数理解析研究所講究録 1330 巻 2003 年 25-39

Conical-shaped travelling fronts in some reaction-diffusion equations

F. Hamel*

Abstract. This paper is devoted to some existence, stability results, and qualitative properties of global solutions of some semilinear elliptic or parabolic equations in the whole space $\mathbb{R}^N$ with conical conditions at infinity. Related free boundary problems are also studied. Applications to models in combustion theory and populations dynamics are given.

1 Conical-shaped fronts in a combustion model

This section is concerned with conical-shaped travelling fronts for reaction-diffusion equations which arise in some combustion models. One especially gives a mathematical analysis of the shape of the premixed Bunsen flames. After a short introduction on the mathematical modelling, one gives some existence, uniqueness and stability results for entire solutions of some semilinear elliptic or parabolic equations in the whole space. One also deals with a singular limit leading to some free boundary problems.

Bunsen flames can be divided into two parts : a diffusion flame and a premixed flame (see Figure 1, and [21], [22], [38], [42], [43], [58], [59], [66]). Here we have chosen to deal with premixed flames, which are themselves divided into two zones : a fresh mixture (fuel and oxidant) and, above, a hot zone made of the burnt gases. For the sake of simplicity, we assume that a single global chemical reaction *fuel + oxidant* $\rightarrow$ *products* takes place in the mixture.

The level sets of the temperature have a conical shape with a curved tip and, far from its axis of symmetry, the flame is asymptotically almost planar. Let us assume that the flame is stabilized and stationary in an upward flow with a uniform intensity c. This uniformity assumption is reasonable at least far from the burner rim. In the classical framework of the thermodiffusive model ([6], [21], [46]) with unit Lewis number, the adimensionalized temperature field $u(x,y)$, which can be assumed to be defined in the whole space $\mathbb{R}^N=\{z=(x,y)\in\mathbb{R}^{N-1}\times\mathbb{R}\}$ because of the invariance of the shape of the flame with respect to the size of the Bunsen burner, satisfies the following reaction-diffusion equation :

$$\Delta u-c\frac{\partial u}{\partial y}+f(u)=0,\quad 0\le u\le 1\text{ in }\mathbb{R}^N. \tag{1.1}$$

*Université Aix-Marseille III, LATP, Faculté Saint-Jérôme, Avenue Escadrille Normandie-Niemen, F–13397 Marseille Cedex 20, France. francois.hamel@univ.u-3mrs.fr

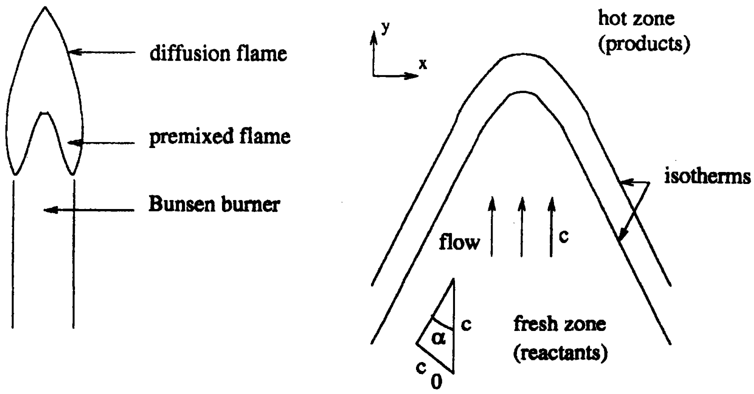

Figure 1: *Bunsen flames,* *premixed flame*

Let $\alpha > 0$ be the angle of the flame (*see* Figure 1). Asymptotic conical conditions like

$$\lim_{y_0 \to -\infty} \sup_{y \leq y_0 - |x| \cot \alpha} u(x,y) = 0, \quad \lim_{y_0 \to +\infty} \inf_{y \geq y_0 - |x| \cot \alpha} u(x,y) = 1 \tag{1.2}$$

are imposed at infinity (other asymptotic conditions have also been considered in [12] and [32]). The normalized temperature u typically ranges in $[0,1]$, the region where u is close to 0 corresponds to the fresh mixture and the region where u is close to 1 corresponds to the burnt gases. In practice, the speed c of the flow at the exit of the Bunsen burner is given and it determines the angle α of the flame. We assume here that the angle α is given and the speed c is unknown. We shall see that these two formulations are equivalent. The nonlinear reaction term $f(u)$ is of the "ignition temperature" type, namely f is assumed to be Lipschitz-continuous in $[0,1]$ and

$$\begin{cases} \exists \theta \in (0,1) \text{ such that } f \equiv 0 \text{ on } [0,\theta] \cup \{1\}, \\ f > 0 \text{ on } (\theta, 1) \text{ and } f'(1) < 0. \end{cases} \tag{1.3}$$

Such a profile can be derived from the Arrhenius kinetics and the law of mass action. The real θ is called an ignition temperature, below which no reaction happens. For mathematical convenience, f is assumed to be extended by 0 outside the interval $[0,1]$.

One points out that the solutions $u(x,y)$ of (1.1) can also be viewed as travelling fronts of the type $v(t,x,y) = u(x, y+ct)$ moving downwards with speed c in a quiescent medium. The function v solves the parabolic reaction-diffusion equation $\partial_t v = \Delta v + f(v)$.

In dimension 1, problem (1.1-1.2) reduces to the equation

$$u'' - cu' + f(u) = 0, \quad u(-\infty) = 0, \; u(+\infty) = 1. \tag{1.4}$$

This problem is known to have a unique solution (c_0, u_0), the function u_0 is increasing and unique up to translation, and the speed c_0 is positive ([2], [5], [9], [39]). These results can be obtained by a shooting method or a study in the phase plane. The above existence, uniqueness and monotonicity results have been generalized by

Berestycki, Larrouturou, Lions [7] and Berestycki, Nirenberg [11] in the multidimensional case of a straight infinite cylinder $\Sigma = \omega \times I\!R = \{z = (x, y),\ x \in \omega,\ y \in I\!R\}$, for equations of the type

$$\left\{\begin{array}{rcl} \Delta u - (c + \beta(x))\partial_y u + f(u) & = & 0 \ \text{ in } \Sigma = \omega \times I\!R \\ \partial_\nu u & = & 0 \ \text{ on } \partial\Sigma \\ u(\cdot, -\infty) \ = \ 0, \quad u(\cdot, +\infty) & = & 1, \end{array}\right. \tag{1.5}$$

where β is a given continuous function defined on the bounded and smooth section $\overline{\omega}$ of the cylinder, and $\partial_\nu u$ denotes the partial derivative of u with respect to the outward unit normal ν on $\partial\Sigma$. Under the above conditions, there exists a unique solution (c, u) of (1.5), and the function $u = u(x, y)$ is increasing in y and unique up to translation in y. Variational formulas for the unique speed exist in the one-dimensional case [29] and in the multidimensional case [30], [37].

Recently, generalizations of the above results have been obtained for *pulsating fronts* in periodic domains and media with periodic coefficients by Berestycki and Hamel [4] and Xin [63], [64].

Let us now come back to problem (1.1) with *conical* conditions (1.2). Note that, although the underlying flow is here uniform, the solutions are nevertheless non-planar, because of the conical conditions (1.2) at infinity. Formal analyses had been done, especially using asymptotic expansions in some singular limits. We here want to establish some existence or uniqueness results for this problem (1.1-1.2) by using PDE methods. We especially want to show the relationship between the speed c of the outgoing flow and the angle α of the flame. In this perspective, the results stated below are the first rigorous analysis of the conical shape of premixed Bunsen flames.

The mathematical difficulties come on the one hand from the fact that the problem is set in the whole space $I\!R^N$ and on the other hand from the *non-standard* conical conditions at infinity. These conditions are rather weak and do not impose anything as far as the behavior of the function u in the directions making an angle α with respect to the unit vector $-e_N = (0, \cdots, 0, -1)$ is concerned. Note that these conditions are very different from the uniform conditions $u(z) \to 0$ as $|z| \to \infty$ which have often been considered for such nonlinear elliptic equations.

In the next subsections, the main results on problem (1.1-1.2) and on related free boundary problems are stated. These results are detailed in some papers by Bonnet, Hamel, Monneau and Roquejoffre in [12], [32], [33] and [34].

1.1 Existence, uniqueness and qualitative properties

In the sequel, $e_N = (0, \cdots, 0, 1)$ is the upward unit vector and, for any vector e and any angle $\varphi \in (0, \pi)$, $\mathcal{C}(e, \varphi)$ denotes the open half-cone directed by e : $\mathcal{C}(e, \varphi) = \{k \in I\!R^N,\ k \cdot e > \|k\|\ \|e\| \cos\varphi\}$.

Let us first deal with the case $N = 2$, which corresponds to Bunsen burners with thin elongated rectangular outlet :

Theorem 1.1 ([12], [32], [34]) *For each $\alpha \in (0, \pi/2]$, there exists a unique solution (c, u) of (1.1-1.2).*[1] *The function u is unique up to translation, and the speed c is uniquely determined by $c = c_0 / \sin\alpha$, where c_0 is the unique planar speed for (1.4).*

Therefore, $c \geq c_0$ and the bigger the speed c is, the smaller the angle α is and the sharper the flame is. The formula for c is pertinent since it can be observed in practice that an increase of the outgoing flow c makes the curvature of the flame tip increase (see [21], [42], [59]). The case $\alpha = \pi/2$ corresponds to the planar fronts $u_0(y)$ (up to translation) with speed c_0.

Moreover, $0 < u < 1$ in $\mathbb{R}^2$, u is decreasing in any direction of $\mathcal{C}(-e_2, \alpha)$ and, up to translation, u is symmetric with respect to the variable x. Lastly, for any sequence $x_n \to \pm\infty$, the functions $u_n(x, y) = u(x + x_n, y - |x_n| \cot\alpha)$ locally converge to a translate of the planar front $u_0(y \sin\alpha \pm x \cos\alpha)$ as $x_n \to \pm\infty$: in other words, u is asymptotically planar along the directions $(\pm\sin\alpha, -\cos\alpha)$ far away from the origin. If the medium were quiescent, the flame front would move with speed c downwards and with speed c_0 in the directions which are asymptotically orthogonal to the level sets of the temperature (see Figure 1); the speed c_0 is then nothing else than the projection of the speed c on the directions $(\pm\cos\alpha, -\sin\alpha)$.

The existence result in Theorem 1.1 can be proved by solving equivalent problems in bounded rectangles such that the ratio between the x-length and the y-length approaches $\tan\alpha$ as the size of the rectangles goes to infinity. One imposes Dirichlet conditions 0 and 1 respectively on the lower and upper sides, and oblique Neumann boundary conditions on the vertical sides. By proving some a priori estimates, one passes to the limit in the whole plane $\mathbb{R}^2$. Furthermore, by using a sliding method similar to the one developped by Berestycki and Nirenberg [10], one can prove that the solutions are decreasing in the directions of the cone $\mathcal{C}(-e_2, \alpha)$. The difficulty is to show the asymptotic conditions at infinity of the type (1.2) and to prove that the level sets of the limit function u are asymptotically planar far away from the axis of symmetry $\{x = 0\}$. One especially makes several uses of the sliding method in several orthogonal directions. One also uses some results on some free boundary problems described below (see [34]).

The following theorem is a non-existence result for angles $\alpha > \pi/2$ in any dimension $N \geq 2$.

Theorem 1.2 ([31], [32]) *In any dimension $N \geq 2$, there is no solution (c, u) of (1.1-1.2) as soon as $\alpha > \pi/2$.*

Thus, despite its simplicity, the mathematical model which is used here to describe premixed Bunsen flames is robust enough and physically meaningful : there cannot be any flame whose tip points downwards if the flow is going upwards. Notice that more general non-existence results with $\alpha > \pi/2$ hold under slightly weaker conical conditions (see [32]).

But the drawback of the strong conditions (1.2) is that there is no solution of (1.1-1.2) in dimension $N \geq 3$, apart from the planar fronts $u_0(y)$ with $\alpha = \pi/2$ (see

[1] Uniqueness was proved in [32]. Existence was proved in [12] with conditions which are slightly weaker than (1.2), and in [34] with conditions (1.2).

[32]). To circumvent this fact, new weaker conditions can be introduced and are described below. Actually, instead of imposing the conditions (1.2), one can just say that the level sets of the temperature have an asymptotic direction with angle α with respect to $-e_N$ at infinity, and that the temperature is close to 0 far below any of its level sets and close to 1 far above. More generally speaking, in any dimension N, one can replace conditions (1.2) with the following ones :

$$\begin{cases} \lim\limits_{y_0 \to -\infty} \sup\limits_{\Omega^-(y_0)} u = 0 \\ \lim\limits_{y_0 \to +\infty} \inf\limits_{\Omega^+(y_0)} u = 1 \end{cases} \tag{1.6}$$

where, for any $y_0 \in \mathbb{R}$, $\Omega^+(y_0) = \{y > y_0 + \phi(x)\}$, $\Omega^-(y_0) = \{y < y_0 + \phi(x)\}$ and $\phi(x)$ is a non-specified, globally Lipschitz function, of class C^1 for large $|x|$, and such that

$$\lim_{|x| \to +\infty} \left(\nabla\phi(x) + \cot\alpha \, \frac{x}{|x|} \right) = 0. \tag{1.7}$$

Conditions (1.2) are a particular case of (1.6) and correspond to the assumption $\sup_{x \in \mathbb{R}^{N-1}} |\phi(x) + |x| \cot\alpha| < +\infty$. But in (1.6), the asymptotic behavior of the graph of ϕ at infinity is not known (this graph represents a zone where the temperature is neither very cold nor very hot). However, the following qualitative properties still hold for (1.6) :

Theorem 1.3 ([31], [32]) *In any dimension $N \geq 2$, if (c, u) solves (1.1) with conditions (1.6-1.7), then $\alpha \leq \pi/2$, $c = c_0/\sin\alpha$ and u is decreasing with respect to any direction of the cone $C(-e_N, \alpha)$. As a consequence, if $\alpha = \pi/2$, then u is planar, it only depends on the variable y and it is unique up to translation.*

The proofs of Theorems 1.2 and 1.3 make an intensive use of sliding methods in various directions, as well as some versions of the maximum principle in $\mathbb{R}^N$ with conical conditions at infinity.

One guesses that these conditions (1.6) are weak enough to guarantee the existence of solutions (c, u) for angles $\alpha < \pi/2$, in any dimension $N \geq 3$. However, this question of the existence is still open, even for axisymmetric functions $u(|x|, y)$. Other open problems concern some models of premixed Bunsen flames with non unit Lewis number (see [58], [59]), with heat losses or with nonconstant density.

1.2 Related free boundary and Serrin type problems

The above subsections dealt with smooth solutions of semilinear elliptic equations (1.1). This subsection is concerned with the so-called limit of high activation energies. In this limit, the source term $f(u)$ vanishes as soon as the temperature is below that of the burnt gases and the zone where the chemical reaction takes place becomes infinitely thin. Below this flame, the gases are not warm enough and the reaction cannot happen, and above the flame, the gases are burnt and the reaction does not happen either because at least one of the reactants has a zero concentration.

More precisely, the following theorem holds :

Theorem 1.4 ([33]) *Let f satisfy (1.3). Let $N=2$ and let $\alpha \in (0,\pi/2]$ be given. The solutions $(c_\varepsilon, u_\varepsilon)$ of (1.1-1.2) with $f_\varepsilon(s)=\varepsilon^{-1}f(1-(1-s)/\varepsilon)$ converge (locally uniformly) to a solution $(c,u)=(c^\alpha,u^\alpha)$ of*

$$\left\{\begin{array}{l} \Delta u - c\partial_y u = 0 \ \text{ in } \Omega=\{0<u<1\}\subset I\!R^2, \quad u=1 \ \text{ in } I\!R^2\backslash\Omega \\ \partial_\nu u = c_0 \ \text{ on } \Gamma := \partial\Omega \ \text{ and } u \text{ is continuous across } \Gamma \\ \limsup\limits_{d(z,\Gamma)\to+\infty,\ z\in\Omega} u(z)=0 \end{array}\right. \tag{1.8}$$

where $c_0=\sqrt{2\int_0^1 f}>0$, $c^\alpha=c_0/\sin\alpha$, $d(z,\Gamma)$ denotes the distance of a point $z\in I\!R^2$ to Γ, and $\partial_\nu u^\alpha$ denotes the normal derivative on Γ of the restriction of the function u^α to $\overline{\Omega}$.

The curve Γ represents the infinitely thin flame front, and it is an analytic conical graph $\{y=\phi(x)\}$ such that $\phi(x)+|x|\cot\alpha \to t^\pm \in I\!R$ as $x\to\pm\infty$. Furthermore, $\Omega=\{y<\phi(x)\}$, u is globally Lipschitz-continuous in $I\!R^2$ and its restriction to $\overline{\Omega}$ is analytic. The condition $\partial_\nu u^\alpha = c_0$ on Γ is a memory of this reaction term and simply means that the normal burning velocity is constant along the flame front (see [3], [9], [15], [16], [17], [18], [19], [21], [22], [23], [65] for other occurences of this type of jump condition in related problems).

This limiting process which consists in considering such functions f_ε comes back to [67] in dimension $N=1$, see also [3] for problems in infinite cylinders with heterogeneous velocity fields.

Theorem 1.4 especially gives a solution to the flame tip problem, which has been set by Buckmaster and Ludford [22]. Problem (1.8) had been studied in various asymptotic formal limits : the case of very sharp flames $\alpha\to 0^+$ with Lewis number close to 1 has been considered by Buckmaster and Ludford (this limit is reduced to a parabolic free boundary problem after a blow-down in the direction y [19], [21], [22]). Multiscale asymptotic expansions have been carried out by Sivashinsky, leading to different shapes of the flame fronts according to the position of the Lewis number with respect to 1 [59] (see also [42], [58] for the three-dimensional case). Another approach has been used by Michelson [47], in the case of a unit Lewis number; namely, Michelson has used the fourth-order Kuramoto-Sivashinsky equation ([14], [28], [60], [61]) for the description of the graph of the flame front and he has obtained the existence and the uniqueness of such graphs for angles α close to 0 (see also [48] for three-dimensional results).

Conversely, problem (1.8) can be viewed as an overdetermined Serrin type problem, for which the domain itself $\Omega=\{u<1\}$ is unknown. Problems of that type have first been considered by Serrin [57] in bounded domains for equations of the type $\Delta u+f(u)=0$, which are invariant by rotation. For such problems it has been proved that, under some conditions on f and u, the domain Ω is necessarily a ball (see also [1], [36], [51] for similar problems in other types of geometries).

For problem (1.8), one cannot expect any radial symmetry because of the first-order term $c\partial_y u$. However, under some smoothness assumptions for Γ, one can prove that, besides the trivial planar solutions, the solutions given in Theorem 1.4 are the only solutions of (1.8) :

Theorem 1.5 ([33]) *Let (c, u, Ω) be a solution of (1.8) such that both Ω and $\mathbb{R}^2 \backslash \Omega$ are not empty, and $\mathbb{R}^2 \backslash \Omega$ has no bounded connected components. Assume that the restriction of u to $\overline{\Omega}$ is C^1, and that the free boundary $\Gamma = \partial\Omega$ is globally $C^{1,1}$ with bounded curvature. Then, even if it means changing (c, u, Ω) into $(-c, u(-x, -y), -\Omega)$, one has $c \geq c_0$ and, if $\alpha \in (0, \pi/2]$ denotes the only solution of $c = c_0/\sin\alpha$, the following two and only two cases occur up to translation and symmetry in x : – either Ω is the half-space $\{y < x\cot\alpha\}$ and $u(x, y) = U_0(y\sin\alpha - x\cos\alpha)$, where $U_0(s) = e^{c_0 s}$ for $s \leq 0$ and $U_0(s) = 1$ for $s \geq 0$, – or $u = u^\alpha$ is the conical solution of (1.8) given in Theorem 1.4 above.*

It follows from Theorems 1.4 and 1.5 that the free boundary problem (1.8), together with the additional assumption that Γ is conical-shaped, is well-posed, in dimension $N = 2$, for any angle $\alpha \in (0, \pi/2]$, whereas no solution exists whenever α is larger than $\pi/2$ or whenever c is smaller than c_0, as for the case with a source term $f(u)$ in Theorem 1.2.

Theorem 1.5 is proved in [33] in several steps. The first step consists in proving that, up to a change of (c, u, Ω) into $(-c, u(-x, -y), -\Omega)$, the domain Ω is a Lipschitz sub-graph. The second step is based on a method of rotation of the domain up to a critical angle, for which the function in the rotated frame is asymptotically planar in a vertical direction. One also uses various versions of the sliding method as well as comparison principles and monotonicity results for solutions of elliptic equations in sub-graphs.

1.3 Stability results

This subsection deals with the global stability of the solutions u of problem (1.1-1.2) in dimension $N = 2$, with angles $\alpha < \pi/2$. The existence of such solutions is given in Theorem 1.1. Another way of formulating this question of the stability is to ask the question of the convergence to the travelling fronts $u(x, y + ct)$, or to some translates of them, for the solutions $v(t, x, y)$ of the Cauchy problem

$$\begin{cases} v_t = \Delta v + f(v), & t > 0,\ (x, y) \in \mathbb{R}^2, \\ v(0, x, y) = v_0(x, y) \text{ given}, & 0 \leq v_0 \leq 1 \end{cases} \tag{1.9}$$

where v_0 is close, in some sense to be defined later, to a translate $\tau_{a,b}u(x, y) = u(x + a, y + b)$ of a solution u of (1.1-1.2).

There are many papers dealing with the stability of the travelling fronts for one-dimensional equations of the type (1.4) with various types of nonlinearities f (see e.g. [2], [13], [26], [39], [55], [56]), or for wrinkled travelling fronts of multidimensional equations in infinite cylinders (see [8], [44], [52], [53], [54]), or lastly for planar fronts in the whole space (see [41], [62]). However, nothing seems to be known about the stability of the solutions of the two-dimensional problem (1.1) under conical conditions of the type (1.2), for $\alpha < \pi/2$. As already emphasized, the travelling fronts $u(x, y + ct)$ are special time-global solutions of (1.9) satisfying, at each time, the conical conditions (1.2) in the frame moving downwards with speed $c = c_0/\sin\alpha$. Therefore, the question of the global stability of these travelling waves and the question of the asymptotic behaviour for large time of the solutions of the Cauchy problem (1.9) starts from the study of the global attractor of equation (1.9) under conical conditions of the type (1.2) in a frame moving downwards with speed c.

The next theorem states that the travelling waves are the only time-global solutions of (1.9) satisfying such conical conditions.

Theorem 1.6 ([34]) *Let f satisfy (1.3) and let $\alpha \in (0, \pi/2)$. Let $0 \leq v(t, x, y) \leq 1$ solve the equation*

$$v_t = \Delta v + f(v) \quad \textit{for all } (x, y) \in \mathbb{R}^2 \textit{ and } t \in \mathbb{R} \tag{1.10}$$

and assume that

$$\left\{ \begin{array}{l} \displaystyle\lim_{y_0 \to -\infty} \sup_{t \in \mathbb{R},\ y \leq y_0 - |x| \cot\alpha} v(t, x, y - ct) = 0 \\ \displaystyle\lim_{y_0 \to +\infty} \inf_{t \in \mathbb{R},\ y \geq y_0 - |x| \cot\alpha} v(t, x, y - ct) = 1. \end{array} \right. \tag{1.11}$$

Then there exists a solution u of (1.1-1.2) such that $v(t, x, y) = u(x, y + ct)$ for all $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2$.

Since the solutions u of (1.1-1.2) are such that $u(x, y) \to 0$ (resp. $\to 1$) uniformly as $y + |x| \cot\alpha \to -\infty$ (resp. $y + |x| \cot\alpha \to +\infty$), it follows that, if $0 \leq v(t, x, y) \leq 1$ is a solution of (1.10) such that $\tau_{a_1, b_1} u(x, y + ct) \leq v(t, x, y) \leq \tau_{a_2, b_2} u(x, y + ct)$ for all $(t, x, y) \in \mathbb{R}^3$, for some solution u of (1.1-1.2) and for sme couples (a_1, b_1) and $(a_2, b_2) \in \mathbb{R}^2$, then the conclusion of Theorem 1.6 holds.

The idea for proving Theorem 1.6 is based on a sliding method (*see* [10]) in the variable t and some versions of the maximum principle for parabolic equations in unbounded domains. Similar methods were used in [54] and [4] to get some monotonicity results for the solutions of some semilinear parabolic equations in various domains.

Theorem 1.6 especially implies the following

Theorem 1.7 ([34]) *Let u be a solution of (1.1-1.2). Let $v(t, x, y)$ be a solution of the Cauchy problem (1.9) such that*

$$\left\{ \begin{array}{l} v_0 \leq u \quad \textit{in } \mathbb{R}^2 \\ \displaystyle\lim_{y_0 \to +\infty} \inf_{y \geq y_0 - |x| \cot\alpha} v_0(x, y) > \theta. \end{array} \right. \tag{1.12}$$

Then, for every sequence $t_n \to +\infty$, there exist a subsequence $t_{n'} \to +\infty$ and $(a, b) \in \mathbb{R}^2$ such that

$$v(t_{n'} + t, x, y - ct_{n'} - ct) \to u(x + a, y + b) \quad \textit{as } n' \to +\infty$$

locally uniformly in $(t, x, y) \in \mathbb{R}^3$.

A consequence of this result is that, if v_0 satisfies (1.12) and if $\omega(v_0)$ is the ω-limit set of v_0 for the semi-group $S(t)$ given by (1.9), then $\omega(v_0)$ is made up of travelling waves. Condition (1.12) is especially satisfied when v_0 lies between two translates of a solution u of (1.1-1.2). But, even under condition (1.12), the ω-limit set $\omega(v_0)$ of v_0 may well be a continuum, and one may ask for sufficient conditions for $\omega(v_0)$ to be a singleton. This is the goal of Theorem 1.8 below.

Theorem 1.8 ([34]) *Choose $\alpha \in (0,\pi/2)$ and let f satisfy (1.3). Let $v(t,x,y)$ be a solution of Cauchy problem (1.9) with initial datum v_0 uniformly continuous and such that $0 \le v_0 \le 1$. Assume the existence of ρ_0, $C_0 > 0$ and of a solution u of (1.1-1.2) such that $|v_0(x,y) - u(x,y)| \le C_0 e^{-\rho_0\sqrt{x^2+y^2}}$ in $\mathbb{R}^2$. Also assume that there exists $(a,b) \in \mathbb{R}^2$ such that $v_0 \le \tau_{a,b}u$ in $\mathbb{R}^2$.*

Then $v(t,x,y-ct)$ converges to u uniformly in (x,y) and exponentially in t, as $t \to +\infty$.

Notice that Theorem 1.8 holds especially if v_0 is uniformly continuous and such that $0 \le v_0 \le 1$ and if there exists a solution u of (1.1-1.2) such that $v_0 - u$ has compact support.

Furthermore, Theorem 1.8 admits the following extension :

Theorem 1.9 ([34]) *Let $\alpha \in (0,\pi/2)$, and f satisfy (1.3). Let $0 \le v(t,x,y) \le 1$ be a solution of the Cauchy problem (1.9) with v_0 bounded in $C^1(\mathbb{R}^2)$ and $0 \le v_0 \le 1$. Assume that $\lim_{y_0\to+\infty} \inf_{y \ge y_0 - |x|\cot\alpha} v_0 > \theta$ and that there exists a solution u of (1.1-1.2) such that $v_0 \le u$ in $\mathbb{R}^2$. Also assume that for some $\rho_0 > 0$*

$$|\partial_{e_\alpha} v_0(x,y)| \le Ce^{\rho_0(y\sin\alpha - x\cos\alpha)}, \quad |\partial_{e'_\alpha} v_0(x,y)| \le Ce^{\rho_0(y\sin\alpha + x\cos\alpha)}$$

for all $(x,y) \in \mathbb{R}^2$, where $e_\alpha = (\sin\alpha, -\cos\alpha)$ and $e'_\alpha = (-\sin\alpha, -\cos\alpha)$.

Then the function $v(t,\cdot,\cdot - ct)$ converges uniformly in $\mathbb{R}^2$, as $t \to +\infty$, to a solution u' of (1.1-1.2).

Remark 1.10 The convergence phenomenon is really governed by the behaviour of the initial datum when the space variable becomes infinite along the directions e_α and e'_α. In that sense, the situation is similar to the KPP situation ; see [44]. It may well happen that, if the initial datum v_0 has no limit in the e_α and e'_α directions, its ω-limit is made up of a continuum of waves.

Let us mention here that similar stability results were obtained by Ninomiya and Taniguchi [50] for curved fronts in singular limits for Allen-Cahn bistable equations. Existence of smooth solutions of problem (1.1-1.2) with bistable nonlinearity f was obtained by Fife [25] for angles $\alpha < \pi/2$ close to $\pi/2$. The approach in [50] complements the one used in this paper because the fronts $\{y = \varphi(x)\}$ are viewed as an interface in a curvature flow ; the function $\varphi(x)$ solves a specific differential equation and is proved to be stable with respect to perturbations. Other stability results were also obtained by Michelson [49] for Bunsen fronts solving the Kuramoto-Sivashinsky equation, in some asymptotic regimes. Formal stability results in the nearly equidiffusional case were also given in [45].

2 Curved fronts for the Fisher-KPP equation

The previous section was concerned with conical-shaped fronts in reaction-diffusion equations with combustion-type nonlinearities f. We emphasized that conical fronts also exist for bistable-type nonlinearities, at least for angles α close to $\pi/2$.

This section deals with another class of nonlinearities f, so-called of Fisher or Kolmogorov-Petrovsky-Piskunov type ([27], [40]). Namely, one assumes that f is of class $C^2([0,1])$ and satisfies :

$$\begin{array}{l} f(0)=f(1)=0,\ f'(0)>0,\ f'(1)<0, \\ f(s)>0 \text{ for any } 0<s<1,\ f \text{ is concave.} \end{array} \tag{2.1}$$

An example of such a function f is the quadratic nonlinearity $f(s)=s(1-s)$. Such profiles arise in models in population dynamics (see [2]).

It is well-known that the equation $v_t=\Delta v+f(v)$ has, in dimension $N\geq 2$, an $N+1$-dimensional manifold of planar travelling waves, namely $v_{\nu,c,h}(x,t)=\varphi_c(x\cdot\nu+ct+h)$ where ν varies in the unit sphere S^{N-1} of $\mathbb{R}^N$, h varies in $\mathbb{R}$ and c varies in $[c^*,+\infty[$ with $c^*=2\sqrt{f'(0)}>0$. In space dimension $N=1$, there are two 2-dimensional manifolds of travelling waves solutions: $v^+_{c,h}(x,t)=\varphi_c(x+ct+h)$ and $v^-_{c,h}(x,t)=\varphi_c(-x+ct+h)$ ([2], [13], [24], [29]). For any $c\geq c^*$, the function φ_c satisfies

$$\varphi_c''-c\varphi_c'+f(\varphi_c)=0 \text{ in } \mathbb{R},\quad \varphi_c(-\infty)=0 \text{ and } \varphi_c(+\infty)=1. \tag{2.2}$$

The function φ_c is increasing and unique up to translation.

Many works have been devoted to the question of the behavior for large time and the convergence to travelling waves for the solutions of the Cauchy problem for $v_t=\Delta v+f(v)$, especially in dimension 1, under a wide class of initial conditions (see *e.g.* Bramson [13]).

However, the question of the existence of *non planar* fronts had been open since recently. Theorem 1.1 above was about conical-shaped travelling fronts for equation (1.1) with combustion-type nonlinearities f satisfying (1.3). Theorem 2.1 below answers the same question, in dimension $N=2$, with KPP type nonlinearities f :

Theorem 2.1 ([35]) *Let f satisfy (2.1) and $N=2$. Let $c>c^*$, let $0<\alpha_1,\alpha_2\leq\pi/2$, $c_1=c\sin\alpha_1$, $c_2=c\sin\alpha_2$, and $\nu_1=(-\cos\alpha_1,\sin\alpha_1)$, $\nu_2=(\cos\alpha_2,\sin\alpha_2)$. Assume that $c_1,c_2\geq c^*$ and that α_1 and α_2 are not both equal to $\pi/2$. Let φ_1 and φ_2 be two solutions of (2.2) with speeds c_1 and c_2. Then there exists a travelling front solution $u(x,y)$ of (1.1) such that*

$$\left\{\begin{array}{ll} u(r\cos\beta,r\sin\beta)\to 0 & \textit{for all } -\pi/2-\alpha_1<\beta<-\pi/2+\alpha_2 \\ u(r\cos\beta,r\sin\beta)\to 1 & \textit{for all } -\pi/2+\alpha_2<\beta<3\pi/2-\alpha_1 \\ u(x-r\sin\alpha_1,y-r\cos\alpha_1)\to\varphi_1(-x\cos\alpha_1+y\sin\alpha_1) & \\ u(x+r\sin\alpha_2,y-r\cos\alpha_2)\to\varphi_1(x\cos\alpha_2+y\sin\alpha_2) & \end{array}\right. \tag{2.3}$$

as $r\to+\infty$. The last two limits in (2.3) hold locally in (x,y).

Therefore, equation (1.1) with a nonlinearity f satisfying (2.1) gives rise to more solutions than the same equation with combustion-type nonlinearities (1.3), as for the one-dimensional case. In particular, the solutions u in Theorem 2.1 are not symmetric, up to shift, with respect to any direction, provided $c_1\neq c_2$. The existence of a larger class of solutions of (1.1) with nonlinearities (2.1) is a consequence of the fact that the speeds c of (2.2) are not unique anymore. Furthermore, given c_1, c_2, α_1, α_2 as in Theorem 2.1, one can prove that there exists an infinity of solutions u of (1.1) fulfilling (2.3), namely having the same asymptotic profile at infinity.

Let us also mention that more general existence results of conical-shaped travelling fronts for (1.1) with nonlinearities f of the type (2.1), as well as fronts with more general shapes, in any dimension $N \geq 2$, have also been obtained in [35]. Namely, given $N \geq 2$, $c > c^*$, given any nonnegative and nonzero Radon measure μ supported in $S_{c,e_N} = \{(\nu,\gamma) \in S^{N-1} \times (c^*, +\infty),\ c\nu \cdot e_N = \gamma\}$, one can prove the existence of a solution u_μ of (1.1) (we denote by S^{N-1} the unit euclidean sphere of $\mathbb{R}^N$, the set S_{c,e_N} is a subset of the sphere with diameter ce_N). Furthermore, the map $\mu \mapsto u_\mu$ is one-to-one and continuous (*see* [35] for details). Therefore, there exists an infinite-dimensional manifold of solutions of (1.1). The proof of this result, given in [35], generalizes that of Theorem 2.1, which is done below, but is much more technical.

The more general question of the description of the set of all time-global solutions $v(t, x_1, \cdots, x_N)$ of $v_t = \Delta v + f(v)$ is also dealt with in [35] (travelling fronts are particular solutions of this problem). There exists an infinite-dimensional manifold of solutions of this problem, given as nonlinear interactions of planar travelling fronts. Furthermore, a partial-uniqueness result is also proved in [35].

Proof of Theorem 2.1. The proof of Theorem 2.1 is actually much easier than the proof of Theorem 1.1, which was concerned with the case of a nonlinearity f of type (1.3).

Under the assumptions of Theorem 2.1, it is straightforward to check that both functions $u_1(x,y) = \varphi_1(-x\cos\alpha_1 + y\sin\alpha_1)$ and $u_2(x,y) = \varphi_2(x\cos\alpha_2 + y\sin\alpha_2)$ solve (1.1). Let now $v(x,t)$ denote the solution of the Cauchy problem

$$\begin{cases} v_t &= \Delta v - c\partial_y v + f(v), \quad t > 0,\ (x,y) \in \mathbb{R}^2 \\ v(0,x,y) &= v_0(x,y) := \max(\varphi_1(-x\cos\alpha_1 + y\sin\alpha_1), \varphi_2(x\cos\alpha_2 + y\sin\alpha_2)). \end{cases}$$

Since $v_0(x,y)$ is a subsolution for (1.1), it follows that $v(t,x,y) \geq v_0(x,y)$ for all $t \geq 0$ and $(x,y) \in \mathbb{R}^2$, and that v is nondecreasing in t. On the other hand, the maximum principle yields that $v \leq 1$. Standard parabolic estimates then imply that $v(t,x,y) \to u(x,y)$ as $t \to +\infty$, where u is a classical solution of (1.1) such that $v_0(x,y) \leq u(x,y) \leq 1$ in $\mathbb{R}^2$.

Let us now extend f by 0 outside the interval $[0,1]$. From the concavity of f on $[0,1]$, it follows that the function $\overline{u}(x,y) := \varphi_1(-x\cos\alpha_1 + y\sin\alpha_1) + \varphi_2(x\cos\alpha_2 + y\sin\alpha_2)$ is a supersolution for (1.1). Furthermore, $v_0 \leq \overline{u}$ since both φ_1 and φ_2 are positive. Therefore, $u \leq \overline{u}$.

As a conclusion, one has

$$\max(\varphi_1(-x\cos\alpha_1 + y\sin\alpha_1), \varphi_2(x\cos\alpha_2 + y\sin\alpha_2))$$

$$\leq u(x,y) \leq \min(\varphi_1(-x\cos\alpha_1 + y\sin\alpha_1) + \varphi_2(x\cos\alpha_2 + y\sin\alpha_2), 1)$$

for all $(x,y) \in \mathbb{R}^2$. It is then easy to check that property (2.3) holds. That completes the proof of Theorem 2.1. $\square$

References

[1] A. Aftalion, J. Busca, *Radial symmetry of overdetermined problems in exterior domains*, Arch. Rat. Mech. Anal. **143** (1998), pp 195-206.

[2] D.G. Aronson, H.F. Weinberger, *Multidimensional nonlinear diffusions arising in population genetics*, Adv. Math. **30** (1978), pp 33-76.

[3] H. Berestycki, L. Caffarelli, L. Nirenberg, *Uniform estimates for regularisation of free boundary problems*, In: Anal. and Part. Diff. Eq., C. Sadosky & M. Decker eds, 1990, 567-617.

[4] H. Berestycki, F. Hamel, *Front propagation in periodic excitable media*, Comm. Pure Appl. Math. **55** (2002), pp 949-1032.

[5] H. Berestycki, B. Larrouturou, *Planar travelling front solutions of reaction-diffusion problems*, preprint.

[6] H. Berestycki, B. Larrouturou, *Quelques aspects mathématiques de la propagation des flammes prémélangées*, In: Nonlinear p.d.e. and their applications, Collège de France seminar **10**, Brézis and Lions eds, Pitman Longman, Harbow, UK, 1990.

[7] H. Berestycki, B. Larrouturou, P.L. Lions, *Multidimensional traveling-wave solutions of a flame propagation model*, Arch. Rat. Mech. Anal. **111** (1990), pp 33-49.

[8] H. Berestycki, B. Larrouturou, J.-M. Roquejoffre, *Stability of travelling fronts in a curved flame model, Part I, Linear analysis*, Arch. Rat. Mech. Anal. **117** (1992), pp 97-117.

[9] H. Berestycki, B. Nicolaenko, B. Scheurer, *Traveling waves solutions to combustion models and their singular limits*, SIAM J. Math. Anal. **16** (1985), pp 1207-1242.

[10] H. Berestycki, L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. da Soc. Braseleira de Matematica **22** (1991), pp 1-37.

[11] H. Berestycki, L. Nirenberg, *Travelling fronts in cylinders*, Ann. Inst. H. Poincaré, Anal. Non Lin. **9** (1992), pp 497-572.

[12] A. Bonnet, F. Hamel, *Existence of non-planar solutions of a simple model of premixed Bunsen flames*, SIAM J. Math. Anal. **31** (1999), 80-118.

[13] M. Bramson, *Convergence of solutions of the Kolmogorov equation to travelling waves*, Memoirs Amer. Math. Soc. **44**, 1983.

[14] C.-M. Brauner, A. Lunardi, *Instability of a free boundary in a two-dimensional combustion model*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), pp 77-81.

[15] C.-M. Brauner, A. Lunardi, Cl. Schmidt-Lainé, *Une nouvelle formulation de modèles de fronts en problèmes totalement non linéaires*, C.R. Acad. Sci. Paris **311** **I** (1990), pp 597-602.

[16] C.-M. Brauner, A. Lunardi, Cl. Schmidt-Lainé, *Stability of travelling waves with interface conditions*, Nonlinear Analysis, Theo. Math. Appl. **19** (1992), pp 455-474.

[17] C.-M. Brauner, A. Lunardi, Cl. Schmidt-Lainé, *Multidimensional stability analysis of planar travelling waves*, Appl. Math. Letters **7** 5 (1994), pp 1-4.

[18] C.-M. Brauner, S. Noor Ebad, Cl. Schmidt-Lainé, *Nonlinear stability analysis of singular travelling waves in combustion: a one-phase problem*, Nonlinear Analysis, Theo. Meth. Appl. **16** (1991), pp 881-892.

[19] J.D. Buckmaster, *A mathematical description of open and closed flame tips*, Comb. Sci. Tech. **20** (1979), 33-40.

[20] J.D. Buckmaster, *Polyhedral flames, an exercise in bimodal bifurcation analysis*, SIAM J. Appl. Math. **44** (1982), pp 40-55.

[21] J.D. Buckmaster, G.S.S. Ludford, *Lectures on Mathematical Combustion*, In: CBMS-NSF Conf. Series in Applied Math. **43**, SIAM, 1983.

[22] J.D. Buckmaster, G.S.S. Ludford, *The mathematics of combustion*, Front. Appl. Math. **2**, SIAM, 1985.

[23] L.A. Caffarelli, J.L. Vazquez, *A free-boundary problem for the heat equation arising in flame propagation*, Trans. Am. Math. Soc. **347** (1995), pp 411-441.

[24] P.C. Fife, *Mathematical aspects of reacting and diffusing systems*, Lecture Notes in Biomathematics, **28**, Springer Verlag, 1979.

[25] P.C. Fife, *Dynamics of internal layers and diffusive interfaces*, Cbms-Nsf Regional Conference, Series in Applied Mathematics 53, 1988.

[26] P.C. Fife, J.B. McLeod, *The approach of solutions of non-linear diffusion equations to traveling front solutions*, Arch. Rat. Mech. Anal. **65** (1977), pp 335-361.

[27] R.A. Fisher, *The advance of advantageous genes*, Ann. Eugenics **7** (1937), pp 335-369.

[28] M.L. Frankel, G.I. Sivashinsky, *On the equation of curved flame front*, Physica D **30** (1988), pp 28-42.

[29] K.P. Hadeler, F. Rothe, *Travelling fronts in nonlinear diffusion equations*, J. Math. Biology **2** (1975), pp 251-263.

[30] F. Hamel, *Formules min-max pour les vitesses d'ondes progressives multidimensionnelles*, Ann. Fac. Sci. Toulouse **8** (1999), pp 259-280.

[31] F. Hamel, R. Monneau, *Solutions d'équations elliptiques semilinéaires dans $\mathbb{R}^N$ ayant des courbes de niveau de forme conique*, C. R. Acad. Sci. Paris **327** I (1998), pp 645-650.

[32] F. Hamel, R. Monneau, *Solutions of semilinear elliptic equations in $\mathbb{R}^N$ with conical-shaped level sets*, Comm. Part. Diff. Equations **25** (2000), 769-819.

[33] F. Hamel, R. Monneau, *Existence and uniqueness for a free boundary problem arising in combustion theory*, Interf. Free Boundaries **4** (2002), 167-210.

[34] F. Hamel, R. Monneau, J.-M. Roquejoffre, *Stability of conical fronts in a combustion model*, preprint.

[35] F. Hamel, N. Nadirashvili, *Travelling waves and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$*, Arch. Ration. Mech. Anal. **157** (2001), 91-163.

[36] A. Henrot, G.A. Philippin, *Some overdetermined boundary value problems with elliptical free boundaries*, SIAM J. Math. Anal. **29** (1998), pp 309-320.

[37] S. Heinze, G. Papanicolaou, A. Stevens, *Variational principles for propagation speeds in inhomogeneous media*, SIAM J. Appl. Math. **62** (2001), pp 129-148.

[38] G. Joulin, *Dynamique des fronts de flammes*, In: Modélisation de la combustion, Images des Mathématiques, CNRS, 1996.

[39] Ya.I. Kanel', *Certain problems of burning-theory equations*, Sov. Math. Dokl. **2** (1961), 48-51.

[40] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov, *Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bulletin Université d'Etat à Moscou (Bjul. Moskowskogo Gos. Univ.), Série internationale **A 1** (1937), pp 1-26.

[41] C.D. Levermore, J.X. Xin, *Multidimensional stability of travelling waves in a bistable reaction-diffusion equation, II*, Comm. Part. Diff. Eq. **17** (1992), pp 1901-1924.

[42] B. Lewis, G. Von Elbe, *Combustion, flames and explosions of gases*, Academic Press, New York, London, 1961.

[43] A. Liñan, *The structure of diffusion flames*, In: Fluid dynamical aspects of combustion theory, Pitman Res. Notes Math. Ser. **223**, Longman Sci. Tech., Harlow, 1991, pp 11-29.

[44] J.-F. Mallordy, J.-M. Roquejoffre, *A parabolic equation of the KPP type in higher dimensions*, SIAM J. Math. Anal. **26** (1995), pp 1-20.

[45] B.J. Matkowsky, D.O. Olagunju, *Pulsations in a burner-stabilized premixed plane flame*, SIAM J. Appl. Math. **40** (1981), pp 551-562.

[46] B.J. Matkowsky, G.I. Sivashinsky, *An asymptotic derivation of two models in flame theory associated with the constant density approximation*, SIAM J. Appl. Math **37** (1979) pp 686-699.

[47] D. Michelson, *Steady solutions of the Kuramoto-Sivashinsky equation*, Phys. D. **19** (1986), 89-111.

[48] D. Michelson, *Bunsen flames as steady solutions of the Kuramoto-Sivashinsky equation*, SIAM J. Math. Anal. **23** (1991), 364-386.

[49] D. Michelson, *Stability of the Bunsen flame profiles in the Kuramoto-Sivashinsky equation*, SIAM J. Math. Anal. **27** (1996), pp 765-781.

[50] H. Ninomiya, M. Taniguchi, *Stability of traveling curved fronts in a curvature flow with driving force*, Meth. Appl. Analysis, to appear.

[51] W. Reichel, *Radial symmetry for elliptic boundary-value problems on exterior problems*, Arch. Rat. Mech. Anal. **137** (1997), pp 381-394.

[52] J.-M. Roquejoffre, *Stability of travelling fronts in a curved flame model, Part II : Non-linear orbital stability*, Arch. Rat. Mech. Anal. **117** (1992), pp. 119-153.

[53] J.-M. Roquejoffre, *Convergence to travelling waves for solutions of a class of semilinear parabolic equation*, J. Diff. Eq. **108** (1994), pp 262-295.

[54] J.-M. Roquejoffre, *Eventual monotonicity and convergence to travelling fronts for the solutions of parabolic equations in cylinders*, Ann. Inst. H. Poincaré, Anal. Non Lin. **14** (1997), pp 499-552.

[55] D.H. Sattinger, *Stability of waves of nonlinear parabolic systems*, Adv. Math. **22** (1976), pp 312-355.

[56] D.H. Sattinger, *Weighted norms for the stability of travelling waves*, J. Diff. Eq. **25** (1977), pp 130-144.

[57] J. Serrin, *A symmetry theorem in potential theory*, Arch. Rat. Mech. Anal. **43** (1971), pp 304-318.

[58] G.I. Sivashinsky, *The diffusion stratification effect in Bunsen flames*, J. Heat Transfer **11** (1974), pp 530-535.

[59] G.I. Sivashinsky, *The structure of Bunsen flames*, J. Chem. Phys. **62** (1975), 638-643.

[60] G.I. Sivashinsky, *Nonlinear analysis of hydrodynamics instability in laminar flames, I: Derivation of basic equations*, Acta Astro. **4** (1977), pp 1177-1206.

[61] G.I. Sivashinsky, *On flame propagation under conditions of stoichiometry*, SIAM J. Appl. Math. **39** (1980), pp 67-82.

[62] J.X. Xin, *Muldimensional stability of travelling waves in a bistable reaction-diffusion equation, I*, Comm. Part. Diff. Eq. **17** (1992), pp 1889-1899.

[63] X. Xin, *Existence of planar flame fronts in convective-diffusive periodic media*, Arch. Rat. Mech. Anal. **121** (1992), pp 205-233.

[64] J.X. Xin, *Analysis and modeling of front propagation in heterogeneous media*, SIAM Review **42** (2000), pp 161-230.

[65] G. Weiss, *A singular limit arising in combustion theory: Fine properties of the free boundary*, Calc. Var. (2003).

[66] F. Williams, *Combustion Theory*, Addison-Wesley, Reading MA, 1983.

[67] J.B. Zeldovich, D.A. Frank-Kamenetskii, *A theory of thermal propagation of flame*, Acta physiochimica URSS **9** (1938), 341-350. English translation: In *Dynamics of curved fronts*, R. Pelcé ed., Perspectives in Physics Series, Academic Press, New York, 1988, 131-140.