiplications by $\xi_p^{\kappa_i}$, $\xi_p^{\mu_i}$ respectively

$$\Longrightarrow \text{Ind}(D_L) = e^{c_1(L)} e^{\frac{c_1(\eta)}{2}} \widehat{A}(TM)[M] \ ,$$

$$\text{Ind}(D_L, h) = \sum_{i=1}^{n} \varepsilon_i \xi_p^{\mu_i} \xi_p^{\frac{\nu_i}{2}} \prod_{j=1}^{m} \frac{1}{1 - \xi_p^{-\tau_{ij}}}$$

where $\widehat{A}$ is the $\widehat{A}$-class, $\varepsilon_i = \pm 1$ and $\nu_i = \kappa_i - \sum_{j=1}^{m} \tau_{ij}$.

($\varepsilon$ , $\kappa_i$ are not determined by the data of the action on $M$)

**Proposition 4**   $M = (M, J)$ : almost complex manifold

$G \subset \text{Aut}_J(M)$ : group of automorphisms of $M$ preserving

the almost complex structure $J$

$h|T_{q_i}M = (\xi_p^{\tau_{i1}}, \cdots, \xi_p^{\tau_{im}})$ : diagonal matrix with respect to $J$

$h$ acts on the fiber $L|q_i$ via multiplication by $\xi_p^{\mu_i}$

$$\Longrightarrow \text{Ind}(D_L) = e^{c_1(L)} \text{Td}(TM)[M] \ , \quad \text{Ind}(D_L, h) = \sum_{i=1}^{n} \xi_p^{\mu_i} \prod_{j=1}^{m} \frac{1}{1 - \xi_p^{-\tau_{ij}}}$$

where Td is the Todd class.

# 2   Finite subgroup of the mapping class group

$\Gamma_\sigma$ : mapping class group of genus $\sigma$ ($\sigma \geq 2$)

$\Sigma_\sigma$ : compact Riemann surface of genus $\sigma$ ($\sigma \geq 2$)

In this section,   $G$-action $\Longleftrightarrow$ $\exists J$; $G \subset \text{Aut}_J(\Sigma_\sigma)$ $\overset{(*)}{\Longleftrightarrow}$ $G \subset \Gamma_\sigma$.

($J$ : (integrable) almost complex structure, (*) : Nielsen realization)

$\mathbf{Z}_p = \langle g \rangle \subset \text{Aut}_J(\Sigma_\sigma)$

$\pi : \Sigma_\sigma \longrightarrow \Sigma_\sigma / \mathbf{Z}_p$ : branched covering with $b$ branch point

$\quad y_1, \cdots, y_b \in \Sigma_\sigma / \mathbf{Z}_p$ of order $(n_1, \cdots, n_b)$

$\pi^{-1}(y_i) = \{q_i, g \cdot q_i, \cdots, g^{r_i - 1} \cdot q_i\}$ : consists of $r_i := p/n_i$ points

For $1 \leq i \leq b$, assume that $g^{r_i}|_{T_{\pi^{-1}(y_i)}\Sigma_\sigma} = \xi_{n_i}^{t_i}$ where $1 \leq t_i \leq n_i - 1$

$\quad$ and $t_i$ is prime to $n_i$

$D_\ell$ : $\otimes^\ell T\Sigma_\sigma$-valued Dirac operator on $\Sigma_\sigma$

### Theorem 2

$$I_{D_\ell}(g^z) - z\, I_{D_\ell}(g) = 0 \iff \varphi_{\ell,z}(t_1, \cdots, t_b) \in \mathbf{Z},$$

$$N\, I_{D_\ell}(g^z) = 0 \iff N\psi_{\ell,z}(t_1, \cdots, t_b) \in \mathbf{Z}$$

for any $z$ ($1 \leq z < p$) which is prime to $p$ and for any $\ell$ ($0 \leq \ell < p$) where

$$\varphi_{\ell,z}(t_1, \cdots, t_b)$$

$$= (1 - z)\frac{p-1}{2p}(1 - \sigma)(2\ell + 1)$$

$$- \sum_{i=1}^b \frac{1}{n_i} \sum_{j=1}^{n_i - 1} \frac{1}{1 - \xi_{n_i}^{-j}} \left( \frac{\xi_{n_i}^{jzt_i\ell}}{1 - \xi_{n_i}^{-jzt_i}} - z\frac{\xi_{n_i}^{jt_i\ell}}{1 - \xi_{n_i}^{-jt_i}} \right),$$

$$I_{D_\ell}(g^k) \overset{\text{mod.}\mathbf{Z}}{\equiv} \psi_{\ell,z}(t_1, \cdots, t_b)$$

$$= \frac{p-1}{2p}(1 - \sigma)(2\ell + 1) - \sum_{i=1}^b \frac{1}{n_i} \sum_{j=1}^{n_i - 1} \frac{\xi_{n_i}^{jzt_i\ell}}{(1 - \xi_{n_i}^{-j})(1 - \xi_{n_i}^{-jzt_i})}$$

and $N$ is a natural number such that $\det(D_\ell, g)^N = 1$.

Precise values of $\varphi$ and $\psi$ are obtained by using the following proposition.

## Proposition 5

$12p\,I_{D_\ell}(g^z)$ is an integer and we have

$$12p\,I_{D_\ell}(g^z)$$

$$\equiv 6(p-1)(1-\sigma)(2\ell+1)$$

$$+\sum_{i=1}^{b} r_i \left\{ zt_i(n_i-1)(7n_i-11) + 6 \sum_{j=\left[\frac{(\ell+1)zt_i}{n_i}\right]+1}^{\left[\frac{(\ell+n_i+1)zt_i}{n_i}\right]} f_{n_i}\left(\left[\frac{jn_i-1}{zt_i}\right]-\ell-1\right) \right\}$$

$$(\bmod\ 12p)$$

where $f_{n_i}(x) = x^2 - (n_i-2)x - (n_i-1)^2$ and $[\ \ ]$ is the Gauss's symbol.

## Riemann-Hurwitz equation

$$2\sigma - 2 = p(2\bar\sigma - 2) + \sum_{i=1}^{b}(p-r_i) \quad \left(r_i = \frac{p}{n_i}\right)$$

where $\bar\sigma$ is the genus of $\Sigma_\sigma/\mathbf{Z}_p$.

**Example 1**  The necessary and sufficient condition of $(\sigma, p)$ for
$\mathbf{Z}_p \subset \Gamma_\sigma$ is known. (W.J.Harvey, Cyclic groups of automorphisms of a
compact Riemann surface, Quart. J. Math. **17**, 86-97 (1966))
We consider one hundred cases where $2 \le \sigma,\ p \le 11$.
Then $\mathbf{Z}_p \not\subset \Gamma_\sigma$ if and only if

$$(\sigma,p) = (2,7),\ (2,11),\ (3,11),\ (4,11),\ (5,7),\ (7,11),\ (8,11),\ (9,11)$$

$\implies$ R-H equation is not satisfied for any $\bar\sigma,\ b,\ r_i$

or

$$(*1)\quad (\sigma,p) = (2,9),\ (3,5),\ (3,10),\ (4,7),\ (5,9),\,(6,11),\ (11,7)$$

(Results of Harvey)

Using the R-H equation and Theorem 2, we can show that $\mathbf{Z}_p \not\subset \Gamma_\sigma$ for $(\sigma,p)$ in $(*1)$.

## Example 2

$G_p$ : finite non-Abelian group generated by $g, h_1, \cdots, h_s$

    $\mathrm{order}(g) = p$, $\mathrm{order}(h_j) = q_j$ $(1 \leq j \leq s)$

Assume that $[G_p, G_p] \ni \exists \gamma$ such that $\gamma$ is expressed as the product

    of $m$ $g$'s $(0 < m < p)$ and $\mu_j$ $h_j$'s $(0 \leq \mu_j < q_j,\ 1 \leq j \leq s)$

    which satisfies that $d := \mathrm{g.c.d.}(p,\ m\beta) < p$

(where $\beta := \ell.c.m(q_1^{\varepsilon_1} \cdots q_s^{\varepsilon_s})$ , $\varepsilon_j = 1$ if $\mu_j > 0$ , $\varepsilon_j = 0$ if $\mu_j = 0$)

Assume moreover that $\mathrm{Aut}_J(\Sigma_\sigma) \supset G_p$ $(2 \leq \sigma \leq 11,\ 3 \leq p \leq 11)$

    $\Longrightarrow$   $\mathrm{Aut}_J(\Sigma_\sigma) \supset \mathbf{Z}_p = \langle g \rangle$

$\Sigma_\sigma \longrightarrow \Sigma_\sigma/\mathbf{Z}_p$ : branched covering with $b$ branch points

        $y_1, \cdots, y_b$ of order $(n_1, \cdots, n_b)$

Then, $[G_p, G_p] \subset \ker \det(D_\ell, \cdot)$ $\Longrightarrow$

$$1 = \det(D_\ell, \gamma^\beta) = \det(D_\ell, g)^{m\beta} \iff \det(D_\ell, g)^d = 1$$

$$\iff 0 = dI_{D_\ell}(g^z) \equiv d\psi_{\ell,z}(t_1, \cdots, t_b) \pmod{\mathbf{Z}}$$

$$\left( \begin{array}{l} \exists t_i : \text{prime to } n_i\ (1 \leq i \leq b) , \\ \forall z\ (1 \leq z < p,\ z : \text{prime to } p)\ ,\ \forall \ell\ (0 \leq \ell < p) \end{array} \right)$$

R-H equation, results of Harvey $\Longrightarrow$

$$(\sigma, p) = (2, 5) \implies (b, \{n_1, \cdots, n_b\}) = (3, \{5, 5, 5\})$$

$$(\sigma, p) = (7, 5) \implies (b, \{n_1, \cdots, n_b\}) = (3, \{5, 5, 5\})$$

$$(\sigma, p) = (3, 9) \implies (b, \{n_1, \cdots, n_b\}) = (3, \{3, 9, 9\})$$

$$(\sigma, p) = (4, 9) \implies (b, \{n_1, \cdots, n_b\}) = (3, \{9, 9, 9\})$$

$$(\sigma, p) = (11, 9) \implies (b, \{n_1, \cdots, n_b\}) = (5, \{3, 9, 9, 9, 9\})$$

$$(\sigma, p) = (7, 10) \implies (b, \{n_1, \cdots, n_b\}) = (4, \{2, 10, 10, 10\}) ,\ (5, \{2, 2, 2, 5, 10\})$$

$$(\sigma, p) = (5, 11) \implies (b, \{n_1, \cdots, n_b\}) = (3, \{11, 11, 11\}) .$$

Using Theorem 2, we have the following results.

$p = 5,\ \sigma = 2,\ 7 \implies (b, \{n_1, \cdots, n_b\}) = (3, \{5, 5, 5\})$

$\psi_{1,1}(t_1, t_2, t_3) \notin \mathbf{Z}\ (\forall t_1,\ t_2,\ t_3 = 1,\ 2,\ 3,\ 4) \implies G_5 \not\subset \mathrm{Aut}_J(\Sigma_2),\ \mathrm{Aut}_J(\Sigma_7)$

$p = 9,\ \sigma = 3 \implies (b, \{n_1, \cdots, n_b\}) = (3, \{3, 9, 9\})$

$3\psi_{1,1}(t_1, t_2, t_3) \notin \mathbf{Z}\ (\forall t_1 = 1,\ 2,\ \forall t_2,\ t_3 = 1,\ 2,\ 4,\ 5,\ 7,\ 8) \implies G_9 \not\subset \mathrm{Aut}_J(\Sigma_3)$

$p = 9,\ \sigma = 4 \implies (b, \{n_1, \cdots, n_b\}) = (3, \{9, 9, 9\})$

$3\psi_{1,1}(t_1, t_2, t_3) \notin \mathbf{Z}\ (\forall t_1,\ t_2,\ t_3 = 1,\ 2,\ 4,\ 5,\ 7,\ 8) \implies G_9 \not\subset \mathrm{Aut}_J(\Sigma_4)$

$p = 9,\ \sigma = 11 \implies (b, \{n_1, \cdots, n_b\}) = (5, \{3, 9, 9, 9, 9\})$

$3\psi_{1,1}(t_1, t_2, t_3, t_4, t_5) \notin \mathbf{Z}\ (\forall t_1 = 1,\ 2,\ ,\ \forall t_2,\ t_3,\ t_4,\ t_5 = 1,\ 2,\ 3,\ 4)$

$\implies G_9 \not\subset \mathrm{Aut}_J(\Sigma_{11})$

$p = 10,\ \sigma = 7 \implies (b, \{n_1, \cdots, n_b\}) = (4, \{2, 10, 10, 10\}),\ (5, \{2, 2, 2, 5, 10\})$

$\begin{cases} k\psi_{1,1}(1, t_2, t_3, t_4) \notin \mathbf{Z}\ (k = 2,\ 5)\ (\forall t_2,\ t_3,\ t_4 = 1,\ 3,\ 7,\ 9) \\ k\psi_{1,1}(1, 1, 1, t_4, t_5) \notin \mathbf{Z}\ (k = 2,\ 5)\ (\forall t_4 = 1,\ 2,\ 3,\ 4,\ \forall t_5 = 1,\ 3,\ 7,\ 9) \end{cases}$

$\implies G_{10} \not\subset \mathrm{Aut}_J(\Sigma_7)$

$p = 11,\ \sigma = 5 \implies (b, \{n_1, \cdots, n_b\}) = (3, \{11, 11, 11\})$

$\{(t_1, t_2, t_3)\,|\,\psi_{1,1}(t_1, t_2, t_3) \in \mathbf{Z}\} \cap \{(t_1, t_2, t_3)\,|\,\psi_{2,1}(t_1, t_2, t_3) \in \mathbf{Z}\} = \phi$

$\implies G_{11} \not\subset \mathrm{Aut}_J(\Sigma_5)$

Hence $G_5 \not\subset \Gamma_2,\ \Gamma_7,\quad G_9 \not\subset \Gamma_3,\ \Gamma_4,\ \Gamma_{11},\quad G_{10} \not\subset \Gamma_7,\quad G_{11} \not\subset \Gamma_5$

In particular,

$G_p = D(2p)$ : dihedral group generated by $g,\ h$ with $g^{-1}h^{-1}gh = g^{p-2}$

g.c.d$(p, p - 2) < p \implies$

$D(2p) \not\subset \Gamma_\sigma$ if $(\sigma, p) = (2, 5),\ (7, 5),\ (3, 9),\ (4, 9),\ (11, 9),\ (7, 10),\ (5, 11)$

Note that $\sigma \equiv 0,\ 1\pmod{p} \implies D(2p) \subset \Gamma_\sigma$.

# 3  0-pseudofree action of cyclic groups of prime order

<u>Definition 1</u>  An action is called "0-peudofree" iff the exceptional orbits are isolated points. (E. Laitinen and P. Traczyk, Pseudofree representations and 2-pseudofree actions on spheres, Proc. Amer. Math. Soc. **97**, 151-157 (1986))

Here 0-pseudofree is simply called "pseudofree".

Pf($M$) : set of orientation-preserving isometries $g$ on $M$

such that the fixed point set Fix($g$) consists of points

When $M = M^{2m}$ has a Spin$^c$-structure,

Pf$^c(M) := \{g \in \text{Pf}(M) \mid g \text{ preserves the Spin}^c\text{-structure}\}$

If $\mathbf{Z}_p = \langle g \rangle$ , $p$ : prime number

$\mathbf{Z}_p$-action is pseudo-free $\Longleftrightarrow$ $\mathbf{Z}_p \subset \text{Pf}(M)$ $\Longleftrightarrow$ Fix($g$) = $\{q_1, q_2, \cdots, q_n\}$ : points

<u>**Theorem 3**</u>  Assume that $\mathbf{Z}_p \subset \text{Pf}(M)$ and suppose that $T_{q_i}M = \oplus_{j=1}^m N(\tau_{ij})$
$(1 \le i \le n)$. Then using $pI_D(g) = 0$ (D : signature operator), we have

$$\sum_{k=1}^{\frac{p-1}{2}} \sum_{i=1}^n \prod_{j=1}^{2s} \cot \frac{\pi k \tau_{ij}}{p} \equiv 0 \pmod{\mathbf{Z}} \text{ if } m = 2s$$

$$\sum_{k=1}^{\frac{p-1}{2}} \cot \frac{\pi k}{p} \sum_{i=1}^n \prod_{j=1}^{2s-1} \cot \frac{\pi k \tau_{ij}}{p} \equiv 0 \pmod{\mathbf{Z}} \text{ if } m = 2s - 1$$

<u>Corollary 1</u>

$\mathbf{Z}_3 \subset \text{Pf}(M) \Longrightarrow n$ : even  or  $n \ge 3^{[(m+1)/2]}$

Assume that $M$ has a Spin$^c$-structure and $\mathbf{Z}_p \subset \text{Pf}^c(M)$

$L$ : (arbitrary) $\mathbf{Z}_p$-line bundle, $D_L$ : $L$-valued Dirac operator

<u>**Theorem 4**</u>  When $p = 2$, $n \ge 2^m$ unless $n$ and $\text{Ind}(D_L)$ ($\forall L$) are even.

**Theorem 5**  When $p = 3$ or $p = 5$,

$$\gamma = \min_{s \in \mathbf{Z}} \left| \frac{p-1}{2} \mathrm{Ind}(D_L) - ps \right| \; : \; \text{distance from } \frac{p-1}{2} \mathrm{Ind}(D_L) \text{ to } p\,\mathbf{Z}$$

$$\implies \; n \geq \frac{\gamma}{3(p-1)} \left( 2\sin\frac{\pi}{p} \right)^{m+1}$$

$$\det(D_L, g) = 1 \implies n \geq \frac{\gamma}{p-1} \left( 2\sin\frac{\pi}{p} \right)^{m+1}$$

**Example 4**

$$M_{pk} = \Sigma_{pk} \times \overbrace{S^2 \times \cdots \times S^2}^{m-1 \text{ times}} \quad (p : \text{prime})$$

has a $\mathrm{Spin}^c$-structure with

$$c_1(\eta) = (2s + 2 - 2pk)y + \sum_{j=1}^{m-1} (2t_j + 2)z_j$$

$$\in H^2(M_{pk}; \mathbf{Z}) \cong H^2(\Sigma_{pk}; \mathbf{Z}) \oplus \oplus_{j=1}^{m-1} H^2(S^2; \mathbf{Z}) \quad (s, \, t_j \in \mathbf{Z})$$

$M_{pk}$ admits an peudo-free $\mathrm{Spin}^c$-action of $\mathbf{Z}_p$ with $2^m$ fixed points if the $\mathrm{Spin}^c$-structure comes from the almost complex structure.

$$\implies \; s, \, t_j\text{'s equal } 0.$$

Since we can show that

$$\mathrm{Ind}(D) = (s + 1 - pk) \prod_{j=1}^{m-1} (t_j + 1) \equiv (s + 1) \prod_{j=1}^{m-1} (t_j + 1) \pmod{p},$$

$s$, $t_j$'s are even and $\mathbf{Z}_2 \subset \mathrm{Pf}^c(M_{2k}) \overset{\text{Theorem 4}}{\implies} n \geq 2^m$

$(s + 1) \prod_{j=1}^{m-1} (t_j + 1) \equiv 1, 2 \pmod{3}$, $\mathbf{Z}_3 \subset \mathrm{Pf}^c(M_{3k})$

$$\implies \gamma = 1 \overset{\text{Theorem 5}}{\implies} n \geq \frac{1}{6} 3^{\frac{m+1}{2}} \; (\sim 0.866^m \cdot 2^m)$$

$(s + 1) \prod_{j=1}^{m-1} (t_j + 1) \equiv 1, 4 \pmod{5}$, $\mathbf{Z}_5 \subset \mathrm{Pf}^c(M_{5k})$

$$\implies \gamma = 2 \overset{\text{Theorem 5}}{\implies} n \geq \frac{1}{6} \left( \frac{5 - \sqrt{5}}{8} \right)^{-\frac{m+1}{2}} \; (\sim 0.294^m \cdot 2^m)$$

$(s + 1) \prod_{j=1}^{m-1} (t_j + 1) \equiv 2, 3 \pmod{5}$, $\mathbf{Z}_5 \subset \mathrm{Pf}^c(M_{5k})$

$$\implies \gamma = 1 \overset{\text{Theorem 5}}{\implies} n \geq \frac{1}{12} \left( \frac{5 - \sqrt{5}}{8} \right)^{-\frac{m+1}{2}} \; (\sim 0.294^m \cdot 2^m)$$

Moreover $\mathbf{Z}_3 \subset \mathrm{Pf}(M_{pk}) \overset{\text{Corollary 1}}{\Longrightarrow} n \neq 1, 3, 5, \cdots, 3^{[(m+1)/2]} - 2$

## Example 5

$M = \mathbf{CP}^2 \times \mathbf{CP}^k$ $(k \geq 3)$

Assume that $\mathrm{Pf}(M) \supset \mathbf{Z}_p = \langle g \rangle$ $(p : \text{odd prime})$

$H^2(M; \mathbf{Z}) = H^2(\mathbf{CP}^2; \mathbf{Z}) \oplus H^2(\mathbf{CP}^k; \mathbf{Z}) = \{\lambda x + \mu y \mid \lambda, \mu \in \mathbf{Z}\} = \mathbf{Z} \oplus \mathbf{Z}$

where $H^2(\mathbf{CP}^2; \mathbf{Z}) \cong \mathbf{Z} = \langle x \rangle$ and $H^2(\mathbf{CP}^k; \mathbf{Z}) \cong \mathbf{Z} = \langle y \rangle$

$g^*(x^2 y^k) = x^2 y^k \overset{(*1)}{\Longleftrightarrow} g^*x = x, \ g^*y = y \overset{(*2)}{\Longrightarrow} \#\mathrm{Fix}(g) = 3(k+1)$

$((*1) : \text{not trivial}, \quad (*2) : \text{Lefschetz fixed point theorem})$

$$\left( \begin{array}{l} \text{For example, if } k < p, \\ \quad \text{pseudo-free action of } \mathbf{Z}_p \text{ on } \mathbf{CP}^j \ (j = 2 \text{ or } k) \text{ defined by} \\ \qquad g : [z_0 : z_1 : z_2 : \cdots : z_j] \longrightarrow [z_0 : \xi_p z_1 : \xi_p^2 z_2 : \cdots : \xi_p^j z_j] \\ \quad \text{has } j + 1 \text{ fixed points} \\ \quad \Longrightarrow \text{ diagonal action of } \mathbf{Z}_p \text{ on } M \text{ has } 3(k+1) \text{ fixed points} \end{array} \right)$$

Almost complex structure $J$ of $M \Longrightarrow \text{Spin}^c\text{-structure } P_J$ of $M$

$$\Longrightarrow c_1(\eta) = 3x + (k+1)y \Longrightarrow g^*c_1(\eta) = c_1(\eta)$$

$H^1(M; \mathbf{Z}) = 0 \Longrightarrow \mathbf{Z}_p\text{-action lifts to an action on the Spin}^c\text{-structure } P_J$

We can show that $\mathrm{Ind}(D) = 1$, and hence

$$\gamma = 1 \ (p = 3), \ \gamma = 2 \ (p = 5) \text{ in Theorem 5} \Longrightarrow$$

$$\left\{ \begin{array}{l} 3(3-1) \cdot 3(k+1) \geq \left(2 \sin \dfrac{\pi}{3}\right)^{2+k+1} \Longleftrightarrow k \leq 5 \quad (p = 3) \\[3mm] 3(5-1) \cdot 3(k+1) \geq 2 \left(2 \sin \dfrac{\pi}{5}\right)^{2+k+1} \Longleftrightarrow k \leq 37 \quad (p = 5) \end{array} \right.$$

$k \geq 3 \Longrightarrow 3(k+1) < 3^{[(3+k+1)/2]} \overset{\text{Corollary 1}}{\Longrightarrow} \mathbf{Z}_3 \not\subset \mathrm{Pf}(M)$ if k is even

If $G_p \subset \mathrm{Pf}^c(M)$ $(p : \text{odd prime})$, then $\det(D, g) = 1$ and hence

$$\gamma = 1 \ (p = 3), \ \gamma = 2 \ (p = 5) \text{ in Theorem 5} \Longrightarrow$$

$$\left\{ \begin{array}{l} (3-1) \cdot 3(k+1) \geq \left(2 \sin \dfrac{\pi}{3}\right)^{2+k+1} \Longleftrightarrow \nexists k \quad (p = 3) \\[3mm] (5-1) \cdot 3(k+1) \geq 2 \left(2 \sin \dfrac{\pi}{5}\right)^{2+k+1} \Longleftrightarrow k \leq 29 \quad (p = 5) \end{array} \right.$$