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# Jørgensen groups of parabolic type II

# (Countable infinite case)

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ABSTRACT. This paper is the second part of the study on Jørgensen groups of parabolic type. We will show that there are countable infinite many Jørgensen groups of parabolic type on a certain cylinder in this case.

## 1. Introduction.

**1.1.** It is one of the most important problems in the theory of Kleinian groups to decide whether or not a subgroup $G$ of the Möbius transformation group is discrete. For this problem there are two important and useful theorems: One is Poincaré's polyhedron theorem, which is a sufficient condition for $G$ to be discrete. The other is Jørgensen's inequality, which is a necessary condition for a two-generator Möbius transformation group $\langle A, B\rangle$ to be discrete. Here we will consider extreme discrete groups (Jørgensen groups) for Jørgensen's inequality. This paper is the second part

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of a series of studies on Jørgensen groups (cf. Li - Oichi - Sato [4]).

**1.2.** Let Möb denote the set of all linear fractional transformations ( Möbius transformations)

$$A(z) = \frac{az+b}{cz+d}$$

of the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, where $a, b, c, d$ are complex numbers and the determinant $ad - bc = 1$. There is an isomorphism between Möb and $PSL(2, \mathbf{C})$. We always write elements of Möb as matrices with determinant 1 in this paper. We recall that Möb $(= PSL(2, \mathbf{C}))$ acts on the upper half space $H^3$ of $\mathbf{R}^3$ as the group of conformal isometries of hyperbolic 3-space.

In this paper we use a Kleinian group in the same meaning as a discrete group. Namely, a Kleinian group is a discrete subgroup of Möb. A Kleinian group $G$ is of *the first kind* if the limit set $\Lambda(G)$ of $G$ is all of the extended complex plane $\hat{\mathbf{C}}$ and it is of *the second kind* otherwise. A subgroup $G$ of Möb is said to be *elementary* if there exists a finite $G$-orbit in $\mathbf{R}^3$.

**1.3.** The *trace* $\mathrm{tr}(A)$ of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad - bc = 1)$$

in $SL(2, \mathbf{C})$ is defined by $\mathrm{tr}(A) = a + d$. We remark that the trace of an element of Möb $(= PSL(2, \mathbf{C}))$ is not well-defined, but Jørgensen number (defined later) is still well-defined after choosing matrix representatives.

**1.4.** Let $A^*$ and $B^*$ be matrices in $SL(2, \mathbf{C})$ representing the Möbius transformations $A$ and $B$, respectively. As $A^*$ and $B^*$ are determined by $A$ and $B$ to within a factor of $-1$, we see that the commutator $A^*B^*(A^*)^{-1}(B^*)^{-1}$ (resp. $(A^*)^2$) are uniquely determined by $A$ and $B$ (resp. $A$). Thus we may write $\mathrm{tr}(ABA^{-1}B^{-1}) = \mathrm{tr}(A^*B^*(A^*)^{-1}(B^*)^{-1})$ and $\mathrm{tr}^2(A) = \mathrm{tr}^2(A^*)$.

In 1976 Jørgensen obtained the following important theorem, which gives a necessary condition for a non-elementary Möbius transformation group $G = \langle A, B\rangle$ to be discrete.

**Theorem A** (Jørgensen [1]). *Suppose that the Möbius transformations $A$ and $B$ generate a non-elementary discrete group. Then*

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

*The lower bound* 1 *is best possible.*

**1.5.** DEFINITION 1. Let $A$ and $B$ be Möbius transformations. The *Jørgensen number* $J(A,B)$ for the ordered pair $(A,B)$ is defined by

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

DEFINITION 2. A subgroup $G$ of Möb is called a *Jørgensen group* if $G$ satisfies the following four conditions:

(1) $G$ is a two-generator group.

(2) $G$ is a discrete group.

(3) $G$ is a non-elementary group.

(4) There exist generators $A$ and $B$ of $G$ such that $J(A,B) = 1$.

1.6 Jørgensen and Kiikka showed the following.

**Theorem B** (Jørgensen-Kiikka [2]). *Let $\langle A, B\rangle$ be a non-elementary discrete group with $J(A,B) = 1$. Then $A$ is elliptic of order at least seven or $A$ is parabolic.*

If $\langle A, B\rangle$ is a Jørgensen group such that $A$ is parabolic and $J(A,B) = 1$, then we call it *a Jørgensen group of parabolic type*. There are infinite many Jørgensen groups of parabolic type (Jørgensen-Lascurain-Pignataro [3], Sato [6]).

Now it gives rise to the following problem.

**Problem 1.** Find all Jørgensen groups of parabolic type.

**1.7.** Let $\langle A, B\rangle$ be a marked two-generator group such that $A$ is parabolic. Then we can normalize $A$ and $B$ as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\sigma,\mu} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix}$$

where $\sigma \in \mathbf{C}\backslash\{0\}$ and $\mu \in \mathbf{C}$. See [4] for this normalization.

We denote by $G_{\sigma,\mu}$ the marked group generated by $A$ and $B_{\sigma,\mu}$: $G_{\sigma,\mu} = \langle A, B_{\sigma,\mu}\rangle$. We say that $(\sigma,\mu) \in \mathbf{C}\backslash\{0\} \times \mathbf{C}$ *is the point representing a marked group* $G_{\sigma,\mu}$ *and that* $G_{\sigma,\mu}$ *is the marked group corresponding to a point* $(\sigma,\mu)$.

**1.8.** In [6], Sato considered the case of $\mu = ik$ ($k \in \mathbf{R}$). Namely, he considered marked two-generator group $G_{\sigma,ik} = \langle A, B_{\sigma,ik}\rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\sigma,ik} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix}$$

where $\sigma \in \mathbf{C}\backslash\{0\}$ and $k \in \mathbf{R}$.

Now we have the following conjecture.

CONJECTURE. For any Jørgensen group $G$ of parabolic type there exists a marked group $G_{\sigma,ik}(\sigma \in \mathbf{C}\backslash\{0\}, k \in \mathbf{R})$ such that $G_{\sigma,ik}$ is conjugate to $G$.

If this conjecture is true, then it is sufficient to consider the case of $\mu = ik$ in order to find all Jørgensen groups of parabolic type. In this paper we only consider the case of $\mu = ik$.

**1.9.** Let $C$ be the following cylinder:

$$C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbf{R}\}.$$

**Theorem C** (Sato [6]). *If a marked two-generator group* $G_{\sigma,ik}(\sigma \in \mathbf{C}\backslash\{0\}, k \in$

**R**) *is a Jørgensen group, then the point* $(\sigma, ik)$ *representing* $G_{\sigma,ik}$ *lies on the cylinder* $C$.

By Theorem C, if $(\theta, k)$ is a point on the cylinder $C$, then we can set $\sigma = -ie^{i\theta}$ $(0 \leq \theta \leq 2\pi)$. If a point $(-ie^{i\theta}, ik)$ on the cylinder $C$ represents a Jørgensen group, then we say that the group is a Jørgensen group of parabolic type $(\theta, k)$.

Now it gives rise to the following problem.

**Problem 2**. Find all Jørgensen groups of parabolic type $(\theta, k)$.

We divide Jørgensen groups of this type into three parts as follows:

Part 1. $|k| \leq \sqrt{3}/2,\ 0 \leq \theta \leq 2\pi$ (finite case).

Part 2. $\sqrt{3}/2 < |k| \leq 1,\ 0 \leq \theta \leq 2\pi$ (countable infinite case).

Part 3. $1 < |k|,\ 0 \leq \theta \leq 2\pi$ (uncountable infinite case).

By some lemmas in [6], it suffices to consider the case of $0 \leq \theta \leq \pi/2$ and $k \geq 0$ in order to find Jørgensen groups of parabolic type $(\theta, k)$.

In the previous paper [4] we find all Jørgensen groups in the case where $0 \leq \theta \leq \pi/2$ and $0 \leq k \leq \sqrt{3}/2$, that is, we obtain the following theorem.

**Theorem D (finite case)** (Li - Oichi - Sato [4]). (i) *There are sixteen Jørgensen groups in* $D = \{(\theta, k) \in \mathbf{R} |\ 0 \leq \theta \leq \pi/2, 0 \leq k \leq \sqrt{3}/2\ \}$.
(ii) *Nine of them are Kleinian groups of the first kind and seven groups are of the second kind.*

**1.10.** For a sufficient condition for a subgroup of the Möbius transformation group to be discrete, the following theorem is well-known.

**Theorem E** (Poincaré's Polyhedron Theorem (Maskit [5, *p*.73])).

*Let* $P$ *be a polyhedron with side pairing transformations satisfying the following conditions* (1) *through* (6). *Then,* $G$, *the group generated by the side pairing trans-*

*formations, is discrete and $P$ is a fundamental polyhedron for $G$, and the reflection relations and cycle relations form a complete set of relations for $G$:*

(1) *For each side $s$ of $P$ , there is a side $s'$ and there is an element $g_s \in G$ satisfying $g_s(s) = s'$ and $g_{s'} = g_s^{-1}$.*

(2) $g_s(P) \cap P = \emptyset$.

(3) *For every point $z \in P^*, p^{-1}(z)$ is a finite set. Here $P^*$ is the space of equivalence classes so that the projection $p : \bar{P}$ (the closure of $P$) $\to P^*$ is continous and open.*

(4) *Let $e$ be an edge and let $h$ be the cycle transformation at $e$. Then for each edge $e$, there is a positive integer $t$ such that $h^t = 1$.*

(5) *Let $\{e_1, e_2, \dots, e_m\}$ be any cycle of edges of $P$ and let $\alpha(e_k)$ $(k = 1, 2, \dots, m)$ be the angle measured from inside $P$ at the edge $e_k$. Let $q$ be the smallest positive integer such that $h^q = 1$, where $h$ is the cycle transformation at $e_1$. Then the equality*

$$\sum_{k=1}^{m} \alpha(e_k) = 2\pi/q$$

*holds.*

(6) *$P^*$ is complete.*

## 2. Theorems.

In this section we will state that we find all Jørgensen groups in Part 2, that is, we obtain the following theorems. The proofs will appear elsewhere.

**Main Theorem.** *There are countable infinite many Jørgensen groups on the region $\{(\theta, k) \mid 0 \leq \theta \leq \pi/2, \sqrt{3}/2 < k \leq 1\}$.*

For simplicity we write $B_{\theta,k}$ for $B_{-ie^{i\theta},ik}$.

This theorem consists of the following Theorem 1 through Theorem 6.

Let $A$ and $B_{\theta,k}$ $(k \in \mathbf{R}, 0 \le \theta \le \pi/2)$ be the following matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\theta,k} = \begin{pmatrix} ke^{i\theta} & ie^{-i\theta}(k^2e^{2i\theta} - 1) \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}$$

We can prove these theorems by using Jørgensen's inequality and Poincaré's polyhedron theorem.

**Theorem 1** (Li - Oichi - Sato [4]). *Let $G_{\theta,k} = \langle A, B_{\theta,k}\rangle$ be the group generated by $A$ and $B_{\theta,k}$. If $0 < \theta < \pi/6$, $\pi/6 < \theta < \pi/4$, $\pi/4 < \theta < \pi/3$, $\pi/3 < \theta < \pi/2$, then $G_{\theta,k} = \langle A, B_{\theta,k}\rangle$ are not Kleinian groups and so not Jørgensen groups for $k \in \mathbf{R}$.*

**Theorem 2** (the case of $\theta = 0$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{0,k} = \begin{pmatrix} k & i(k^2 - 1) \\ -i & k \end{pmatrix} \quad (k \in \mathbf{R}),$$

*and let $G_k = \langle A, B_k\rangle$. Then the following hold.*

(i) *In the case where $\cos(\pi/2m) < k < \cos(\pi/(2m+2))$ and $k \ne \cos(\pi/(2m+1))$ $(m = 3, 4, \cdots)$, $G_k$ are not Kleinian groups and not Jørgensen groups.*

(ii) *In the case of $k = 1$, $G_k$ is a Kleinian group of the second kind and a Jørgensen group, and $\Omega(G_k)/G_k$ is a union of two Riemann surfaces with signature $(0; 2, 3, \infty)$.*

(iii) *In the case of $k = \cos(\pi/n)$ $(n = 7, 8, \dots)$, $G_k$ is a Kleinian group of the second kind and a Jørgensen group, and $\Omega(G_k)/G_k$ is a union of two Riemann surfaces with signatures $(0; 2, 3, n)$ and $(0; 2, 3, \infty)$.*

**Theorem 3** (the case of $\theta = \pi/6$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/6,k} = \begin{pmatrix} ke^{\pi i/6} & i(k^2e^{\pi i/6} - e^{-\pi i/6}) \\ -ie^{\pi i/6} & ke^{\pi i/6} \end{pmatrix} \quad (k \in \mathbf{R}),$$

*and let* $G_k = \langle A, B_k \rangle$. *Then* $G_k$ *are not Kleinian groups and not Jørgensen groups for* $k$ *with* $\sqrt{3}/2 < k \leq 1$.

**Theorem 4** (the case of $\theta = \pi/4$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/4,k} = \begin{pmatrix} ke^{\pi i/4} & i(k^2 e^{\pi i/4} - e^{-\pi i/4}) \\ -ie^{\pi i/4} & ke^{\pi i/4} \end{pmatrix} \quad (k \in \mathbf{R}),$$

*and let* $G_k = \langle A, B_k \rangle$. *Then the following hold.*

(i) *In case of* $\sqrt{3}/2 < k < 1$, $G_k$ *are not Kleinian groups and not Jørgensen groups.*

(ii) *In the case of* $k = 1$, $G_k$ *is a Kleinian group of the first kind and a Jørgensen group. The volume* $V(G_{\pi/4,1})$ *of the 3-orbifold for* $G_{\pi/4,1}$ *is*

$$V(G_{\pi/4,1}) = 8[2L(\pi/4) - L(\pi/12) - L(5\pi/12)],$$

*where* $L(\theta)$ *is the Lobachevskiĭ function:*

$$L(\theta) = -\int_0^\theta \log|2\sin u|\, du.$$

**Theorem 5** (the case of $\theta = \pi/3$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/3,k} = \begin{pmatrix} ke^{\pi i/3} & i(k^2 e^{\pi i/3} - e^{-\pi i/3}) \\ -ie^{\pi i/3} & ke^{\pi i/3} \end{pmatrix} \quad (k \in \mathbf{R}),$$

*and let* $G_k = \langle A, B_k \rangle$. *Then* $G_k$ *are not Kleinian groups and not Jørgensen groups for* $k$ *with* $\sqrt{3}/2 < k \leq 1$.

**Theorem 6** (the case of $\theta = \pi/2$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/2,k} = \begin{pmatrix} ik & -(k^2+1) \\ 1 & ik \end{pmatrix} \quad (k \in \mathbf{R}),$$

*and let* $G_k = \langle A, B_k \rangle$. *Then the following hold.*

(i) *In the case where* $\cos(\pi/(2n-1)) < k < \cos(\pi/(2n+1))$ *and* $k \neq \cos(\pi/2n)$ ( $n = 3, 4, \dots$), $G_k$ *are not Kleinian groups and not Jørgensen groups.*

(ii) *In the case of* $k = 1$, $G_k$ *is a Kleinian group of the second kind and a Jørgensen group, and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 3, 3, \infty)$.

(iii) *In the case of* $k = \cos(\pi/n)$ $(n = 7, 8, \dots)$, $G_k$ *are Kleinian groups of the second kind and Jørgensen groups, and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 3, 3, n)$.

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