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Bifurcation of helical wave from traveling wave

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1 Introduction

In the present paper, we show that helical waves can bifurcate directly from planar traveling waves by using a simple mathematical model.

A helical wave is observed in self-propagating high-temperature syntheses (abbr. SHS), for instance. The SHS is a synthetic method creating refractory ceramics, intermetalic compounds, composites and so on ([9]). One can create a very-high-quality uniform product by the SHS when a combustion wave keeps its profile and propagates at a constant velocity, that is, combustion waves of steady-state mode (planar traveling waves) bring high quality products. When experimental conditions are changed, however, the planar traveling wave may lose its stability and give place some non-uniform waves. Actually, a planar pulsating wave appears through the Hopf bifurcation of planar traveling wave. We also observe a wave that propagates in the form of spiral encircling the cylindrical sample with several reaction spots ([9], [3] for instance). In the present paper, this wave is called a helical wave since it has been shown by our three-dimensional numerical simulation ([11]) that the isothermal surface of the wave has some wings and it helically rotates down as time passes on. It is also observed that the number of wings is the same as that of reaction spots on the cylindrical surface. Similar helical waves are observed also in propagation fronts of polymerizations in laboratory ([13]) and they are obtained also by numerical simulation of some autocatalytic reactions ([8]) as well as the SHS.

We have been interested in the existing condition of stable helical wave and the transition process of wave patterns from steady-state mode to pulsating mode and/or helical mode. For this purpose, we have studied the following mathematical model exhibiting helical waves:

$$\frac{\partial u}{\partial t} = \Delta u + vf(u;\mu), \quad \frac{\partial v}{\partial t} = -vf(u;\mu) \qquad (t>0,\ \boldsymbol{x}\in\Omega), \tag{1.1}$$

where the domain Ω is a line $\mathbf{R}$, a two-dimensional band domain $\Omega_2 = \{(x,y); x\in \mathbf{R},\ y\in(0,L)\}$ or a three-dimensional cylindrical domain $\Omega_3 = \{(x,y,z)); x\in \mathbf{R},\ y^2+z^2<R^2\}$ (L: band width, R: radius). The unknowns are $u(t,\boldsymbol{x})$ and $v(t,\boldsymbol{x})$ and a reaction term $f(u;\mu)$ has some parameter μ. In the case of the SHS, $u(t,\boldsymbol{x})$ and $v(t,\boldsymbol{x})$ stand for the non-dimensional temperature and reactant concentration, respectively, and the reaction term is given as

$$f(u;e_{app}) = 0 \quad (u<u_{ig}), \qquad f(u;e_{app}) = \exp(-\frac{e_{app}}{u+u_0}) \quad (u>u_{ig}) \tag{1.2}$$

by the use of the non-dimensional ignition temperature u_{ig}, the non-dimensional apparent activation energy e_{app} and a positive constant u_0. For the autocatalytic reaction $m\mathrm{A} + \mathrm{B} \to (m+1)\mathrm{A}$, $u(t, \boldsymbol{x})$ and $v(t, \boldsymbol{x})$ correspond to the density of A and B, respectively, and

$$f(u; m) = u^m. \tag{1.3}$$

Through various numerical simulation ([11]) we have observed the following propagation patterns:

1) a planar traveling wave is stable when e_{app} in (1.2) or m in (1.3) is small,

2) for the one-dimensional problem, there appears a pulsating wave via the Hopf bifurcation of the traveling wave when the parameter becomes large,

3) if the parameter is set so that the pulsating wave exists stably for the one-dimensional problem, then the planar pulsating wave is still stable in the band and cylindrical domains when L and R are small while a helical wave takes the place of the pulsating wave when L and R become larger.

The above observation indicates the existence of bifurcation branch connecting a planar traveling wave and a helical wave, however, it is not clear whether planar pulsating wave bifurcates first from a planar traveling wave and a helical wave takes the place of a planar pulsating wave when some parameter varies or a helical wave bifurcates directly from a planar traveling wave under some suitable condition.

Quite recently we have found a similar behavior of solution for

$$f(u; u_{ig}) = \frac{1}{2}(1 + \tanh \frac{u - u_{ig}}{\delta}) \tag{1.4}$$

where $0 < u_{ig} < 1$ is a parameter and $0 < \delta \ll 1$ is a constant, Moreover, we have succeeded in detailed mathematical analysis by adopting a reaction term given by a step function

$$f(u; u_{ig}) = 0 \quad (u < u_{ig}), \qquad f(u; u_{ig}) = 1 \quad (u > u_{ig}) \tag{1.5}$$

$(0 < u_{ig} < 1)$. In the present paper, we report the following results obtained by using the reaction-diffusion system (1.1) with (1.5):

(1) A stable helical wave can bifurcate directly from a planar traveling wave.

(2) Even if a traveling wave is stable in $\mathbf{R}$, the corresponding planar traveling wave can be unstable in the band domain as well as in the cylindrical domain, and a helical wave takes the place of planar traveling wave.

(3) There are no stable helical wave when L is small or R is small.

(4) Helical waves with different numbers of reaction spots can coexist stably.

2 Planar traveling wave

In our mathematical analysis, we begin with

$$\frac{\partial u}{\partial t} = \Delta u + \alpha\beta v f(u), \quad \frac{\partial v}{\partial t} = d\Delta v - \alpha v f(u) \qquad (t > 0,\ \boldsymbol{x} \in \Omega) \tag{2.1}$$

where α and β are positive constants, and $f(u)$ is the same as $f(u; u_{ig})$ given by (1.5). Since we assume that a chemical reaction propagates from left to right, we subject the following boundary condition at $|x| \to \infty$:

$$\lim_{x\to-\infty} u(t,\boldsymbol{x}) = u_-, \lim_{x\to-\infty} v(t,\boldsymbol{x}) = 0, \lim_{x\to\infty} u(t,\boldsymbol{x}) = u_+, \lim_{x\to\infty} v(t,\boldsymbol{x}) = v_+ \tag{2.2}$$

on $u(t,\boldsymbol{x})$ and $v(t,\boldsymbol{x})$, where positive constants u_+, v_+ and u_- satisfy

$$u_- = u_+ + \beta v_+, \qquad u_+ < u_{ig} < u_- \tag{2.3}$$

because u_+ and u_- respectively correspond to the temperature before and after synthesis and v_+ means the initial concentration of reactant. The diffusion coefficient d is assumed to be non-negative in this paper. Terman [16] dealt with the gas-solid combustion ([7]) where $f(u)$ is given by the Arrhenius kinetic with high activation energy, and he discussed the stability of planar traveling wave in the specific case of $d \cong 1$. To return to our subject, reactants do not diffuse in the gasless synthesis system. We can construct a planar traveling wave even if $d = 0$, however, we have few mathematical tool for studying its stability in this case. For this reason, we will study the stability of planar traveling wave and routes to a helical wave in the case of $0 < d \ll 1$ ([2] and [17]) and will try to obtain results in the case of $d = 0$ by letting $d \to 0$.

Changing variables

$$\tilde{t} = \alpha t, \quad \tilde{\boldsymbol{x}} = \sqrt{\alpha}\boldsymbol{x}, \quad \tilde{u} = \frac{u - u_+}{\beta v_+}, \quad \tilde{v} = \frac{v}{v_+}$$

and denoting $\tilde{t}$, $\tilde{\boldsymbol{x}}$, $\tilde{u}$, $\tilde{v}$ and $\dfrac{u_{ig} - u_+}{\beta v_+}$ respectively by t, $\boldsymbol{x}$, u, v and u_{ig} again, we rewrite (2.1), (1.5), (2.2), (2.3) as

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + v f(u), \quad \frac{\partial v}{\partial t} = d\Delta v - v f(u) \qquad (t > 0,\ \boldsymbol{x} \in \Omega) \\ &f(u) = 0 \quad (u < u_{ig}), \quad f(u) = 1 \quad (u > u_{ig}), \qquad 0 < u_{ig} < 1 \\ &\lim_{x\to-\infty} u(t,\boldsymbol{x}) = 1, \lim_{x\to-\infty} v(t,\boldsymbol{x}) = 0, \lim_{x\to\infty} u(t,\boldsymbol{x}) = 0, \lim_{x\to\infty} v(t,\boldsymbol{x}) = 1. \end{aligned} \tag{2.4}$$

We first construct a traveling wave solution of (2.4) on $\mathbf{R}$.

Let $d > 0$. We denote the velocity of traveling wave by s, which is an unknown non-negative constant. By the use of the moving coordinate system with velocity s, the equations in (2.4) are expressed as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + s\frac{\partial u}{\partial x} + vf(u), \quad \frac{\partial v}{\partial t} = d\frac{\partial^2 v}{\partial x^2} + s\frac{\partial v}{\partial x} - vf(u). \tag{2.5}$$

A traveling wave $(U(x), V(x))$ is a stationary solution of (2.5) satisfying

$$U(-\infty) = 1, \quad V(-\infty) = 0, \quad U(\infty) = 0, \quad V(\infty) = 1. \tag{2.6}$$

Since we cover a traveling wave with the property $U'(x) \leq 0$ alone $(' = \frac{d}{dx})$, we may fix $U(0) = u_{ig}$. Thus, $(U(x), V(x))$ has to fulfill

$$U'' + sU' + V = 0, \quad dV'' + sV' - V = 0 \quad (x < 0), \tag{2.7}$$

$$U'' + sU' = 0, \quad dV'' + sV' = 0 \quad (x > 0). \tag{2.8}$$

The solution of (2.7) subject to (2.6) is

$$V(x) = v_{ig}e^{\alpha x}, \quad U(x) = 1 - \frac{v_{ig}}{\alpha^2 + s\alpha}e^{\alpha x} \quad (x < 0), \tag{2.9}$$

where $v_{ig} = V(0)$ and α is the positive root of $dx^2 + sx - 1 = 0$. Noting $U(0) = u_{ig}$ and $V(0) = v_{ig}$, we express the solution of (2.8) subject to (2.6) as

$$V(x) = 1 - (1 - v_{ig})e^{-(s/d)x}, \quad U(x) = u_{ig}e^{-sx} \quad (x > 0). \tag{2.10}$$

Since both $U(x)$ and $V(x)$ are of class C^1 at $x = 0$, it should be satisfied that

$$1 - \frac{v_{ig}}{\alpha^2 + s\alpha} = u_{ig}, \quad -\frac{v_{ig}}{\alpha + s} = -su_{ig}, \quad \alpha v_{ig} = \frac{s}{d}(1 - v_{ig}). \tag{2.11}$$

Multiplying the third equality by $d\alpha$ and noting $d\alpha^2 + s\alpha - 1 = 0$, we have $v_{ig} = s\alpha$. By the substitution of $v_{ig} = s\alpha$ the first and second equalities of (2.11) are respectively rewritten as

$$1 - \frac{s}{\alpha + s} = u_{ig} \quad \text{and} \quad \frac{\alpha}{\alpha + s} = u_{ig},$$

which are equivalent each other. Hence,

$$s^2 = \frac{(1 - u_{ig})^2}{(1 - u_{ig} + du_{ig})u_{ig}}, \quad \alpha = \frac{su_{ig}}{1 - u_{ig}}, \quad v_{ig} = s\alpha. \tag{2.12}$$

Substituting these into (2.9) and (2.10), we obtain a traveling wave solution of (2.4) with $d > 0$. We note that

$$\lim_{d \to 0} s = \sqrt{\frac{1 - u_{ig}}{u_{ig}}} \equiv s^*, \quad \lim_{d \to 0} \alpha = \frac{1}{s^*}, \quad \lim_{d \to 0} v_{ig} = 1. \tag{2.13}$$

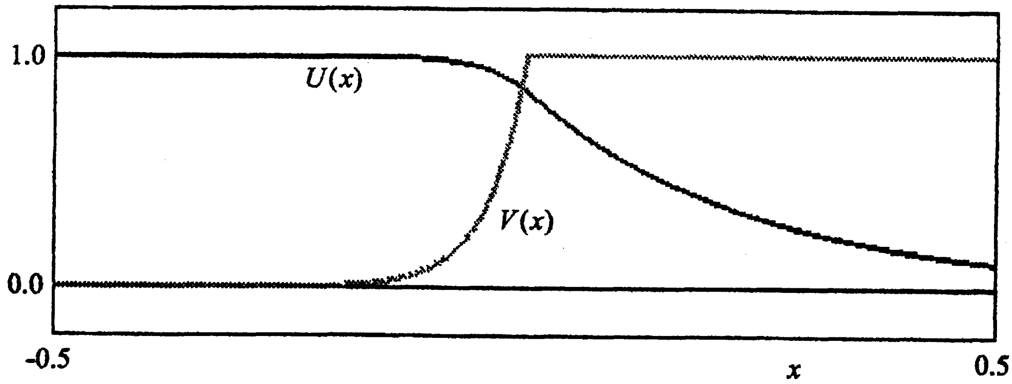

Figure 2.1: A traveling wave solution ($u_{ig} = 0853555$)

A traveling wave solution of (2.4) with $d = 0$ is similarly constructed. The solutions of (2.7) and (2.8) with $d = 0$ subject to (2.6) are given by

$$V(x) = v_{ig}e^{\alpha x}, \quad U(x) = 1 - \frac{v_{ig}}{\alpha^2 + s\alpha}e^{\alpha x} \quad (x < 0), \tag{2.14}$$

$$V(x) = 1, \quad U(x) = u_{ig}e^{-sx} \quad (x > 0), \tag{2.15}$$

respectively, where $s > 0$ and $\alpha = \frac{1}{s}$. Since $U(x)$ is of C^1 and $V(x)$ is of class C^0 at $x = 0$, s and v_{ig} have to satisfy

$$1 - \frac{v_{ig}}{\alpha^2 + s\alpha} = u_{ig}, \quad -\frac{v_{ig}}{\alpha + s} = -su_{ig}, \quad v_{ig} = 1. \tag{2.16}$$

We thus obtain

$$s^2 = \frac{1 - u_{ig}}{u_{ig}}, \quad \alpha = \frac{1}{s}, \quad v_{ig} = 1. \tag{2.17}$$

The profile of traveling wave with $d = 0$ is shown in Figure 2.1. As derived from (2.9)~(2.10), (2.13), (2.14)~(2.15) and (2.17), the traveling wave for $d > 0$ tends to that for $d = 0$ as $d \to 0$.

We now proceed to a planar traveling wave in the band domain Ω_2 and the cylindrical domain Ω_3. We employ the periodic boundary condition in the y-direction on the boundary of Ω_2 and the no-flux condition $\frac{\partial u}{\partial \boldsymbol{n}} = \frac{\partial v}{\partial \boldsymbol{n}} = 0$ on the boundary of Ω_3 where $\boldsymbol{n}$ denotes the unit outer normal vector on the boundary. Hence, $(u(t,x,y), v(t,x,y)) = (U(x-st), V(x-st))$ and $(u(t,x,y,z), v(t,x,y,z)) = (U(x-st), V(x-st))$ are a planar traveling wave in Ω_2 and Ω_3, respectively, where $(U(x-st), V(x-st))$ is the traveling wave on $\mathbf{R}$.

3 Linearized equation around the planar traveling wave

We first deal with the one-dimensional problem. The inner product of $L^2(\mathbf{R})$ is denoted by $(\xi,\zeta) = \int_{\mathbf{R}} \xi\zeta dx$, and the moving coordinate $x - st$ is simply expressed by x again. Let $d > 0$. Then, the linearized equation of (2.5) around the traveling wave $(U(x), V(x))$ is given by

$$
\begin{aligned}
(\frac{\partial\phi}{\partial t},\xi) &= -(\frac{\partial\phi}{\partial x},\frac{\partial\xi}{\partial x}) + s(\frac{\partial\phi}{\partial x},\xi) + (\psi f(U),\xi) - \omega\phi(t,0)\xi(0) \\
&\qquad\qquad \forall\xi \in H^1(\mathbf{R}), \\
(\frac{\partial\psi}{\partial t},\zeta) &= -d(\frac{\partial\psi}{\partial x},\frac{\partial\zeta}{\partial x}) + s(\frac{\partial\psi}{\partial x},\zeta) - (\psi f(U),\zeta) + \omega\phi(t,0)\zeta(0) \\
&\qquad\qquad \forall\zeta \in H^1(\mathbf{R})
\end{aligned}
\tag{3.1}
$$

in the weak form, where

$$
\omega = \frac{V(0)}{U'(0)} = -\frac{\alpha}{u_{ig}} = -\frac{s}{1-u_{ig}} < 0. \tag{3.2}
$$

We define an operator $\mathcal{L}$ by the right-hand side of (3.1). More precisely, $\mathcal{L}(\phi,\psi)^T \equiv (L_1(\phi,\psi), L_2(\phi,\psi))^T$ is defined by

$$
\begin{aligned}
(L_1(\phi,\psi),\xi) &= -(\frac{\partial\phi}{\partial x},\frac{\partial\xi}{\partial x}) + s(\frac{\partial\phi}{\partial x},\xi) + (\psi f(U),\xi) - \omega\phi(t,0)\xi(0) \\
&\qquad\qquad \forall\xi \in H^1(\mathbf{R}), \\
(L_2(\phi,\psi),\zeta) &= -d(\frac{\partial\psi}{\partial x},\frac{\partial\zeta}{\partial x}) + s(\frac{\partial\psi}{\partial x},\zeta) - (\psi f(U),\zeta) + \omega\phi(t,0)\zeta(0) \\
&\qquad\qquad \forall\zeta \in H^1(\mathbf{R}).
\end{aligned}
\tag{3.3}
$$

Let us consider the spectrum of $\mathcal{L}$ in the weighted Banach space

$$
X_{(w_\phi,w_\psi)}(\mathbf{R}) = \{(\phi,\psi); \phi w_\phi \in H^1(\mathbf{R}), \psi w_\psi \in H^1(\mathbf{R})\} \tag{3.4}
$$

where $w_\phi(x)$ and $w_\psi(x)$ are smooth weight-functions satisfying

$$
w_\phi(x) = e^{sx/2} \ (x \in \mathbf{R}), \tag{3.5}
$$

$$
w_\psi(x) = 1 \ (x < 0,\ |x| \gg 1), \quad w_\psi(x) = e^{sx/2d} \ (x > 0,\ |x| \gg 1). \tag{3.6}
$$

Then, as described in [2], [10] and [12] for instance, $\mathcal{L}$ defines a sectorial operator and its essential spectrum lies in the left half complex plane bounded away from the imaginary axis. We thus arrive the eigenvalue problem around the traveling wave $(U(x), V(x))$:

$$
\begin{aligned}
&\text{Find } \lambda \in \mathbf{C} \text{ and } (\phi,\psi) \in X_{(w_\phi,w_\psi)}(\mathbf{R}) \text{ such that} \\
&\mathcal{L}(\phi,\psi)^T = \lambda(\phi,\psi)^T,
\end{aligned}
\tag{3.7}
$$

which is equivalent to

$$
\begin{aligned}
&\text{Find } \lambda \in \mathbf{C} \text{ and } (\phi,\psi) \in X_{(w_\phi,w_\psi)}(\mathbf{R}) \text{ such that}\\
&\phi'' + s\phi' + \psi = \lambda\phi, \quad d\psi'' + s\psi' - \psi = \lambda\psi \qquad (x<0),\\
&\phi'' + s\phi' = \lambda\phi, \quad d\psi'' + s\psi' = \lambda\psi \qquad (x>0),\\
&\phi'(+0) - \phi'(-0) = \omega\phi(0), \quad d\psi'(+0) - d\psi'(-0)) = -\omega\phi(0).
\end{aligned} \tag{3.8}
$$

We now turn to a planar traveling wave in the band domain Ω_2. Denote the inner product of $L^2(\Omega_2)$ by $(\xi,\zeta) = \int_{\Omega_2} \xi\zeta dx$ and let $X(\Omega_2) = \{\xi \in H^1(\Omega_2) : \xi(x,0) = \xi(x,L) \text{ for } x \in \mathbf{R}\}$ because of the periodic boundary condition in the y-direction. Then, the linearized equation around the planar traveling wave is expressed as

$$
\begin{aligned}
(\frac{\partial\phi}{\partial t},\xi) &= -(\frac{\partial\phi}{\partial x},\frac{\partial\xi}{\partial x}) - (\frac{\partial\phi}{\partial y},\frac{\partial\xi}{\partial y}) + s(\frac{\partial\phi}{\partial x},\xi) + (\psi f(U),\xi)\\
&\qquad - \omega\int_0^L \phi(t,0,y)\xi(0,y)dy \qquad \forall\xi \in X(\Omega_2),\\
(\frac{\partial\psi}{\partial t},\zeta) &= -d(\frac{\partial\psi}{\partial x},\frac{\partial\zeta}{\partial x}) - d(\frac{\partial\psi}{\partial y},\frac{\partial\zeta}{\partial y}) + s(\frac{\partial\psi}{\partial x},\zeta) - (\psi f(U),\zeta)\\
&\qquad + \omega\int_0^L \phi(t,0,y)\zeta(0,y)dy \qquad \forall\zeta \in X(\Omega_2)
\end{aligned} \tag{3.9}
$$

in the weak form, and the eigenvalue problem is given by

$$
\begin{aligned}
&\Delta\phi + s\frac{\partial\phi}{\partial x} + \psi = \lambda\phi, \quad d\Delta\psi + s\frac{\partial\psi}{\partial x} - \psi = \lambda\psi \quad (x<0,\ y \in (0,L)),\\
&\Delta\phi + s\frac{\partial\phi}{\partial x} = \lambda\phi, \quad d\Delta\psi + s\frac{\partial\psi}{\partial x} = \lambda\psi \quad (x>0,\ y \in (0,L)),\\
&\frac{\partial\phi}{\partial x}(+0,y) - \frac{\partial\phi}{\partial x}(-0,y) = \omega\phi(0,y) \quad (y \in (0,L)),\\
&d\frac{\partial\psi}{\partial x}(+0,y) - d\frac{\partial\psi}{\partial x}(-0,y) = -\omega\phi(0,y) \quad (y \in (0,L)).
\end{aligned} \tag{3.10}
$$

Applying the Fourier expansion to the eigenfunctions ([15] for rigorous treatment), we look for a solution of (3.10) in the form

$$
\phi(x,y) = e^{i\frac{2\pi n}{L}y}\tilde{\phi}(x), \quad \psi(x,y) = e^{i\frac{2\pi n}{L}y}\tilde{\psi}(x) \quad n = 0,1,2,\cdots \tag{3.11}
$$

where $i = \sqrt{-1}$. We put $k = \dfrac{2\pi n}{L}$ and denote $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ in (3.11) by $\phi(x)$

and $\psi(x)$, respectively. Then, (3.10) together with (3.11) is expressed as

$$\begin{aligned}&\text{Find } \lambda \in \mathbf{C} \text{ and } (\phi, \psi) \in X_{(w_\phi, w_\psi)}(\mathbf{R}) \text{ such that}\\ &\phi'' + s\phi' - k^2\phi + \psi = \lambda\phi, \quad d\psi'' + s\psi' - (1 + dk^2)\psi = \lambda\psi \quad (x < 0),\\ &\phi'' + s\phi' - k^2\phi = \lambda\phi, \quad d\psi'' + s\psi' - dk^2\psi = \lambda\psi \quad (x > 0),\\ &\phi'(+0) - \phi'(-0) = \omega\phi(0), \quad d(\psi'(+0) - \psi'(-0)) = -\omega\phi(0).\end{aligned} \tag{3.12}$$

The problem (3.12) with $k = 0$ is the same as the one-dimensional eigenvalue problem (3.8).

We apply the Fourier-Bessel expansion to eigenfunctions with the polar coordinate in the cylindrical domain $\Omega_3 = \{(x, r, \theta)); x \in \mathbf{R}, r < R, 0 \leq \theta < 2\pi\}$:

$$\phi(x, r, \theta) = e^{in\theta} J_n(\frac{R_{nm}}{R} r)\tilde{\phi}(x), \quad \psi(x, r, \theta) = e^{in\theta} J_n(\frac{R_{nm}}{R} r)\tilde{\psi}(x),$$

where $J_n(r)$ is the Bessel function of order n ($n = 0, 1, \cdots$) and R_{nm} is the m-th positive root of

$$\frac{dJ_n}{dr}(r) \equiv \frac{n}{r} J_n(r) - J_{n+1}(r) = 0$$

($m = 1, 2 \cdots$). Each R_{nm} is determined so that $\phi(x, r, \theta)$ and $\psi(x, r, \theta)$ satisfy the no-flux boundary condition at $r = R$, and it holds that

$$R_{11} < R_{21} < R_{01} < R_{31} < \cdots.$$

Thus the eigenvalue problem in the cylindrical domain is given by the same form as (3.12) except the wave number k defined by $k = \dfrac{R_{nm}}{R}$ ($n = 0, 1, \cdots, m = 1, 2, \cdots$).

4 Computation of eigenvalues

In this section, we discuss the way to solve the eigenvalue problem (3.12), where an eigenvalue λ can be assumed to satisfy

$$\mathrm{Re}\lambda > -1/2. \tag{4.1}$$

Step 1 (general solution of $d\psi'' + s\psi' - (\lambda + 1 + dk^2)\psi = 0$ in $x < 0$) Let γ_1 and γ be two roots of $dx^2 + sx - (\lambda + 1 + dk^2) = 0$ ($\mathrm{Re}\gamma_1 \leq \mathrm{Re}\gamma$). Then, $e^{\gamma_1 x}$ is eliminated because $\mathrm{Re}\gamma_1 \leq -s/(2d)$ from $\gamma_1 + \gamma = -s/d$. On the other hand, $\mathrm{Re}\gamma > 0$. Actually, comparing the real part of the relation between roots and coefficients

$$(\lambda + 1 + dk^2)/d = -\gamma_1\gamma = -(-s/d - \gamma)\gamma = (s/d + \gamma)\gamma,$$

we obtain $(\mathrm{Re}\lambda + 1 + dk^2)/d = (s/d + \mathrm{Re}\gamma)\mathrm{Re}\gamma - (\mathrm{Im}\gamma)^2$, which together with $\mathrm{Re}\gamma \geq -s/(2d)$ and (4.1) implies $\mathrm{Re}\gamma > 0$. We thus obtain the general solution with an integration constant C

$$\psi(x) = Ce^{\gamma x} \qquad (x < 0). \tag{4.2}$$

Step 2 (particular solution of $\phi'' + s\phi' - (\lambda + k^2)\phi = -\psi$ in $x < 0$) Generically, it holds that $\gamma^2 + s\gamma - \lambda - k^2 \neq 0$ and the particular solution is

$$-\frac{Ce^{\gamma x}}{\gamma^2 + s\gamma - \lambda - k^2}.$$

The exceptional case of $\gamma^2 + s\gamma - \lambda - k^2 = 0$ will be discussed in Step 8.
Step 3 (general solution of $\phi'' + s\phi' - (\lambda + k^2)\phi = -\psi$ in $x < 0$) Let κ_1 and κ_2 be two roots of $x^2 + sx - (\lambda + k^2) = 0$ ($\mathrm{Re}\kappa_1 \leq \mathrm{Re}\kappa_2$). Since $\mathrm{Re}\kappa_1 \leq -s/2$ from $\kappa_1 + \kappa_2 = -s$, $e^{\kappa_1 x}$ cannot be a ϕ-component of a function pair belonging to $X_{(w_\phi, w_\psi)}(\mathbf{R})$. Hence, the general solution of $\phi'' + s\phi' - (\lambda + k^2)\phi = 0$ in $x < 0$ is $\phi(x) = Ae^{\kappa_2 x}$ (A: an integration constant), and the general solution of $\phi'' + s\phi' - (\lambda + k^2)\phi = -\psi$ in $x < 0$ is given by

$$\phi(x) = Ae^{\kappa_2 x} - \frac{Ce^{\gamma x}}{\gamma^2 + s\gamma - \lambda - k^2} \qquad (x < 0). \tag{4.3}$$

Step 4 (general solution of $\phi'' + s\phi' - (\lambda + k^2)\phi = 0$ in $x > 0$) The general solution is given by

$$\phi(x) = Be^{\kappa_1 x} \qquad (x > 0) \tag{4.4}$$

with an integration constant B since $e^{\kappa_2 x}$ is eliminated because of $\mathrm{Re}\kappa_2 \geq -s/2$.
Step 5 (general solution of $d\psi'' + s\psi' - (\lambda + dk^2)\psi = 0$ in $x > 0$) Let δ and δ_2 be two roots of $dx^2 + sx - (\lambda + dk^2) = 0$ ($\mathrm{Re}\delta \leq \mathrm{Re}\delta_2$). Since $\mathrm{Re}\delta_2 \geq -s/(2d)$ from $\delta + \delta_2 = -s/d$, $e^{\delta_2 x}$ cannot be a ψ-component of a function pair belonging to $X_{(w_\phi, w_\psi)}(\mathbf{R})$. Hence the general solution is given by

$$\psi(x) = De^{\delta x} \qquad (x > 0) \tag{4.5}$$

with an integration constant D.
Step 6 (continuity and jump conditions) At $x = 0$, $\phi(x)$ and $\psi(x)$ are continuous and their derivatives satisfy the jump condition in (3.12), it holds that

$$\begin{aligned} &A - \frac{C}{\gamma^2 + s\gamma - \lambda - k^2} = B, \quad C = D, \\ &\kappa_1 B - \kappa_2 A + \frac{\gamma C}{\gamma^2 + s\gamma - \lambda - k^2} = \omega B, \quad d(\delta D - \gamma C) = -\omega B. \end{aligned} \tag{4.6}$$

Step 7 (algebraic equation determining eigenvalues) Removing A, B and D from (4.6), we obtain

$$\begin{aligned}\frac{\kappa_1}{\omega}d(\gamma-\delta)C-\frac{\kappa_2}{\omega}d(\gamma-\delta)C-\frac{\kappa_2 C}{\gamma^2+s\gamma-\lambda-k^2}\\+\frac{\gamma C}{\gamma^2+s\gamma-\lambda-k^2}=d(\gamma-\delta)C,\end{aligned}\tag{4.7}$$

which results in the following algebraic equation determining eigenvalues:

$$\frac{d(\gamma-\delta)}{\omega}(\kappa_1-\kappa_2-\omega)+\frac{1}{\gamma-\kappa_1}=0.\tag{4.8}$$

Step 8 (exceptional case of $\gamma^2+s\gamma-\lambda-k^2=0$) In this case, the particular solution of $\phi''+s\phi'-(\lambda+k^2)\phi=-\psi$ in $x<0$ is $-\dfrac{C}{s+2\gamma}xe^{\gamma x}$. After calculations similar to Steps 3 $\sim$ 7, we obtain

$$\begin{aligned}&A=B,\quad C=D,\\&\kappa_1 B-\kappa_2 A+\frac{C}{s+2\gamma}=\omega B,\quad d(\delta D-\gamma C)=-\omega B\end{aligned}\tag{4.9}$$

corresponding to (4.6), which results in

$$\frac{d(\gamma-\delta)}{\omega}(\kappa_1-\kappa_2-\omega)+\frac{1}{s+2\gamma}=0.\tag{4.10}$$

Since $\gamma=\kappa_2$ and $\kappa_1+\kappa_2=-s$ from $\gamma^2+s\gamma-\lambda-k^2=0$, we have $s+2\gamma=\gamma-\kappa_1$. Hence, (4.10) is the same as (4.8) and all eigenvalues satisfying (4.1) are determined by (4.8).

5 Linearized equation for the case of $d=0$

The linearized equation around the planar traveling wave for the case of $d=0$ is derived by the method similar to the case of $d>0$. In the band domain Ω_2 for instance, it is expressed by

$$\begin{aligned}(\frac{\partial\phi}{\partial t},\xi)&=-(\frac{\partial\phi}{\partial x},\frac{\partial\xi}{\partial x})-(\frac{\partial\phi}{\partial y},\frac{\partial\xi}{\partial y})+s(\frac{\partial\phi}{\partial x},\xi)+(\psi f(U),\xi)\\&\quad-\omega\int_0^L\phi(t,0,y)\xi(0,y)dy\qquad\forall\xi\in X(\Omega_2),\\(\frac{\partial\psi}{\partial t},\zeta)&=-s(\psi,\frac{\partial\zeta}{\partial x})-(\psi f(U),\zeta)\\&\quad+\omega\int_0^L\phi(t,0,y)\zeta(0,y)dy\qquad\forall\zeta\in H^1(\Omega_2)\end{aligned}\tag{5.1}$$

in the weak form, where $(\xi, \zeta) = \int_{\Omega_2} \xi\zeta dx$ and

$$\omega = \frac{V(0)}{U'(0)} = -\frac{1}{su_{ig}} = -\frac{s}{1-u_{ig}} < 0. \tag{5.2}$$

It is natural to consider the weighted Banach space $X_{(w_\phi, w_\psi)}(\mathbf{R})$ tends to

$$X_{(w_\phi, *)}(\mathbf{R}) = \{(\phi, \psi); \phi w_\phi \in H^1(\mathbf{R}), \psi|_{\mathbf{R}_-} \in H^1(\mathbf{R}_-), \psi(x) = 0 \text{ for } x > 0\}$$

as $d \to 0$ ($\mathbf{R}_- = (-\infty, 0)$). Based on this understanding, we get the following *formal* eigenvalue problem

$$\begin{array}{l}
\text{Find } \lambda \in \mathbf{C} \text{ and } (\phi, \psi) \in X_{(w_\phi, *)}(\mathbf{R}) \text{ such that} \\
\phi'' + s\phi' - k^2\phi + \psi = \lambda\phi, \quad s\psi' - \psi = \lambda\psi \qquad (x < 0), \\
\phi'' + s\phi' - k^2\phi = \lambda\phi, \quad s\psi' = \lambda\psi \qquad (x > 0), \\
\phi'(+0) - \phi'(-0) = \omega\phi(0), \quad s(\psi(+0) - \psi(-0)) = -\omega\phi(0),
\end{array} \tag{5.3}$$

where $k = \frac{2n\pi}{L}$ $(n = 0, 1, \cdots)$. The *formal* eigenvalue problem in the one-dimensional space and the cylindrical domain Ω_3 is given by (5.3) with $k = 0$ and $k = \frac{R_{nm}}{R}$ $(n = 0, 1, \cdots, m = 1, 2, \cdots)$, respectively.

The problem (5.3) subject to (4.1) is solved as follows. Put $\gamma = (\lambda + 1)/s$ and denote by κ_1 and κ_2 two roots of $x^2 + sx - (\lambda + k^2) = 0$ ($\mathrm{Re}\kappa_1 \leq \mathrm{Re}\kappa_2$). It is shown that there exists no eigenvalue satisfying $\gamma^2 + s\gamma - \lambda - k^2 = 0$. Hence, we obtain

$$\begin{array}{l}
\psi(x) = Ce^{\gamma x}, \quad \phi(x) = Ae^{\kappa_2 x} - \dfrac{Ce^{\gamma x}}{\gamma^2 + s\gamma - \lambda - k^2} \qquad (x < 0), \\
\phi(x) = Be^{\kappa_1 x}, \quad \psi(x) = 0 \qquad (x > 0)
\end{array} \tag{5.4}$$

corresponding to (4.2) $\sim$ (4.5). It follows from the continuity of ϕ at $x = 0$ and the jump condition in (5.3) that

$$\begin{array}{l}
A - \dfrac{C}{\gamma^2 + s\gamma - \lambda - k^2} = B, \\
\kappa_1 B - \kappa_2 A + \dfrac{\gamma C}{\gamma^2 + s\gamma - \lambda - k^2} = \omega B, \quad -sC = -\omega B,
\end{array} \tag{5.5}$$

which leads to the algebraic equation

$$\frac{s}{\omega}(\kappa_1 - \kappa_2 - \omega) + \frac{1}{\gamma - \kappa_1} = 0. \tag{5.6}$$

6 Appearance of helical waves

We solve (4.8) by using the Newton method since its solution is not be given explicitly. We first fix d and u_{ig} suitably. Calculating the left-hand side of (4.8) and checking the signs of its real and imaginary parts for various λ ($\mathrm{Re}\lambda > -1/2$), we find some approximate eigenvalues. Then, employing these approximate eigenvalues as initial values, we get eigenvalues by the Newton method. Moreover, the dependency of eigenvalue on d and u_{ig} is studied also by the Newton method. Since the terms $s^2+4d(\lambda+1+dk^2)$, $s^2+4d(\lambda+dk^2)$ and $s^2+4(\lambda+k^2)$ included in the left-hand side of (4.8) never take a negative real value, their square roots with positive real part are denoted by using the notation $\sqrt{}$. For $d > 0$, $d(\gamma-\delta)$ is expressed as

$$d(\gamma-\delta) = \frac{1}{2}\sqrt{s^2+4d(\lambda+1+dk^2)} + \frac{1}{2}\sqrt{s^2+4d(\lambda+dk^2)}, \qquad (6.1)$$

which implies $d(\gamma-\delta)$ tends to s *formally* as $d \to 0$. In this sense, (4.8) tends (5.6) as $d \to 0$.

The following behaviors of eigenvalues are proved.

Proposition 1 When $k = 0$, $\lambda = 0$ is a simple eigenvalue and the eigenfunction is $(U'(x), V'(x))$.

Proposition 2 If $d = 0$, $\lambda = 0$ is not an eigenvalue for $k \neq 0$ and $\frac{\partial\lambda}{\partial k^2}|_{k=0,\lambda=0} < 0$.

In the case of $d = 0$ and $k = 0$, we can get more precise information on solutions of (5.6). Put $\kappa_2 = s\kappa$. We note that $\mathrm{Re}\kappa > -1/2$ from (3.5). Substituting $\kappa_1 = -s-\kappa_2 = -(1+\kappa)s$, $\lambda = -\kappa_1\kappa_2 - k^2 = \kappa(1+\kappa)s^2 - k^2$ and $s^2 = (1-u_{ig})/u_{ig}$ into (5.6) and multiplying it by $s\omega(\gamma-\kappa_1){u_{ig}}^2$, we reduce (5.6) to the cubic equation

$$\{(1-u_{ig})(\kappa+1)^2 + u_{ig}(1-k^2)\}\{1-(1-u_{ig})(2\kappa+1)\} - u_{ig} = 0 \qquad (6.2)$$

with respect to κ. We here let $k = 0$. Then, (6.2) is factorized as

$$-(1-u_{ig})\kappa\{2(1-u_{ig})\kappa^2 + (4-5u_{ig})\kappa + 2(1-u_{ig})\} = 0. \qquad (6.3)$$

Since $D = u_{ig}(9u_{ig}-8)$ is the discriminant of the quadratic equation $2(1-u_{ig})\kappa^2 + (4-5u_{ig})\kappa + 2(1-u_{ig}) = 0$, its solutions are complex conjugate $\kappa = \dfrac{5u_{ig}-4 \pm i\sqrt{-D}}{4(1-u_{ig})}$ when $u_{ig} < 8/9$ and they are positive real when $u_{ig} \geq 8/9$. The condition $\mathrm{Re}\kappa > -1/2$ is satisfied by $u_{ig} > 2/3$. Hence, the eigenvalue problem (5.3) subject to (4.1) has just one solution $\lambda = 0$ when $u_{ig} \leq 2/3$, three solutions $\lambda = 0$ and a pair of complex conjugate numbers when $2/3 < u_{ig} < 8/9$

and also three solutions $\lambda = 0$ and two positive numbers when $u_{ig} \geq 8/9$. The real part of the pair of complex conjugate eigenvalues is expressed as

$$\mathrm{Re}\lambda = \mathrm{Re}\kappa(1 + \mathrm{Re}\kappa)s^2 - (\mathrm{Im}\kappa)^2 s^2 = \frac{7u_{ig} - 6}{8(1 - u_{ig})}, \tag{6.4}$$

which is negative for small u_{ig} and increases with u_{ig}. The above expression together with $|\mathrm{Re}\lambda| = \dfrac{(3u_{ig} - 2)\sqrt{u_{ig}(8 - 9u_{ig})}}{8u_{ig}(1 - u_{ig})}$ implies that the complex conjugate eigenvalues cross the imaginary axis transversely at the critical $u_{ig} = 6/7$ and they move in the complex plane with positive real part for larger u_{ig} and arrive at the point $\lambda = 1/4$ on the real axis when $u_{ig} = 8/9$.

Solving the algebraic equations (4.8) and (6.2) by the Newton method and the bisection method, we clarify the following properties of eigenvalues. (In the description of properties, we neglect two positive real eigenvalues which may appear when u_{ig} is very near to the unity.)

Property 1 (number of eigenvalues) The problem (3.12) subject to (4.1) as well as (5.3) subject to (4.1) has one real eigenvalue and a pair of complex conjugate eigenvalues at most.

Property 2 (real eigenvalue) There exists a continuous function $\bar{k}(u_{ig}; d)$ of u_{ig} and d such that the problem has a real eigenvalue if and only if $k < \bar{k}(u_{ig}; d)$. The real eigenvalue λ is equal to zero for $k = 0$ and is negative for $k \neq 0$.

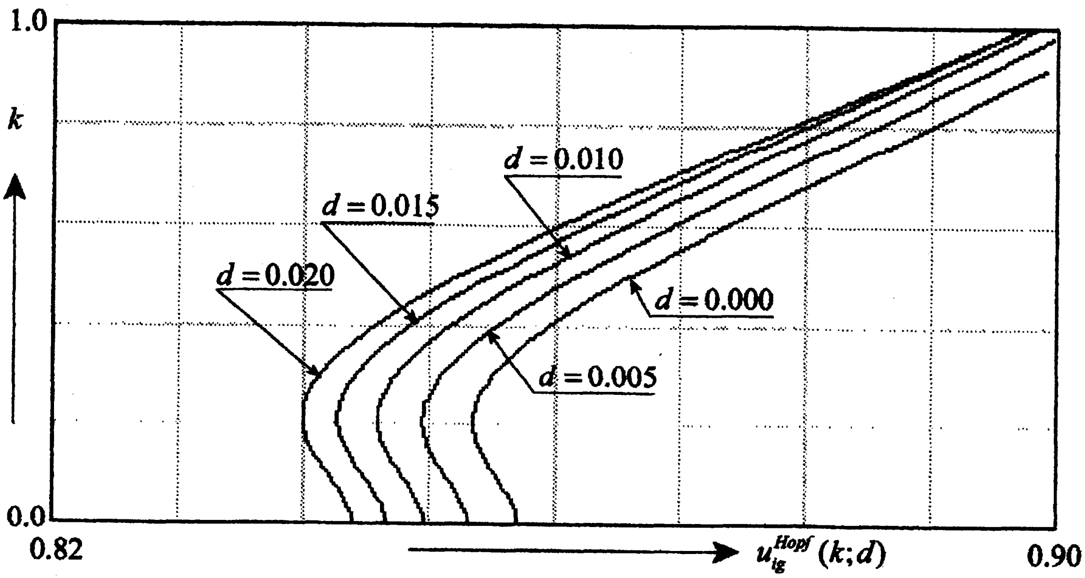

Figure 6.1: Hopf bifurcation point $u_{ig}^{Hopf}(k; d)$ of planar traveling wave

Property 3 (a pair of complex conjugate eigenvalues and the Hopf bifurcation) Let $d = 0$ or $d \ll 1$. There exists an interval $J \subset (0, 1)$ such that the problem has a pair of complex conjugate eigenvalues for $u_{ig} \in J$. Its real part is negative for small u_{ig} and increases with u_{ig}, and the pair crosses the imaginary axis transversely at a critical value $u_{ig} = u_{ig}^{Hopf}(k; d)$.

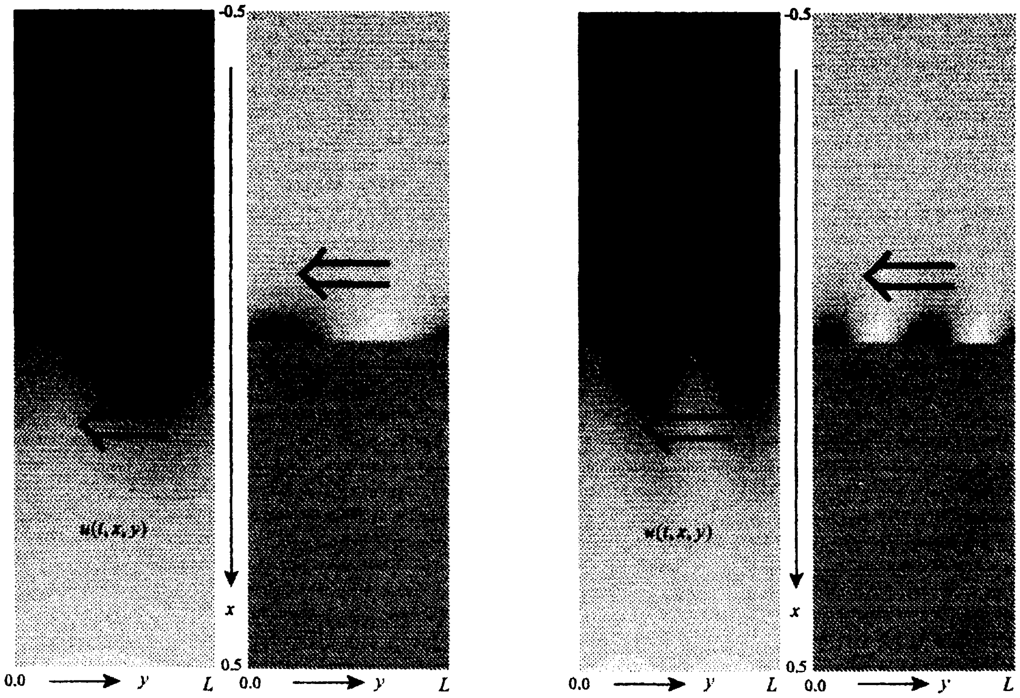

Figure 6.2: An approximate solution at the Hopf bifurcation point for $k = 0.21$ ($t = 0.0$, $u_{ig}^{Hopf}(k; 0) \simeq 0.853555$, $\lambda = i\sigma$ with $\sigma \simeq 0.407945$, $L = 2\pi/k$ for the left pair while $L = 4\pi/k$ for the right one)

Property 4 (relation between the wave number k and the Hopf bifurcation point $u_{ig}^{Hopf}(k; d)$) For each d, there exists $k^*(d) > 0$ such that $u_{ig}^{Hopf}(k; d)$ decreases with increasing k for $k < k^*(d)$ and $u_{ig}^{Hopf}(k; d)$ increases with k for $k > k^*(d)$ as shown in Figure 6.4.

As stated in Property 3 in the above, a planar traveling wave lose its stability by the Hopf bifurcation at $u_{ig} = u_{ig}^{Hopf}(k; d)$. Then, what solution emerges through the Hopf bifurcation ? The planar pulsating wave takes place of the planar traveling wave clearly when $k = 0$, however, what kind of oscillatory solution appears when $k \neq 0$? The bifurcated solution in the band domain Ω_2 is approximated by a suitable sum of planar traveling wave and solution of linearized equation (5.1) at the Hopf bifurcation point $u_{ig} = u_{ig}^{Hopf}(k; d)$. A solution of (5.1) with $u_{ig} = u_{ig}^{Hopf}(k; d)$ is given by

$$\begin{pmatrix} \mathrm{Re}\phi(x) & \mathrm{Im}\phi(x) \\ \mathrm{Re}\psi(x) & \mathrm{Im}\psi(x) \end{pmatrix} \begin{pmatrix} \cos(ky + \sigma t) & \sin(ky + \sigma t) \\ -\sin(ky + \sigma t) & \cos(ky + \sigma t) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{6.5}$$

with arbitrary constant a and b, where σ denotes the imaginary part of eigenvalue and $(\phi(x), \psi(x))$ is the eigenfunction associated with $i\sigma$. In Figure 6.5 we draw the distribution of the sum of planar traveling wave and the above solution with $a = b = 1$ with the view to checking the type of bifurcated solution. The wave number k equals 0.21 for the both left and right pairs of figures. The band width L equals $2\pi \cdot 1/k$ for the left pair and it does $2\pi \cdot 2/k$ for the right one. The planar traveling wave propagates downward, and the left and right figures of each pair display the distribution of $u(0, x, y)$ and $v(0, x, y)$ in a grey scale, respectively. The expression (6.5) implies that the distributions of $u(t, x, y)$ and $v(t, x, y)$ rotate left or right as time t goes on, and tells us the appearance of helical wave via the Hopf bifurcation of planar traveling wave.

Combining the above discussion, we roughly summarize the results at the end of Section 1. We note that there are reported bifurcation diagrams similar to Figure 6.4 in [5], [1] and [6] among others, where the appearance of spin waves (= helical waves) are discussed based on a reduced model called the two-phase model of Margolis ([4]). Sivashinsky [14] also obtains a similar bifurcation diagram by using another reduced system.

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