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Approximations of reaction-diffusion equations by interface equations

- boundary-interior layer -

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1 Introduction.

We deal with transition layers of the following scalar reaction-diffusion equation

$$
\text{(1.1)} \qquad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f(u) & (\text{in } \Omega,\, t>0) \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & (\text{on } \partial\Omega,\, t>0) \end{cases}
$$

with the homogeneous Neumann boundary conditions. This system, called the Allen-Cahn equation, has been studied extensively for bistable reaction kinetics. A typical example of the nonlinearity f is a cubic polynomial $f(u) = u - u^3$. In general, we assume that the nonlinearity f is obtained from a double-well potential $F(u)$ of equal depth. Namely, $f(u) = -F'(u)$ with $F(u) \geq 0$ attains its absolute minimum at exactly two *non-degenerate* critical points $u = \pm 1$ (non-degereracy here means that $F''(\pm 1) > 0$). These conditions ensure the existence of a special solution $Q(z)$ $(z \in \mathbb{R})$, called a standing wave solution, which satisfies

$$
\text{(S-W)} \qquad \frac{d^2 Q}{dz^2} + f(Q) = 0, \quad z \in \mathbb{R}, \qquad \lim_{z \to \pm\infty} Q(z) = \pm 1, \quad Q(0) = 0.
$$

The function $Q(z)$ will play important roles in this paper.

The domain Ω is a smooth, bounded one in $\mathbb{R}^N$, $\mathbf{n}$ stands for the unit *inward* normal vector on $\partial\Omega$, and the parameter $\varepsilon > 0$ is small.

Our main concern in this paper is to show the existence of internal transition layers which exhibit a sharp transition from $u \approx -1$ to $u \approx +1$ across such a hypersurface Γ that intersects the boundary of the domain; $\overline{\Gamma} \cap \partial\Omega \neq \emptyset$. We call this kind of internal transition layer a *boundary-interior layer.* We also analyze the stability property of boundary-interior layers by using some geometric information of $\Gamma, \partial\Omega$ and $\partial\Gamma \subset \partial\Omega$.

When $\varepsilon > 0$ is small, the solutions of (1.1) for a class of initial functions are known to develop transition layers within a short time scale of $O(\varepsilon^2 |\log \varepsilon|)$ [3]. This phenomenon is caused by the strong bistability of the ordinary differential equation

$u_t = \frac{1}{\varepsilon^2} f(u)$ with $u = \pm 1$ being stable equilibria. According to the sign of the value of the initial function, the solution is quickly attracted to either $u = +1$ or $u = -1$, thus creating a sharp transition from $u \approx -1$ to $u \approx 1$ near the set, called an interface,

$$\Gamma(t) := \{x \in \Omega \mid u^\varepsilon(u,t) = 0\}.$$

The interface divides Ω into two sub-domains $\Omega^\pm(t)$ (cf. Figure 1) defined by $\Omega^\pm(t) := \{x \in \Omega \mid \pm u^\varepsilon(x,t) > 0\}$. When $x \in \Omega^\pm(t)$, $u^\varepsilon(x,t) \to \pm 1$ as $\varepsilon \to 0$. Such solutions with sharp transition are called *transition layer* solutions.

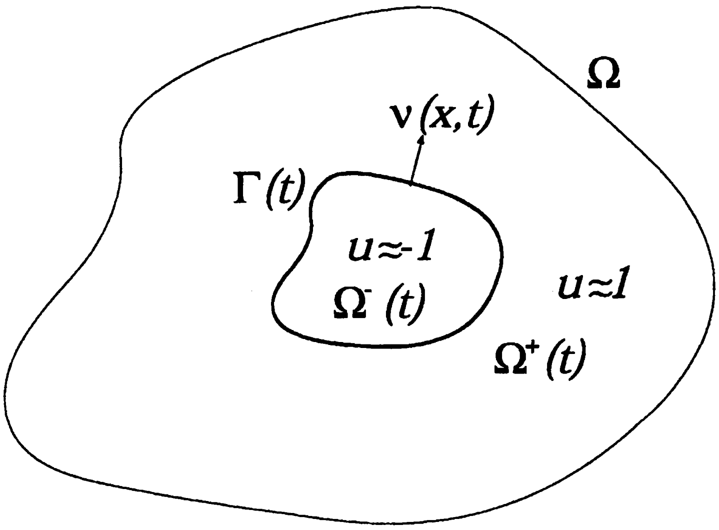


Figure 1: The i terface $\Gamma(t)$ and the normal vector $\nu(x,t)$.

It is also well known (cf. [3], for instance) that the interface $\Gamma(t)$ evolves according to its mean curvature:

$$\mathbf{V}_{\Gamma(t)}(x) = -\kappa(x;\Gamma(t)) \quad (x \in \Gamma(t),\ t > 0\ ) \tag{1.2}$$

where $\mathbf{V}_{\Gamma(t)}(x)$ is the speed of the interface measured along the unit normal $\nu(x,t)$ of $\Gamma(t)$ at x (ν points to the $\Omega^+(t)$-side, cf. Figure 1) and $\kappa(x;\Gamma)$ stands for the sum of the principal curvatures of Γ at $x \in \Gamma$. Hereafter, κ is simply called the mean curvature and the equation (1.2) is referred to as the mean curvature flow. To be precise about the sign of κ (which is the opposite to geometers' convention), let us extend the unit normal vector ν to a neighbourhood of Γ. Then our mean curvature is defined as the *divergence* of ν;

$$\kappa(x;\Gamma) = \operatorname{div} \nu(x), \quad x \in \Gamma.$$

When the interface $\Gamma(t)$ stays away from the boundary $\partial\Omega$, the dynamics of (1.2) has been studied rather extensively ([6, 8]). In such a case, the interface governed by the mean curvature flow (1.2) does not feel the presence of the boundary $\partial\Omega$. Therefore, the domain Ω does not play any role in the dynamics of (1.2).

Our concern in this paper, on the other hand, is the case where the interface $\Gamma(t)$ intersects the boundary $\partial\Omega$ (cf. Figure 2). The motion of $\Gamma(t)$ in such a situation is still described by the mean curvature flow (1.2) to the lowest order approximation. Main questions we raise in this article are:

> When (1.2) has an equilibrium interface, does it give rise to an equilibrium boundary-interiorlayer for (1.1)? If the answer is affirmative, what is it that determines the stability of the layer?

The dynamics of such interfaces intersecting the boundary of domain has been studied by several authors ([2, 13, 4, 5, 12, 15, 10]).

Since we have identified $\Gamma(t)$ as the 0-level set of the solution to (1.1), the homogeneous Neumann boundary conditions demand that $\Gamma(t)$ be perpendicular to $\partial\Omega$ at the intersection $\partial\Gamma(t) = \overline{\Gamma(t)} \cap \partial\Omega$. Therefore, the interface $\Gamma(t)$ immediately feels the presence of the boundary, and the geometry of $\partial\Omega$ influences the dynamics of (1.2).

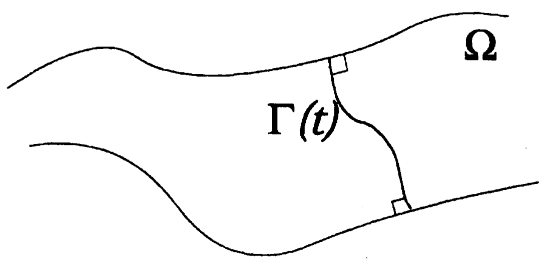


Figure 2: The interface intersecting the boundary.

The existence of energy-minimising solutions (of (1.1)) with interface intersecting the boundary was first rigorously established in [15] by a variational method. For competition-diffusion systems, stable internal layers intersecting the boundary was established in [12] for rotationally symmetric domains. Exponentially slow motions of flat interfaces are discussed in [2, 13], where interfaces intersect flat parallel part of the boundary. Motions of interfaces with contact angle was treated in [4] for a generalized mean curvature flow. Dynamics of flat interfaces in a strip-like domain was discussed in [5], where the speed of the interface is of order $O(\epsilon^2)$ with respect to the time scale of (1.1). In [10], the existence and stability of equilibrium boundary-interior layers with flat interfaces were established. Recently, the same results as [10] have been obtained by [14] via different methods. In all of these works, the geometry of the boundary $\partial\Omega$ has essential effects on the dynamics of (1.1).

The purpose of this article is to extend the results in [10] and [14] to higher-dimensional domains.

2 Review of two-dimensional results.

In this section we assume $N = 2$, and review know results according to [10].

In order to describe an equilibrium interface Γ of (1.2), let us consider the distance function L;

$$L : \partial\Omega \times \partial\Omega \longrightarrow [0, \infty), \qquad L(p, q) = \mathrm{dist}(p, q).$$

Theorem 2.1 (Equilibrium Interface). *If $(p_*, q_*) \in \partial\Omega \times \partial\Omega$ satisfies following conditions:*

(i) *(p_*, q_*) is a critical point of L;*

(ii) *$L_* := L(p_*, q_*) > 0$;*

(iii) *the open straight line-segment $\overline{p_* q_*}$ is contained in Ω,*

then, $\Gamma = \overline{p_* q_*}$ is an equilibrium interface of (1.2).

Conversely, any equilibrium of (1.2) *is characterized by these properties.*

Proof. In the two-dimensional case, $\kappa = 0$ is equivalent to Γ being a straight line. It is verified that $\partial L(p_*, q_*)/\partial p = 0$ is equivalent to $\overline{p_* q_*} \perp_{p_*} \partial\Omega$. Also, $\partial L(p_*, q_*)/\partial q = 0$ is equivalent to $\overline{p_* q_*} \perp_{q_*} \partial\Omega$. Now, the statements of the theorem follow. $\square$

We now define the curvature of $\partial\Omega$ with respect to its *inward* unit normal $\mathbf{n}$ by

$$\overline{\kappa}_p = \mathrm{div}\ \mathbf{n}(p), \quad p \in \partial\Omega.$$

Let us denote by $\overline{\kappa}_{p_*}$ and $\overline{\kappa}_{q_*}$ the cruvature of $\partial\Omega$ at the two end points of the equilibrium interface $\Gamma = \overline{p_* q_*}$. Let us define $\mathbf{D}$ and $\mathbf{T}$ by

$$\mathbf{D} := \overline{\kappa}_{p_*} + \overline{\kappa}_{q_*} + L_* \overline{\kappa}_{p_*} \overline{\kappa}_{q_*},$$
$$\mathbf{T} := 2 + L_*(\overline{\kappa}_{p_*} + \overline{\kappa}_{q_*}).$$

These quantities are related to the second variation of L. Namely, $\mathbf{T}/L_*$ and $\mathbf{D}/L_*$ are, respectively, the *trace* and *determinant* of the Hessian matrix (i.e., the second variation) of L at $(p, q) = (p_*, q_*)$.

Theorem 2.2 (Existence of boundary-interior layers). *Assume that the following non-degeneracy condition is satisfied*

$$\text{(a)} \qquad \mathbf{D} \neq 0.$$

Then there exist an $\varepsilon_* > 0$ *and a family of equilibrium solutions* $U^\varepsilon(x)$ *of* (1.1) *for* $\varepsilon \in (0, \varepsilon_*]$, *enjoying the following properties:*

(i) *For each* $\delta > 0$,

$$\lim_{\varepsilon \to 0} U^\varepsilon(x) = \begin{cases} 1 \\ -1 \end{cases} \textit{uniformly in } \begin{cases} x \in \overline{\Omega}^+, \ \mathrm{dist}(x, \Gamma) > \delta \\ x \in \overline{\Omega}^-, \ \mathrm{dist}(x, \Gamma) > \delta. \end{cases}$$

(ii) *Near the interface* Γ, *the solution* $U^\varepsilon(x)$ *has the asymptotic characterization:*

$$U^\varepsilon(x) \approx Q\left(\frac{\mathrm{dist}(x, \Gamma)}{\varepsilon}\right),$$

where $Q(z)$ $(z \in \mathbb{R})$ *is the standing wave solution in* (S-W).

Let us call such a solution as in Theorem 2.2 a *boundary-interior layer.* The stability properties of the boundary-interior layer is given in the following theorem. In Theorem 2.3 below, the *Morse index* of an equilibrium solution to (1.1) means the number of unstable (positive) eigenvalues for the eigenvalue problem

$$\text{(2.1)} \qquad \lambda\phi = \Delta\phi + \frac{1}{\varepsilon^2} f'(U^\varepsilon)\phi \quad \text{in } \Omega, \qquad \frac{\partial \phi}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega,$$

associated with the linearized operator around the equilibrium U^ε of (1.1). Also, in the same theorem, the Morse index of a critical point (p_*, q_*) of L is the number of positive eigenvalues of the Hessian matrix of $-L$ at $(p, q) = (p_*, q_*)$.

Theorem 2.3 (Stability property of boundary-interior layers). *Let* $U^\varepsilon(x)$ *be the solution in Theorem* 2.2. *As an equilibrium solution of* (1.1), *it is*

(1) *stable (the Morse index* 0*), if* $\mathbf{D} > 0$ *and* $\mathbf{T} > 0$,

(2) *unstable, if otherwise, with*

  (2-i) *the Morse index* 1 *if* $\mathbf{D} < 0$

  (2-ii) *the Morse index* 2 *if* $\mathbf{D} > 0$ *and* $\mathbf{T} < 0$.

(3) *The Morse index of the equilibrium solution described in items* (1) *and* (2) *are the same as the Morse index of the corresponding critical point* (p_*, q_*) *for the function* $L(p, q)$.

Theorems 2.2 and 2.3 says that the dynamics of boundary-interior layers are qualitatively described by the gradient system of the function L. It is easy to see that Theorems 2.2 and 2.3 are restatements of Theorems 1.3 and 1.4 in [10].

In order to gain some insights for Theorem 2.3, it is illuminating to consider the following eigenvalue problem

$$\begin{cases} \phi_{\tau\tau}(\tau) = \lambda\phi(\tau), & \tau \in (-\frac{L_*}{2}, \frac{L_*}{2}), \\ \phi_\tau(-\frac{L_*}{2}) - \overline{\kappa}_{q_*}\phi(-\frac{L_*}{2}) = 0 \\ -\phi_\tau(\frac{L_*}{2}) - \overline{\kappa}_{p_*}\phi(\frac{L_*}{2}) = 0. \end{cases} \tag{2.2}$$

This is an eigenvalue problem associated with the linearization of (1.2) on the equilibrium interface $\Gamma = \overline{p_* q_*}$. It was shown in [10] that non-critical eigenvalues of (2.1) go to $-\infty$ as $\varepsilon \to 0$ and that critical eigenvalues of (2.1) converge to the eigenvalues of (2.2). It is rather elementary to show that (2.2) has

1. no positive eigenvalues and no 0-eigenvalue if $\mathbf{D} > 0$ and $\mathbf{T} > 0$;
2. one positive eigenvalue and no 0-eigenvalue if $\mathbf{D} < 0$;
3. two positive eigenvalues and no 0-eigenvalue if $\mathbf{D} > 0$ and $\mathbf{T} < 0$;
4. one 0-eigenvalue and no positive eigenvalue if $\mathbf{D} = 0$ and $\mathbf{T} > 0$;
5. one 0-eigenvalue and one positive eigenvalue if $\mathbf{D} = 0$ and $\mathbf{T} < 0$.

Note that it is impossible to have both $\mathbf{D} = 0$ and $\mathbf{T} = 0$ satisfied. We have thus classified the stability property of the boundary-interior layer in terms of the *singular limit* (1.2) (of (1.1)) and its *linearization* (2.2).

A question naturally suggests itself;

What happens when $\mathbf{D} = 0$?

The answer seems to be:

> **Bifurcation of Boundary-Interior Layers.** *Purterbing the boundary of domain $\partial\Omega$ as a* bifurcation parameter, *static bifurcations occur from the equilibrium boundary-interior layer at* $(\mathbf{D} = 0, \mathbf{T} > 0)$ *and* $(\mathbf{D} = 0, \mathbf{T} < 0)$.

We have confirmed in [10] by numerical simulations that the last statement may be true. Its rigorous proof will be treated in a separate work.

There is another way of looking at the problem (2.2). Let us consider a *Dirichlet-Neumann map* Π_{L_*} on the interface $\Gamma = \{\tau \in \mathbb{R} \mid |\tau| < L_*/2\}$. This map sends the *Dirichlet data* $(\phi(-L_*/2), \phi(L_*/2)) = (a_-, a_+) \in \mathbb{R}^2$ to the *inward Neumann data* $(\phi'(-L_*/2), -\phi'(L_*/2)) \in \mathbb{R}^2$, where $\phi(\tau)$ is *harmonic* on Γ, i.e. $\phi_{\tau\tau} \equiv 0$. Elementary computations yield that

$$\Pi_{L_*} : \begin{pmatrix} a_- \\ a_+ \end{pmatrix} \longmapsto \frac{1}{L_*} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_- \\ a_+ \end{pmatrix}.$$

Eigenvalues of a Dirichlet-Neumann map have a close relation to the eigenvalue problem for the *Laplacian* with boundary conditions of the third type (Robin type boundary conditions). In the present situation, since the boundary of Γ is *not* connected, we can consider a little more general eigenvalue problem for Π_{L_*}. We call $(\mu^-, \mu^+) \in \mathbb{R}^2$ an *eigenvalue-pair* of Π_{L_*} if the linear equation

$$\Pi_{L_*} \begin{pmatrix} a_- \\ a_+ \end{pmatrix} = \begin{pmatrix} \mu^- & 0 \\ 0 & \mu^+ \end{pmatrix} \begin{pmatrix} a_- \\ a_+ \end{pmatrix}$$

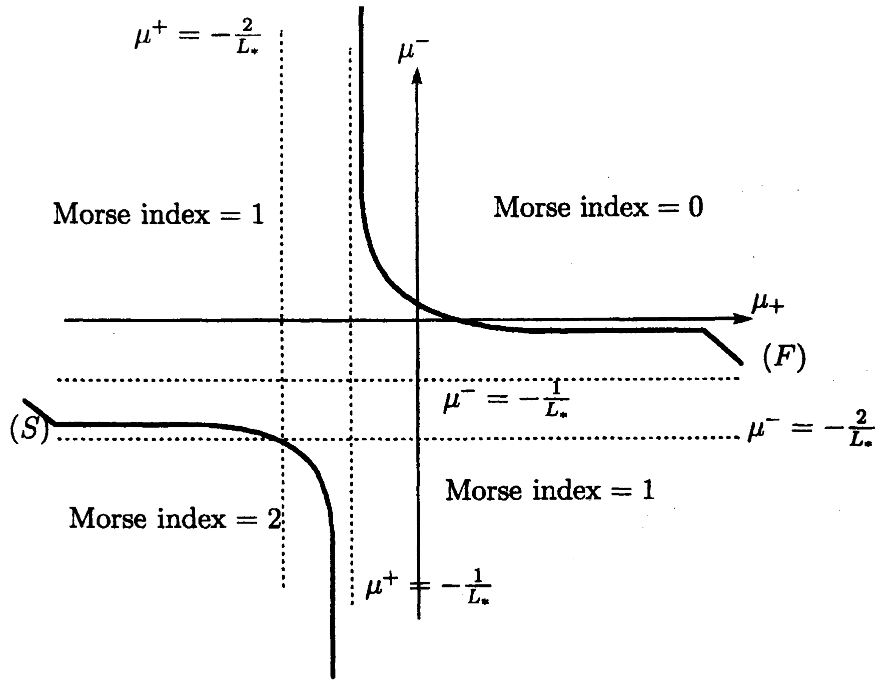
has a non-trivial solution $(a_-, a_+) \neq (0,0)$. By elementray computations, again, one can easily find that (μ^-, μ^+) is an eigenvalue-pair of Π_{L_*} if and only if

$$\text{(D)} \qquad \mathbf{D}(\mu^-, \mu^+) := \mu^- + \mu^+ + L_* \mu^- \mu^+ = 0.$$

One can immediately see that

$$\mathbf{D} = \mathbf{D}(\overline{\kappa}_{q_*}, \overline{\kappa}_{p_*}).$$

In the μ^--μ^+ plane, the equation $\mathbf{D}(\mu^-, \mu^+) = 0$ defines a hyperbola. The hyperbola has two branches, one passing through $(\mu^-, \mu^+) = (0,0)$ (call it (F)) and another passing through $(\mu^-, \mu^+) = (-2/L_*, -2/L_*)$ (call it (S)). Theorems 2.2 and 2.3 apply when the point $(\overline{\kappa}_{q_*}, \overline{\kappa}_{p_*})$ is neither on (F) nor on (S). When the point is above the (F) branch, then Theorem 2.3 (1) applies. If the point is between (F) and (S) branches, Theorem 2.3 (2-i) applies, while if it is below (S) branch, then Theorem 2.3 (2-ii) applies. As mentioned earlier, when the boundary $\partial\Omega$ is deformed so that the point $(\overline{\kappa}_{q_*}, \overline{\kappa}_{p_*})$ crosses either (F) or (S) branches, we expect that bifurcations of boundary-interior layers would occur.



Morse indices of the boundary-interior layer for 2-dimensinal domains.

3 Main results in 3-dimensional domains.

We will establish results similar to Theorems 2.1, 2.2, and 2.3 for 3-dimensional domains. It turns out that to prove an analogue of Theorem 2.1 is the most difficult part for 3-dimensional domains. We will show that once an analogue of Theorem 2.1 is obtained then counterparts of Theorems 2.2 and 2.3 will follow rather easily by the method employed in [10].

3.1 Rotationary-symmetric domains.

We first consider a special class of domains; *rotationally symmetric domains.* Let the axis of rotation be in x-direction ($x \in \mathbb{R}$ here and below within §3.1), and consider a domain $\Omega \subset \mathbb{R}^3$ which (or, part of which) is obtained by rotating the graph of a *positive* function $\psi(x)$ around x-axis:

(3.1) $$\Omega = \{(x, y) \in \mathbb{R}^3 \mid y \in \mathbb{R}^2, \ |y| < \psi(x)\}.$$

In this situation it is easy to find an equilibrium to (1.2).

Theorem 3.1 (Existence of flat disk-type interfaces). *Let* $x_0 \in \mathbb{R}$ *satisfy*

$\psi'(x_0) = 0$. *Then the disk* $\Gamma = \{(x_0, y) \mid |y| < \psi(x_0)\}$ *is an equilibrium solution of* (1.2).

In order to state counterparts of Theorems 2.2 and 2.3, let us define the **Dirichlet-to-Neumann map** Π for the Laplacian:

$$\Pi : C^{2+\alpha}(S_0) \longrightarrow C^{1+\alpha}(S_0); \qquad \Pi\phi(y) := \frac{\partial v}{\partial \mathbf{n}}(y), \quad y \in S_0, \tag{3.2}$$

where $S_0 := \{y \in \mathbb{R}^2 \mid |y| = \psi(x_0)\}$ and $v(y)$ is the unique solution of the boundary value problem:

$$\Delta_y v = 0, \quad y \in \omega := \{|y| < \psi(x_0)\}, \qquad v(y) = \phi(y), \quad y \in S_0. \tag{3.3}$$

To a given *Dirichlet data* $\phi \in C^{2+\alpha}(S_0)$ on S_0, the map asigns the *Neumann data* $\partial v/\partial \mathbf{n}$ of the harmonic extension v of ϕ. It is known that the map Π is a first order *elliptic* operator on S_0. The operator is approximately given by

$$\Pi \approx -\sqrt{-\Delta^{S_0}},$$

and extends to an unbounded operator on $L^2(S_0)$. Let us denote by $\sigma(\Pi)$ the set of eigenvalues of Π:

$$\sigma(\Pi) = \{\mu_j\}_{j=0}^{\infty}; \qquad 0 = \mu_0 > \mu_1 > \ldots > \mu_j > \ldots \to -\infty, \tag{3.4}$$

where we only listed *distinct* eigenvalues. We denote by m_j the *multiplicity* of μ_j. In the present situation one can easily compute these eigenvalues; $\mu_j = -j/\psi(x_0)$ $(j \geq 0)$ and $m_0 = 1$, $m_j = 2$ $(j \geq 1)$.

We are ready to state:

Theorem 3.2 (Existence of boundary-interior layers). *Assume that* x_0 *is such that* $\psi'(x_0) = 0$ *and the following non-degeneracy condition is satisfied*

$$\text{(a):} \qquad \psi''(x_0) \notin \sigma(\Pi).$$

Then there exist an $\varepsilon_* > 0$ *and a family of equilibrium solutions* $U^\varepsilon(x, y)$ *of* (1.1) *for* $\varepsilon \in (0, \varepsilon_*]$, *enjoying the following properties:*

(i) *For each* $\delta > 0$,

$$\lim_{\varepsilon \to 0} U^\varepsilon(x, y) = \begin{cases} 1 \\ -1 \end{cases} \textit{uniformly in} \begin{cases} (x, y) \in \bar{\Omega},\ x \leq x_0 - \delta, \\ (x, y) \in \bar{\Omega},\ x \geq x_0 + \delta. \end{cases}$$

(ii) *Near $x = x_0$, the solution $U^\varepsilon(x, y)$ has the asymptotic characterization:*

$$U^\varepsilon(x,y) \approx Q\left(\frac{x - x_0}{\varepsilon}\right).$$

As for the stability property of the solution, we have:

Theorem 3.3 (Stability Property of boundary-interior layers). *Let $U^\varepsilon(x, y)$ be the solution in Theorem 3.2. As an equilibrium solution of* (1.1), *it is*

(1) *stable if $\psi''(x_0) > 0 = \mu_0$,*

(2) *unstable if $\mu_j > \psi''(x_0) > \mu_{j+1}$ with the Morse index equal to $\sum_{k=0}^{j} m_k$.*

An outline of our proof for Theorems 3.2 and 3.3 is as follows (a rigorous proof will be given later in a context of a general situation).

We consider an eigenvalue problem:

$$\tag{3.5} \begin{cases} \Delta_y \phi = \lambda\phi & \text{in } \omega, \\ \partial\phi/\partial\mathbf{n} - \psi''(x_0)\phi = 0 & \text{on } S_0. \end{cases}$$

We show that if (3.5) has no 0-eigenvalue, then it is possible to construct approximate solutions to a boundary-interior layer along Γ, with as high accuracy as we wish. On the other hand, it is readily shown that (3.5) has no 0-eigenvalue if and only if $\psi''(x_0) \notin \sigma(\Pi)$. This is the source of the nondegeneracy condition (a) in Theorem 3.2. If the approximation is accurate enough, a perturbation argument works and the existence of a boundary-interior solution follows.

It is also shown that (3.5) determines the stability property of the boundary-interior layer. In fact, the critical eigenvalues of (2.1) for the domain Ω as in (3.1) approach the eigenvalues of (3.5) which is an eigenvalue problem associated with the linearization of (1.2) around the disk $\Gamma = \{(x_0, y) \mid |y| < \psi(x_0)\}$ for the domain Ω in (3.1).

Notice that $\psi''(x_0)$ is equal to the curvature $\overline{\kappa}$ of the generating curve $(x, \psi(x), 0) \in \mathbb{R}^3$ of the boundary $\partial\Omega$. If we denote by $\mathbf{n}(x, y)$ the inward unit normal vector of $\partial\Omega$ at $(x, y) = (x, \psi(x)\cos\theta, \psi(x)\sin\theta) \in \partial\Omega$, the curvature of the generating curve has another expression:

$$\overline{\kappa} = \left\langle \frac{\partial\mathbf{n}(x,y)}{\partial x}\bigg|_{x=x_0}, \nu \right\rangle = \left\langle \frac{\partial\mathbf{n}(x,y)}{\partial \nu}\bigg|_{x=x_0}, \nu \right\rangle \qquad \text{(independent of } \theta\text{)},$$

where $\nu = (1, 0, 0)$. The geometric significance of this expression will become clear in the subsequent discussion, when we deal with a general situation.

3.2 General domains

The most difficult part of all to obtain results similar to Theorems 3.2 and 3.3 for general 3-dimensional domains is to find a minimal surface that intersects $\partial\Omega$ in the right angle. We therefore assume the existence of such a minimal surface and prove the counterparts of of these theorems for general domains.

> **(A1)**: Assume that there exists a minimal interface Γ that intersects $\partial\Omega$ in the right angle along a curve $\partial\Gamma = \overline{\Gamma} \cap \partial\Omega$.

In order to state a non-degeneracy condition on Γ, let us consider an eigenvalue problem defined on Γ:

$$\begin{cases} \Delta^\Gamma v + (\kappa_1^2 + \kappa_2^2) v = \lambda v & \text{in } \Gamma, \\ \partial v(y)/\partial \mathbf{n} - \overline{\kappa}(y) v(y) = 0 & \text{on } \partial\Gamma, \end{cases} \tag{3.6}$$

where Δ^Γ is the Laplace-Beltrami operator on Γ, κ_j $(j = 1, 2)$ the principal curvatures of Γ, and

$$\overline{\kappa}(y) = \left\langle \frac{\partial \mathbf{n}}{\partial \nu}, \nu \right\rangle, \quad y \in \partial\Gamma \subset \partial\Omega. \tag{3.7}$$

We recall again that $\mathbf{n}$ is the *inward* unit normal vector on $\partial\Omega$. Since a curve on $\partial\Omega$ is a geodesics if and only if its normal vector is parallel to the normal vector $\mathbf{n}$ of $\partial\Omega$. Therefore, $\overline{\kappa}(y)$ is the curvature of the geodesics on $\partial\Omega$ passing through $y \in \partial\Gamma$ in the direction $\nu(y)$.

Let us denote by σ_Γ the set of eigenvalues for (3.6);

$$\sigma_\Gamma = \{\lambda_j\}_{j=0}^\infty, \quad \lambda_0 > \lambda_1 > \ldots > \lambda_j > \ldots \to -\infty,$$

where we listed only distinct ones. The multiplicity of λ_j is denoted by m_j.

The non-degeneracy condition for Γ is:

> **(A2)**: $\quad 0 \notin \sigma_\Gamma$.

Our main result is the following.

Theorem 3.4 (Existence and stability of boundary-interior layers). *Assume that conditions* **(A1)** *and* **(A2)** *are satisfied. Then there exist an $\varepsilon_* > 0$ and a family of equilibrium solutions $U^\varepsilon(x)$ of* (1.1) *defined for $\varepsilon \in (0, \varepsilon_*]$ with the following properties.*

(i) *For each $\delta > 0$,*

$$\lim_{\varepsilon \to 0} U^\varepsilon(x) = \begin{cases} 1 \\ -1 \end{cases} \textit{uniformly in} \quad \begin{cases} x \in \Omega^+ \backslash \Gamma^\delta, \\ x \in \Omega^- \backslash \Gamma^\delta, \end{cases}$$

where $\Gamma^\delta = \{x \in \Omega \mid \operatorname{dist}(x, \Gamma) < \delta\}$.

(ii) *Near the interface* Γ, *the solution* U^ε *has the following behavior*

$$U^\varepsilon(x) \approx Q\left(\frac{\mathrm{dist}(x,\Gamma)}{\varepsilon}\right).$$

(iii) *If* $0 > \lambda_0$, *then* U^ε *is stable.*

(iv) *If there exits* $j \geq 0$ *satisfying* $\lambda_j > 0 > \lambda_{j+1}$, *then* U^ε *is unstable with Morse index equal to* $\sum_{k=0}^{j} m_k$.

The structure of the contents of Theorem 3.4 is dipicted in the following diagram.

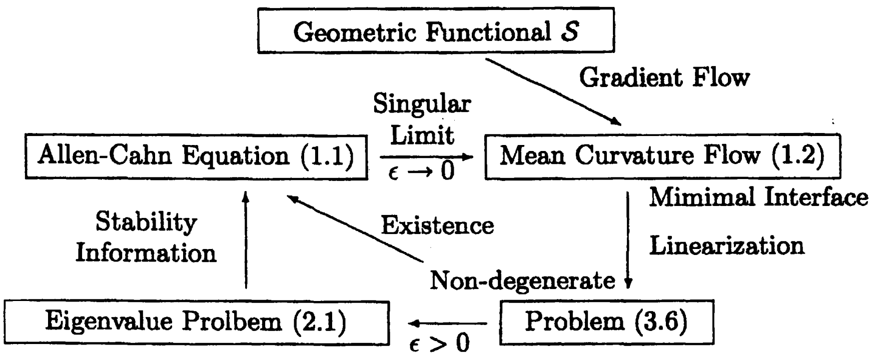


Figure 3: Non-degenerate critical point of $\mathcal{S}$ give rise to boundary-interior layers.

It is illuminating to put the results of Theorem 3.4 in a variational formulation. Let us define the class of admissible interfaces;

$$\mathcal{A}_\Omega := \{\Gamma \mid \overline{\Gamma} \text{ is a } C^2 \text{ surface with } \overline{\Gamma} \cap \partial\Omega = \partial\Gamma \text{ and } \Gamma \subset \Omega\}.$$

Let $\mathcal{S} : \mathcal{A}_\Omega \to \mathbb{R}$ be the surface area. The problem (1.2) is nothing but the gradient flow with respect to the *energy* functional $\mathcal{S}(\Gamma)$;

$$\frac{\partial\Gamma}{\partial t} = -\frac{\delta\mathcal{S}(\Gamma)}{\delta\Gamma} = -\kappa(x;\Gamma),$$

where the interface Γ varies within the class $\mathcal{A}_\Omega$ of admissible surfaces. Critical points of $\mathcal{S}(\Gamma)$ are characterized as

$$\kappa(x;\Gamma) \equiv 0 \quad \text{and} \quad \Gamma \perp_{\partial\Gamma} \partial\Omega. \tag{3.8}$$

Moreover, (3.6) is an eigenvalue problem associated with the second variation of the functional $\mathcal{S}$ at the critical point $\Gamma \in \mathcal{A}_\Omega$ in (3.8). Therefore we may restate Theorem 3.4 as follows:

A non-degenerate critical point $\Gamma \in \mathcal{A}$ of $\mathcal{S}$ gives rise to an equilibrium boundary-interior layer of (1.1). *The Morse index of the boundary-interior layer is the same as that of Γ with respect to the area functional $\mathcal{S}$.*

An interesting implication of Theorem 3.4 is that the boundary-interior layer with transition layers occurring near any minimal hypersurface $\Gamma \in \mathcal{A}_\Omega$, with $\Gamma \perp_{\partial\Gamma} \partial\Omega$, can be made stable by deforming the boundary $\partial\Omega$ near $\partial\Gamma$ so that

$$\inf_{y\in\partial\Gamma} \overline{\kappa}(y) =: \overline{\kappa}_0 \gg 1.$$

To see this, let $K := \sup\{\kappa_1^2(y) + \kappa_2^2(y) \mid y \in \overline{\Gamma}\}$. Note that $\overline{\kappa}_0$ can be made as large as one like, without influencing the magnitude of K, since we are deforming $\partial\Omega$ near $\partial\Gamma$ with Γ being fixed. For the L^2-normalized first eigenpair (λ_0, ϕ_0) of the problem (3.6), we can estimate the eigenvalue as follows;

$$\begin{aligned}
\lambda_0 &= -\int_{\partial\Gamma} \phi_0 \frac{\partial\phi_0}{\partial\mathbf{n}} dS_y^{\partial\Gamma} + \int_\Gamma (\kappa_1^2 + \kappa_2^2)\phi_0^2 dS_y^\Gamma \\
&= -\int_{\partial\Gamma} \overline{\kappa}(y)\phi_0^2 dS_y^{\partial\Gamma} + \int_\Gamma (\kappa_1^2 + \kappa_2^2)\phi_0^2 dS_y^\Gamma \\
&\leq -\overline{\kappa}_0 \int_{\partial\Gamma} \phi_0^2 dS_y^{\partial\Gamma} + K|\Gamma| < 0,
\end{aligned}$$

showing the stability of U^ϵ thanks to Theorem 3.4.

As a direct consequence of Theorem 3.4, we obtain a generalization of Theorems 3.2 and 3.3. In order to present such a generalization, let $\psi(x)$ be a smooth positive function ($x \in \mathbb{R}$ here and within §3.3) and $\omega \subset \mathbb{R}^2$ a bounded smooth domain. We consider a three-dimensional domain Ω defined by

$$\Omega = \{(x,y) \in \mathbb{R} \times \mathbb{R}^2 \mid \frac{1}{\psi(x)} y \in \omega\}. \tag{3.9}$$

If $\psi'(x_0) = 0$, then

$$\Gamma = \{(x_0, y) \in \Omega \mid \frac{1}{\psi(x_0)} y \in \omega\}$$

is an equilibrium interface of (1.2). Since the inward normal vector on the boundary of the domain in (3.9) is given for $(x,y) \in \mathbb{R} \times \partial\omega$ by

$$\mathbf{n}(x,y) = \frac{1}{\sqrt{1 + (\psi'(x))^2 |\langle y, \mathbf{n}_\omega(y)\rangle|^2}} \Big(-\psi'(x)\langle y, \mathbf{n}_\omega(y)\rangle,\ \mathbf{n}_\omega(y)\Big),$$

where $\mathbf{n}_\omega$ is the unit inward normal vector on $\partial\omega$, the eigenvalue problem (3.6) reduces to

$$\left\{\begin{aligned} \Delta v(y) &= \psi(x_0)^2\lambda v(y), & y\in\omega,\\ \frac{\partial v(y)}{\partial \mathbf{n}_\omega} + (\psi''(x_0)\psi(x_0)\langle y,\mathbf{n}_\omega\rangle)v(y) &= 0, & y\in\partial\omega, \end{aligned}\right. \tag{3.10}$$

where the interface Γ is scaled down to ω. We denote by $\sigma_\omega^{\psi(x_0)}$ the eigenvalues of (3.10);

$$\sigma_\omega^{\psi(x_0)} = \{\lambda_j\}_{j=0}^{\infty}; \quad \lambda_0 > \lambda_1 > \ldots > \lambda_j > \ldots \to -\infty,$$

where we listed only distinct eigenvalues and the multiplicity of λ_j is m_j.

Corollary 3.1. *Suppose that $\psi'(x_0) = 0$ and $0 \notin \sigma_\omega^{\psi(x_0)}$. Then for the domain Ω in* (3.9), *the statements in Theorem 3.2 are valid. Moreover, the boundary-interior layer is*

(i) *stable, if $0 > \lambda_0$, and*

(ii) *unstable with the Morse index equal to $\sum_{k=0}^{j} m_k$, if $\lambda_j > 0 > \lambda_{j+1}$.*

The results presented in this acrticle will be rigorously proven in [16]

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