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Remarks on the Perturbed Euler equations

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Abstract

We consider two type of perturbations of the Euler equations for inviscid incompressible fluid flows in $\mathbb{R}^n$, $n \geq 2$. We present global well-posedness result of these perturbed Euler system in the Triebel-Lizorkin spaces for intial vorticity which is small in the critical Triebel-Lizorkin norms. Comparison type of theorems are obtained between the Euler system and its perturbations.

1 Introduction and Main Results

We are concerned with the perturbations of the following Euler equations for the homogeneous incompressible fluid flows.

$$
\text{(E)} \begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p, \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \\ \quad \text{div } v = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \\ v(x,0) = v_0(x), \quad x \in \mathbb{R}^n \end{cases}
$$

where $v = (v_1, \cdots, v_n)$, $v_j = v_j(x,t)$, $j = 1, \cdots, n$, is the velocity of the flow, $p = p(x,t)$ is the scalar pressure, and v_0 is the given initial velocity, satisfying div $v_0 = 0$. The local well-posedness of solution is established by many authors in various function spaces[14, 15, 16, 7, 21, 22, 3, 4, 5]. The question of finite(or infinite) time blow-up of such local regular solution of (E) is an outstanding open problem in the mathematical fluid mechanics. One of the most significant achievements in this direction is the celebrated Beale-Kato-Majda(BKM) criterion for the blow-up of solutions[2], which states

$$\limsup_{t\nearrow T_*} \|v(t)\|_{H^m} = \infty \qquad \text{if and only if} \qquad \int_0^{T_*} \|\omega(s)\|_{L^\infty} ds = \infty,$$

where ω =curl v is the vorticity of the flows. Bahouri and Dehman also obtained similar blow-up criterion in the Hölder space[1]. Recently the BKM criterion has been refined by Kozono and Taniuch[17], replacing the L^∞ norm of vorticity by the BMO norm, and by the author of this paper[3], replacing the L^∞ norm of vorticity by $\dot{F}^0_{\infty,\infty}$ norm and the Sobolev norm $\|u(t)\|_{H^m}$ by the Triebel-Lizorkin norm $\|u(t)\|_{F^s_{p,q}}$ respectively. We note here that $L^\infty \hookrightarrow BMO \hookrightarrow \dot{F}^0_{\infty,\infty}$, and $H^m(\mathbb{R}^n) = F^m_{2,2}$. We also mention that there is a geometric type of blow-up criterion, using deep structure of the nonlinear term of the Euler equation[10].
In this paper we study the well-posednes/blow-up problems for perturbations of the Euler equations, which are supposed to closer to the original Euler system than the usual Navier-Stokes perturbation. In order to optimize the results we use the Triebel-Lizorkin spaces.

Our first perturbation of (E) is the following:

$$\text{(aE)} \begin{cases} \dfrac{\partial u}{\partial t} + a(t)(u\cdot\nabla)u = -\nabla q, \quad (x,t)\in\mathbb{R}^n\times(0,\infty) \\ \text{div } u = 0, \quad (x,t)\in\mathbb{R}^n\times(0,\infty) \\ u(x,0) = u_0(x), \quad x\in\mathbb{R}^n \end{cases}$$

where $u(x,t), q = q(x,t)$ are similar to the above, and u_0 is a given initial vector field satisfying div $u_0 = 0$. $a(t) > 0$ is a given continuous real valued function on $[0,\infty)$. If we set $a(t) \equiv 1$, then the system (aE) reduces to the well-known Euler equations for homogeneous incompressible fluid flows. Below we will impose the condition that $a(\cdot) \in L^1(0,\infty)$. We observe that if we choose e.g. $a(t) = 1$ for $t \in [0,t_0]$, and $a(t) = \frac{1}{1+(t-t_0)^2}$ for $t \in (t_0,\infty)$, then the system (aE) coincides with (E) during the time interval $[0,t_0]$, and distorts from (E) after that. For the system (aE) we have the following small data global existence result. For an introduction to the function spaces we use below, we summarized basic facts about about the Triebel-Lizorkin and the Besov spaces in the Appendix. The detailed proofs of the results below are in [6].

Theorem 1.1 *Let $s > n/p$, with $(p,q) \in [1,\infty]^2$, or $s = n$ with $p = 1$, $q \in [0,\infty]$. Suppose $a(\cdot) \in L^1(0,\infty)$. There exists an absolute constant $C_0 > 0$ such that if initial vorticity $\omega_0 \in F^s_{p,q}$ satisfies*

$$\|\omega_0\|_{\dot{B}^0_{\infty,1}} < \left(C_0 \int_0^\infty a(t)dt\right)^{-1},$$

then a global unique solution $u \in C([0,\infty); F^{s+1}_{p,q})$ *of (aE) exists. Moreover, the solution satisfies the estimate*

$$\sup_{0\le t<\infty} \|\omega(t)\|_{F^s_{p,q}} \le \|\omega_0\|_{F^s_{p,q}} \exp\left(\frac{C_0 \int_0^\infty a(t)dt \|\omega_0\|_{\dot{B}^0_{\infty,1}}}{1 - C_0 \int_0^\infty a(t)dt \|\omega_0\|_{\dot{B}^0_{\infty,1}}} \right). \tag{1.1}$$

Remark 1.1 Since $W^{s,p}(\mathbb{R}^n) = F^s_{p,2}$ is the usual fractional order Sobolev space, Theorem 1.1 implies immediately the global well-posedness of (aE) in $W^{s,p}(\mathbb{R}^n)$ for initial data $u_0 \in W^{s,p}(\mathbb{R}^n)$ with $\|\omega_0\|_{\dot{B}^0_{\infty,1}}$ sufficiently small. We emphasize here that we need smallness only for $\dot{F}^0_{\infty,1}$ norm of vorticity. In view of the embedding $\dot{F}^0_{\infty,1} \hookrightarrow L^\infty$(see Lemma 2.1 below), it would be interesting to extend the above result to the case with smallness assumption on $\|\omega_0\|_{L^\infty}$.

The following theorem states the equivalence of local existence of the Euler system with the global existence of the perturbed system with suitable modification of initial data.

Theorem 1.2 *The solution* v^E *of the Euler system (E) with the initial data* v^E_0 *blows up at* $t = T_* < \infty$ *in* $F^s_{p,q}$, *namely*

$$\limsup_{t\to T_*} \|v^E(t)\|_{F^s_{p,q}} = \infty, \tag{1.2}$$

if and only if for solution u *of* (E_ε) *associated with the initial data*

$$u_0(x) = \frac{T_*}{\int_0^\infty a(s)ds} v^E_0(x)$$

we have

$$\int_0^\infty \|\omega(t)\|_{\|\dot{B}^0_{\infty,1}} a(t)dt = \infty \tag{1.3}$$

for $s > n/p + 1$, $(p,q) \in [1,\infty]^2$, *while*

$$\int_0^\infty \|\omega(t)\|_{\dot{B}^0_{\infty,1}} a(t)dt = \infty \tag{1.4}$$

for $s = n+1$, $p = 1$, $q \in [1,\infty]$ *respectively.*

Remark 1.2 As in Remark 1.1 we can replace $\|v^E(t)\|_{F^s_{p,q}}$ by $\|v^E(t)\|_{W^{s,p}(\mathbb{R}^n)}$ in (1.2). Also, since $L^\infty \hookrightarrow BMO \hookrightarrow \|_{\dot{B}^0_{\infty,1}}$, we can replace the norm, $\|\omega(t)\|_{bdn}$ by $\|\omega(t)\|_{BMO}$, or $\|\omega(t)\|_{L^\infty}$ in (1.3).

Remark 1.3 By following exactly the same procedure as in [3] and [4] it is easy to find that the following blow-up criterion holds for the system (aE): The solution $u(t)$ of the system (aE) blows up at $t = T_* < \infty$ in $F^s_{p,q}$, namely

$$\limsup_{t\to T_*} \|u(t)\|_{F^s_{p,q}} = \infty, \tag{1.5}$$

if and only if

$$\int_0^{T_*} \|\omega(t)\|_{\|\dot{B}^0_{\infty,1}} a(t)dt = \infty \tag{1.6}$$

for $s > n/p + 1$, $(p, q) \in [1, \infty]^2$, while

$$\int_0^{T_*} \|\omega(t)\|_{\dot{B}^0_{\infty,1}} a(t)dt = \infty \tag{1.7}$$

for $s = n + 1$, $p = 1$, $q \in [1, \infty]$ respectively. Thus, the conditions (1.3) and (1.4), in turn, are equivalent to the blow-up of solution $u(t)$ of (aE) at infinite time, namely

$$\limsup_{t\to\infty} \|u(t)\|_{F^s_{p,q}} = \infty. \tag{1.8}$$

Next, we consider the following 'damping' perturbation of the Euler equations:

$$(\mathrm{E})_\varepsilon \begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla q - \varepsilon u, \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \\ \quad \operatorname{div}\ u = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n \end{cases}$$

with $\varepsilon > 0$, which could be considered as a 'milder' perturbation of the Euler system than the usual Navier-Stokes system. We will see below that the system $(\mathrm{E})_\varepsilon$ can be treated as a special case of (aE). Applying Theorem 1.1 and 1.2, we establish the following two results regarding $(\mathrm{E})_\varepsilon$.

Corollary 1.1 *Let $s > n/p$, with $(p, q) \in [1, \infty]^2$, or $s = n$ with $p = 1$, $q \in [0, \infty]$. There exists an absolute constant $C_1 > 0$ such that if initial vorticity $\omega_0 \in F^s_{p,q}$ and the 'viscosity' ε satisfies*

$$\|\omega_0\|_{\dot{B}^0_{\infty,1}} < \frac{\varepsilon}{C_1},$$

then global unique solution $u \in C([0, \infty); F^{s+1}_{p,q})$ of $(E)_\varepsilon$ exists. Moreover the solution satisfies the estimate

$$\sup_{0\le t<\infty} \|\omega(t)\|_{F^s_{p,q}} \le \|\omega_0\|_{F^s_{p,q}} \exp\left(\frac{C_1\|\omega_0\|_{\dot{B}^0_{\infty,1}}}{\varepsilon - C_1\|\omega_0\|_{\dot{B}^0_{\infty,1}}}\right). \tag{1.9}$$

Similar remark to Remark 1.1, concerning the changes of the function spaces into to the more familiar spaces such as $W^{s,p}(\mathbb{R}^n)$, also holds for Corollary 1.1.

Corollary 1.2 *The solution v^E of the Euler system (E) blows up at $t = T_* < \infty$ in $F^s_{p,q}$, namely*

$$\limsup_{t\to T_*} \|v^E(t)\|_{F^s_{p,q}} = \infty, \tag{1.10}$$

if and only if for solution u of (E_ε) with $\varepsilon = \frac{\lambda}{T_*}$ we have

$$\int_0^\infty \|\omega(t)\|_{\|\dot{B}^0_{\infty,1}} dt = \infty \tag{1.11}$$

for $s > n/p + 1$, $(p,q) \in [1,\infty]^2$, while

$$\int_0^\infty \|\omega(t)\|_{\dot{B}^0_{\infty,1}} dt = \infty \tag{1.12}$$

for $s = n+1$, $p = 1$, $q \in [1,\infty]$ respectively.

Remark 1.4 In terms of the usual Sobolev spaces, $H^m(\mathbb{R}^n)$ with $m > \frac{n}{2}+1$, Corollary 1.2 implies that if we have local solution $v^E \in C([0,T]; H^m(\mathbb{R}^n))$ to the problem (E) with initial data v_0^E, then necessarily we have global solution $u \in C([0,\infty); H^m(\mathbb{R}^n))$ of $(\mathrm{E})_\varepsilon$ with the initial data $u_0 = \lambda v_0^E$, and $\varepsilon = \frac{\lambda}{T}$. This resembles the comparison type of result between the Euler equations and the Navier-Stokes equations obtained by Constantin(See Theorem 1.1[9]).

As a model problem of the perturbed Euler equation we also consider the Constantin-Lax-Majda equation[11] first considered in [11]:

$$(\mathrm{CLM}) \begin{cases} \omega_t - H(\omega)\omega = 0 & (x,t) \in \mathbb{R}\times\mathbb{R}^+ \\ \omega(x,0) = \omega_0(x) & x \in \mathbb{R} \end{cases}$$

with $\omega = \omega(x,t)$ a scalar function, and $H(f)$ is the Hilbert transform of f defined by

$$H(f) = \frac{1}{\pi} PV \int \frac{f(y)}{x-y} dy. \tag{1.13}$$

For the problem (CLM), Constantin-Lax-Majda derived the following explicit solution[11](see also Section 5.2.1 of [19]):

$$\omega(x,t) = \frac{4\omega_0(x)}{(2 - tH\omega_0(x))^2 + t^2\omega_0^2(x)}. \tag{1.14}$$

The perturbed equation we are concerned is

$$(\mathrm{CLM})_\varepsilon \begin{cases} \sigma_t - H(\sigma)\sigma = -\varepsilon\sigma & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ \sigma(x,0) = \sigma_0(x) & x \in \mathbb{R} \end{cases}$$

We have the following relation between the two solutions:

$$\begin{aligned} \sigma(x,t) &= e^{-\varepsilon t}\omega\left(x, \frac{1}{\varepsilon}(1-e^{-\varepsilon t})\right), \\ \sigma_0(x) &= \omega_0(x). \end{aligned} \tag{1.15}$$

Combining (1.15) with (1.14), we easily obtain the following explicit solution of $(CLM)_\varepsilon$:

$$\sigma(x,t) = \frac{4\varepsilon^2\sigma_0(x)e^{-\varepsilon t}}{(2\varepsilon - H\sigma_0(x)(1-e^{-\varepsilon t}))^2 + (1-e^{-\varepsilon t})^2\sigma_0^2(x)}. \tag{1.16}$$

The formula leads us to the following proposition:

Proposition 1.1 *In case $H\sigma_0(x) \le 0$ for all $x \in \mathbb{R}$ there is no blow up of solution. Otherwise, we consider the three cases. Let us put $S = \{x \in \mathbb{R} : \sigma_0(x) = 0, H\sigma_0(x) > 0\}$.*

(i) If $\varepsilon > \frac{1}{2}\sup_{x\in S} H\sigma_0(x)$, then there is no blow-up.

(ii) If there exists $x \in S$ such that $\varepsilon < \frac{1}{2}H\sigma_0(x)$, then solution blows up at T_* given by

$$T_* = \frac{1}{\varepsilon}\ln\left(1 - \frac{2\varepsilon}{\sup_{x\in S} H\sigma_0(x)}\right)^{-1}. \tag{1.17}$$

(iii) If the set $S_1 = \{x \in \mathbb{R} : \sigma_0(x) = 0,\ \varepsilon = \frac{1}{2}H\sigma_0(x)\}$ is nonempty, and if for all $x \in \mathbb{R}\backslash S_1$ we have $\varepsilon > \frac{1}{2}H\sigma_0(x)$, or $\sigma_0(x) > 0$, then the solution blows up at $t = +\infty$.

Remark 4.1 We note that (i),(iii) above are the new phenomena of $(\mathrm{CLM})_\varepsilon$ not occurred in (CLM).

2 Appendix: Function spaces

We first set our notations, and recall definitions of the Triebel-Lizorkin spaces. We follow [20]. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}$, its Fourier transform $\mathcal{F}(f) = \hat{f}$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

We consider $\varphi \in \mathcal{S}$ satisfying $\operatorname{Supp}\hat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid \frac{1}{2} \le |\xi| \le 2\}$, and $\hat{\varphi}(\xi) > 0$ if $\frac{1}{2} < |\xi| < 2$. Setting $\hat{\varphi}_j = \hat{\varphi}(2^{-j}\xi)$ (In other words, $\varphi_j(x) = 2^{jn}\varphi(2^j x)$.), we can adjust the normalization constant in front of $\hat{\varphi}$ so that

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given $k \in \mathbb{Z}$, we define the function $S_k \in \mathcal{S}$ by its Fourier transform

$$\hat{S}_k(\xi) = 1 - \sum_{j\ge k+1} \hat{\varphi}_j(\xi).$$

Let $s \in \mathbb{R}$, $p, q \in [0,\infty]$. Given $f \in \mathcal{S}'$, we denote $\Delta_j f = \varphi_j * f$, and then the homogeneous Triebel-Lizorkin semi-norm $\|f\|_{\dot{F}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{F}^s_{p,q}} = \begin{cases} \left\| \left(\sum_{j\in\mathbb{Z}} 2^{jqs} |\Delta_j f(\cdot)|^q \right) \right\|_{L^p} & \text{if } q \in [1,\infty) \\ \left\| \sup_{j\in\mathbb{Z}} (2^{js} |\Delta_j f(\cdot)|) \right\|_{L^p} & \text{if } q = \infty \end{cases}.$$

The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ is a qausi-normed space with the quasi-norm given by $\|\cdot\|_{\dot{F}^s_{p,q}}$. For $s > 0$, $(p,q) \in [1,\infty]^2$ we define the inhomogeneous Triebel-Lizorkin space norm $\|f\|_{F^s_{p,q}}$ of $f \in \mathcal{S}'$ as

$$\|f\|_{F^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{F}^s_{p,q}}.$$

The inhomogeneous Triebel-Lizorkin space is a Banach space equipped with the norm, $\|\cdot\|_{\dot{B}^0_{\infty,1}}$. Similarly, the homogeneous Besov norm $\|f\|_{\|\dot{\cdot}^s B_{p,q}}$ is defined by

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left[\sum_{-\infty}^{\infty} 2^{jqs} \|\varphi_j * f\|^q_{L^p} \right]^{\frac{1}{q}} & \text{if } q \in [1,\infty) \\ \sup_j \left[2^{js} \|\varphi_j * f\|_{L^p} \right] & \text{if } q = \infty \end{cases}.$$

The homogeneous Besov space $\dot{B}^s_{p,q}$ is a quasi-normed space with the quasi-norm given by $\|\cdot\|_{\dot{B}^s_{p,q}}$. For $s > 0$ we define the inhomogeneous Besov space norm $\|f\|_{B^s_{p,q}}$ of $f \in \mathcal{S}'$ as $\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}$.

Lemma 2.1 *Let $s \in (0,n)$, $p, q \in [1,\infty]$ and $sp = n$. Then the following sequence of continuous embeddings hold.*

$$\dot{F}^n_{1,q} \hookrightarrow \dot{B}^s_{p,1} \hookrightarrow \dot{B}^0_{\infty,1} \hookrightarrow \dot{F}^0_{\infty,1} \hookrightarrow L^\infty. \tag{2.1}$$

The first imbedding of Lemma 2.1 is proved in [12], while the second one is proved in [4]. The others are obvious from the definitions of the corresponding norms.

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References

[1] H. Bahouri and B. Dehman, *Remarques sur l'apparition de singularités dans les écoulements Eulériens incompressibles à donnée initiale Höldérienne,* J. Math. Pures Appl., **73**, (1994), pp. 335-346.

[2] J.T. Beale, T. Kato, and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations,* Commun. Math. Phys., **94**, (1984), pp. 61-66.

[3] D. Chae, *On the well-posedness of the Euler equations in the Triebel-Lizorkin spaces,* Commun. Pure and Appl. Math. **55**, no. 5, (2002), pp. 654-678.

[4] D. Chae, *The quasi-geostrophic equation in the Triebel-Lizorkin spaces,* Nonlinearity, **16**, (2003), pp. 479-495.

[5] D. Chae, *On the Euler equations in the critical Triebel-Lizorkin spaces,* to appear.

[6] D. Chae, *On the incompressible Euler system and its perturbations,* RIM-GARC Preprint 02-8.

[7] J. Y. Chemin, *Régularité des trajectoires des particules d'un fluide incompressible remplissant l'espace,* J. Math. Pures Appl., **71**(5) , (1992), pp. 407-417.

[8] J. Y. Chemin, *Perfect incompressible fluids,* Clarendon Press, Oxford, (1998).

[9] P. Constantin, *Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations,* Comm. Math. Phys. **104**, (1986), pp. 311-326.

[10] P. Constantin, C. Fefferman, and A. Majda, *Geometric constraints on potential singularity formation in the 3-D Euler equations,* Commun. P. D. E., **21**, (3-4), (1996), pp. 559-571.

[11] P. Constantin, P. D. Lax, and A. Majda, *A simple one-dimensional model for the three-dimensional vorticity equations,* Commun. Pure Appl. Math. **38**, (1985), pp. 715-724.

[12] M. Frazier, R. Torres, and G. Weiss, *The boundedness of Caleron-Zygmund operators on the spaces $\dot{F}_p^{\alpha,q}$,* Revista Mathemática Ibero-Americana, **4**, no. 1, (1988), pp. 41-72.

[13] B. Jawerth, *Some observations on Besov and Lizorkin-Triebel Spaces,* Math. Sacand., **40**, (1977), pp. 94-104.

[14] T. Kato, *Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$*, J. Functional Anal. **9**, (1972), pp. 296-305.

[15] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations,* Commun. Pure Appl. Math., **41**, (1988), pp. 891-907.

[16] T. Kato and G. Ponce, *Well posedness of the Euler and Navier-Stokes equations in Lebesgue spaces $L_p^s(\mathbb{R}^2)$*, Revista Mathemática Ibero-Americana, **2**, (1986), pp. 73-88.

[17] H. Kozono and Y. Taniuchi, *Limiting case of the Sobolev inequality in BMO, with applications to the Euler equations,* Comm. Math. Phys., **214**, (2000), pp. 191-200.

[18] H. Kozono, T. Ogawa, and T. Taniuchi, *The critical Sobolev inequalities in Besov spaces and regularity criterion to some semilinear evolution equations,* to appear in Math Z. (2002).

[19] A. Majda and A. Bertozzi *Vorticity and Incompressible Flow,* Cambridge Univ. Press,(2002).

[20] H. Triebel, *Theory of Function Spaces,* Birkhäuser, (1983).

[21] M. Vishik, *Hydrodynamics in Besov spaces,* Arch. Rational Mech. Anal, **145**, (1998), pp. 197-214.

[22] M. Vishik, *Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type,* Ann. Scient. Éc. Norm. Sup., 4^e série, t. 32, (1999), pp. 769-812.