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Stable Solutions to the Ginzburg-Landau Equation in a Thin Domain

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1 Introduction

We study the Ginzburg-Landau equation which was proposed as a model in the theory of superconductivity ([8]). In the last decade many mathematical works appeared and the existence of solutions expressing the characteristic features of the superconductivity were studied. For instance see [1], [3], [4], [7],[18, 19, 20] for nucleation of surface superconductivity, [10], [15], [16, 17], [23] for permanent current and [24] [25] for vortices. Such phenomena can be observed in different values of an applied magnetic field. For instance the permanent current is a typical phenomenon in the absence of applied magnetic fields while the other ones take place in appropriate regimes of strength of the field.

In this paper we deal with the Ginzburg-Landau equation and the associate energy functional with no applied magnetic fields. Our aim is to show the existence of stable solutions expressing permanent currents. Here the permanent current can be realized by a stable nonconstant solution to the equation. By physical intuition it seems natural to study this problem in a non-simply connected domain such as a donuts-like domain or a multiply connected domain. Actually the existence of stable nonconstant solutions in a non-simply connected domain were shown in [9], [16], [23]. Among other things Jimbo-Zhai [16] proved that any non-simply connected 3-dimensional domain allows stable solutions with complicated topological structure associated with the topology of the domain by taking λ large. On the other hand the existence of nonconstant stable solutions were proved in [13], [16] under no constraint of the topological condition. They instead used some domain perturbation arguments. The former one showed it in a 3-dimensional thin domain while in the latter one they proved it by filling the holes of a 3-dimensional non-simply connected domain with thin pancake-like domains.

The purpose of the present study is to develop the domain perturbation argument used in [13] to prove the existence of stable nontrivial solutions in more variety of domains than proved in [13]. Here we assume that the thin domain whose thickness is controlled by a small positive parameter ϵ,

$$\Omega(\epsilon) := \{(x', x_3) := (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_3 < \epsilon a(x'), x' \in D\}, \tag{1.1}$$

where D is a 2-dimensional domain with smooth boundary ∂D and $a(x')$ is a smooth positive function describing the variable thickness of $\Omega(\epsilon)$. We consider the Ginzburg-Landau energy given by

$$\mathcal{G}_\epsilon(\Psi, A) := \int_{\Omega(\epsilon)} \left\{ \frac{1}{2}|(\nabla - iA)\Psi|^2 + \frac{\lambda}{4}(1 - |\Psi|^2)^2 \right\} dx + \frac{1}{2}\int_{\mathbb{R}^3} |\mathrm{curl} A|^2 dx. \tag{1.2}$$

where Ψ is complex-valued, A is a magnetic potential and λ is a positive parameter ($\sqrt{\lambda}$ is the Ginzburg-Landau parameter). Then the Ginzburg-Landau equation is obtained by the Euler-Lagrange equation of the above energy functional,

$$\begin{cases} (\nabla - iA)^2\Psi + \lambda(1-|\Psi|^2)\Psi = 0 & \text{in } \Omega(\epsilon), \\ \dfrac{\partial \Psi}{\partial \nu} = i(A\cdot\nu)\Psi & \text{on } \partial\Omega(\epsilon), \\ \mathrm{curlcurl}A = \begin{cases} J & \text{in } \Omega(\epsilon), \\ 0 & \text{in } \mathbb{R}^3\setminus\Omega(\epsilon), \end{cases} \end{cases} \tag{1.3}$$

where

$$J := \frac{1}{2i}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - |\Psi|^2 A, \tag{1.4}$$

ν denotes the outer normal vector on $\partial\Omega(\epsilon)$ and Ψ^* is the complex conjugate of Ψ.

Since the equation is invariant under the gauge transformation

$$\Psi \longmapsto \Psi e^{i\rho}, \quad A \longmapsto A + \nabla\rho$$

(ρ: a smooth scalar function), we can choose a gauge so that

$$\mathrm{div}A = 0 \quad \text{in } \mathbb{R}^3 \tag{1.5}$$

holds. Then $\mathrm{curlcurl}A = -\Delta A$.

We easily see that (1.3) has a constant solution $(\Psi, A) = (e^{ic}, 0)$ (c: any real number) which is a (global) minimizer of (1.2). Thus our interest is to seek a nontrivial stable solution, that is, a (nontrivial) local minimizer of the energy functional.

In [13] the existence of a nontrivial stable solution of (1.3) (or a local minimizer to (1.2)) is studied by considering the reduced equation as $\epsilon \to 0$,

$$\begin{cases} \dfrac{1}{a(x')}\mathrm{div}_{x'}(a(x')\nabla_{x'}\psi) + \lambda(1-|\psi|^2)\psi = 0 & \text{in } D, \\ \dfrac{\partial\psi}{\partial\nu_{x'}} = 0 & \text{on } \partial D \end{cases} \tag{1.6}$$

which is the Euler-Lagrange equation of

$$G(\psi) := \int_D \left\{ \frac{1}{2}|\nabla_{x'}\psi|^2 + \frac{\lambda}{4}(1-|\psi|^2)^2 \right\} a(x')dx'. \tag{1.7}$$

More precisely it is shown that if the reduced equation (1.6) has a 'nondegenerate' stable solution, then the original equation also has a stable solution near the solution of the reduced equation. As an application the existence of a stable solution with zeros is proved with the aid of the result of [12] when the domain D is disk and $a(x')$ satisfies an appropriate condition. Here the 'nondegenerate' stable solution to (1.6) implies a stable solution at which the linearized operator allows a simple zero eigenvalue and negative ones. We note that since (1.6) is invariant under the transformation $\psi \mapsto \psi e^{i\xi}$ (ξ: a real number), the linearized operator always has a zero eigenvalue.

Our aim is to establish the existence of a nontrivial stable solution of (1.3) for a more general or a geometrically complicated domain. In order to carry out it we improve the argument used in [13]. We consider the domain D containing a family of disjoint domains $\{D_j\}_{j=1..N}$ and show the existence of a local minimizer of (1.2) for sufficiently small ϵ if each domain D_j allows a stable nondegenerate solution ψ_j to (1.6) and if the quantity of

$$\alpha_I := \int_{D\setminus\cup_{j=1}^N D_j} a(x')dx' \tag{1.8}$$

is small enough. As applications we can consider the following two types of domains:

(i) each D_j is a disk and the remaining subset $D \setminus \cup_{j=1}^N D_j$ consists of thin channels connecting two of $\{D_j\}$;

(ii) $N = 1$ and $D \setminus D_1$ consists of a family of disjoint closed disks $\{\overline{B}_\rho(p_j)\}_{j=1..m}$, where $B_\rho(p_j) := \{x' : |x' - p_j| < \rho\}$.

In the first case (i) the volume of the channels must be controlled so that the quantity α_I of (1.8) is small enough. We see from [12] that there are nondegenerate stable vortex solution with degree 1 or -1 in each disk D_j. Thus counting the constant solution in D_j, we can obtain 3^N types of local minimizers to (1.2) (including the minimizer) by applying the main theorem presented in the following section.

On the other hand α_I could be small by an appropriate choice of $a(x')$ in the second case (ii). Since D_1 is not simply connected, by virtue of [14] we can construct a nondegenerate stable solution in each homotopy class $C^0(\overline{D_1}; S^1)$; thus the existence of a local minimizer of (1.2) with the zero set localized in $\{(x', x_3) : 0 < x_3 < \epsilon a(x'), x' \in D \setminus \overline{D_1}\}$ can be assured with the aid of the main theorem. This result shows a pinning of vortices by the inhomogeneity of the surface of the domain.

2 Assumptions and main theorem

We identify a complex-valued function $\Psi(x) = u_1(x) + iu_2(x)$ with a vector-valued one $u(x) = (u_1(x), u_2(x))^T$. Thus

$$L^2(\Omega(\epsilon); \mathbb{C}) = L^2(\Omega(\epsilon); \mathbb{R}^2), \quad H^1(\Omega(\epsilon); \mathbb{C}) = H^1(\Omega(\epsilon); \mathbb{R}^2), \quad \text{etc.}$$

As in [13] and [16], define a Banach space

$$Y := \{B \in L^6(\mathbb{R}^3; \mathbb{R}^3) : \nabla B \in L^2(\mathbb{R}^3; \mathbb{R}^{3\times 3})\}, \tag{2.1}$$

with norm

$$\|B\|_Y := \|\nabla B\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}.$$

If a solution of (1.3) is a local minimizer of the functional $\mathcal{G}_\epsilon$ in the space $H^1(\Omega(\epsilon);\mathbb{C})\times Y$, we call it a stable solution.

Let $\{D_j\}_{j=1..N}$ be a family of domains such that

$$\overline{D_j}\cap\overline{D_k}=\emptyset \quad (j\neq k), \tag{2.2}$$

and the boundary of each domain D_j is sufficiently smooth (at least C^3). We suppose that the domain D satisfies

$$D\supset\cup_{j=1}^N D_j \tag{2.3}$$

and that if $N=1$, there are $y_k\in D, \rho_k>0$ $(k=1,..,m)$ such as

$$D\setminus D_1=\cup_{k=1}^m\overline{B_{\rho_k}(y_k)},\quad \overline{B_{\rho_k}(y_k)}\cup\overline{B_{\rho_\ell}(y_\ell)}=\emptyset \quad (k\neq\ell), \tag{2.4}$$

where $B_{\rho_k}(y_k):=\{x':|x'-y_k|<\rho_k\}$.

We let $a(x')$ be a smooth positive function defined on D. Then we may assume

$$\sup_{x'\in D} a(x')=1 \tag{2.5}$$

by normalization. We also write $a=a_j(x')$ for $x'\in D_j$ and define

$$\|\psi\|_{L^2_a(D_j)}:=\left(\int_{D_j}|\psi(x')|^2a_j(x')dx'\right)^{1/2}$$

and by $L^2_a(D_j;\mathbb{C})$ the space of square integrable functions with the norm $\|\psi\|_{L^2_a(D_j)}$. We also define

$$\|\psi\|_{H^1_a(D_j)}=(\|\psi\|^2_{L^2_a(D_j)}+\|\nabla_{x'}\psi\|^2_{L^2_a(D_j)})^{1/2}$$

and $H^1_a(D_j;\mathbb{C})$. Similarly we can define $\|\psi\|_{L^2_a(D)}$, $\|\psi\|_{H^1_a(D)}$, $L^2_a(D;\mathbb{C})$ and $H^1_a(D;\mathbb{C})$ respectively.

Let $\psi_j(x')$ be a solution to (1.6) for $a(x')=a_j(x')$ and $D=D_j$. The linearized operator around ψ_j is given by

$$\begin{cases} \hat{L}_j[\varphi]:=\dfrac{1}{a_j(x')}\mathrm{div}_{x'}(a_j(x')\nabla_{x'}\varphi)+\lambda(1-|\psi_j|^2)\varphi-2\lambda\mathrm{Re}(\psi_j^*\varphi)\psi_j, \\ \\ \mathrm{Dom}(\hat{L}_j):=\{\varphi\in L^2_a(D_j;\mathbb{C}):\varphi\in H^2(D_j;\mathbb{C}),\ \partial\varphi/\partial\nu_{x'}=0 \text{ on } \partial D_j\} \end{cases} \tag{2.6}$$

Note that

$$\hat{L}_j[i\psi_j]=\frac{1}{a_j(x')}\mathrm{div}_{x'}(a_j(x')\nabla_{x'}(i\psi_j))+\lambda(1-|\psi_j|^2)(i\psi_j)=0,$$

thus $\varphi=i\psi_j$ is an eigenfunction corresponding to zero eigenvalue of $\hat{L}_j$. One can also check that $\hat{L}_j$ is a self-adjoint operator with respect to the inner product

$$\langle\psi,\varphi\rangle_{L^2_a(D_j)}:=\mathrm{Re}\int_{D_j}\psi(x')\varphi^*(x')a_j(x')dx' \tag{2.7}$$

(recall $\mathbb{C}$ is identified with $\mathbb{R}^2$), thus the spectrum of $\hat{L}_j$ consists of only real eigenvalues. We call ψ_j a nondegenerate stable solution if the following holds:

(A) Zero is a simple eigenvalue of $\hat{L}_j$ and the remaining eigenvalues are negative.

We write by Ψ_0 a $(C^0(\overline{D};\mathbb{C}) \cap H^1_a(D;\mathbb{C}))$-extension of $\psi_j(x'), x' \in D_j$ $(j = 1,..,N)$, that is,

$$\Psi_0 \in C^0(\overline{D};\mathbb{C}) \cap H^1_a(D;\mathbb{C}), \qquad \Psi_0(x') = \psi_j(x') \text{ in } D_j \quad (j = 1,..,N). \tag{2.8}$$

We denote by

$$\tilde{\Psi}(x',z) := \Psi(x', \epsilon a(x')z) \qquad ((x',z) \in D \times (0,1))$$

the transformed function of $\Psi(x)$ $(x \in \Omega(\epsilon))$ and denote the norm by

$$\|\tilde{\Psi}\|_{L^2_a(D\times(0,1);\mathbb{C})} := \left\{ \int_D \int_0^1 |\tilde{\Psi}(x',z)|^2 a(x')dx'dz \right\}^{1/2}.$$

Now we state the main theorem of this paper.

Theorem 2.1 *Consider* (1.3) *for* (1.1) *with D satisfying* (2.2) – (2.3) *or* (2.3) – (2.4). *For each j, $1 \le j \le N$, suppose that ψ_j is a solution to* (1.6) *with $a = a_j$ and $D = D_j$ and that it satisfies* (A). *Write by Ψ_0 the extension as in* (2.8) *and set*

$$\alpha_0 := \min_{x' \in D \setminus \cup_{j=1}^m D_j} a(x').$$

Then there exist a number $M > 0$ and a small number $\delta_1 > 0$ such that if $\delta \in (0, \delta_1)$ and $a(x')$ satisfies

$$\alpha_I = \int_{D\setminus \cup_{j=1}^N D_j} a(x')dx' \le M\delta^2,$$

there is a small $\epsilon_0 = \epsilon_0(\delta, \alpha_0, D) > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ (1.2) *has a local minimizer $(\Psi_\epsilon, A_\epsilon)$ (in $H^1(\Omega(\epsilon);\mathbb{C}) \times Y$) satisfying*

$$\|\tilde{\Psi}_\epsilon - \Psi_0 e^{i\hat{c}_j}\|_{H^1_a(D_j\times(0,1);\mathbb{C})} < \delta, \quad j = 1, \cdots, N, \tag{2.9}$$

where each $\hat{c}_j$ is the number given by

$$\|\tilde{\Psi}_\epsilon - \Psi_0 e^{i\hat{c}_j}\|_{L^2_a(D_j\times(0,1);\mathbb{C})} = \inf_{0\le c\le 2\pi} \|\tilde{\Psi}_\epsilon - \Psi_0 e^{ic}\|_{L^2_a(D_j\times(0,1);\mathbb{C})} \tag{2.10}$$

For the proof of the above theorem see [21].

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