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A duality theorem for fractal sets and Cuntz algebras and their central extensions

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Abstract

In this paper we introduce concepts of "flower type" and "branch type" for fractal sets, at first and the concept of the central extension of Cuntz algebras secondly and we give the following results:

(1)There exists a duality theorem between fractal sets of flower type and branch type, which is called "flower-branch duality"(Theorem I).
(2)We introduce a concept of central extensions of Cuntz algebras(which will be called Zunk algebra) and make a Fock representation on a fractal set of branch type(Proposition B).
(3)The flower-branch duality induces the duality theorem between the representations of Cuntz algebras and Zunk algebras(Theorem II).

It is suggested how the duality theorem can be applied to several topics both in mathematics and physics. The details will be given in the forthcoming papers.([1],[6] and [7]

1 Introduction

In papers([8],[9]) we have treated fractal sets by use of representations of Cuntz algebras and give the criterion whether they are equivalent or not. In this paper we introduce two kinds of fractal sets, the one is called of flower type which describes the condensating objects and clusters. The other one is called of branch type which describes the growing objects, developing cities, tree leaves, baccterias. At first we show that there exists a one to one correspondence between these two classes, which will be called "flower-branch duality"(Theorem I). Next we will be concerned with the representations. We have constructed representations on fractal sets of flower type. Here we will make the Fock representations on fractal sets of branch type. For this purpose we have to make a central extension of Cuntz algebras, which is called Zunk algebras. Here we have to make its central extension, when we want to make the Fock representation of the Cuntz algebra. Because we have to prepare central elements for the vaccum state. Hence we are led to an introduction of central extensions of Cuntz algebras at first and then we will make representations of Zunk algebras on fractal sets of branch type(Proposition B). Then we can apply the duality theorem between these representations and can discuss their equivalences(Theorem II).

2 A duality theorem for fractal sets of flower type and branch type

In this section we prepare two kinds of fractal sets which are called "of flower type" and "of branch type" and prove Theorem I. At first we recall some basic facts on fractal sets([9]). For the simplicity sake we restrict ourselves only to a system of *piecewise affine* mappings $\{\sigma_j : j = 1, 2, .., N\}$ between a compact set K_0. Then we see that there exist non-negative numbers $\{\lambda_i\}$ and positive numbers $\{\Lambda_i\}$ $(0 \leq \lambda_i \leq \Lambda_i)$ satisfying the condition:

$$\lambda_i d(x, y) \leq d(\sigma_i(x), \sigma_i(y)) \leq \Lambda_i d(x, y)(i = 1, 2, .., N).$$

Here we assume that these conditions are extremal, namely they attain the equalities in the both sides exactly. Here we will be concerned with mappings σ_i with $\Lambda_i \leq 1$, which we call *contractible* mapping simply. A contractible mapping σ_i with $\Lambda_i < 1$(resp. $\Lambda_i = 1$) is called *essentially proper*(resp. *partially isometric*). Moreover, an essentially proper mapping σ_i is called *proper*, if $\lambda_i = \Lambda_i$ holds. We call the mapping σ_i in the case where $\lambda_i = 0$ *degenerate* and in the other case *non-degenerate* respectively. Next we define a self similar fractal set. Here we make a comment on the construction of fractal sets. Although the construction of the fractal sets defined by essentially proper mappings is unique, we still have not definite construction principles of fractal sets for general contractible mappings. For a system of $\sigma_j : K_0 \mapsto K_0(j = 1, 2, 3, 4)$, where we need not necesarry assume that they are contractible, we put

$$K = \cup_{i=0}^{\infty}(\cap_{n=i}^{\infty} K_n), \text{ where } K_n = \cup \sigma_j(K_{n-1})(n = 1, 2, ...). \tag{2.1}$$

We notice the following invariant condition([4]):

$$\cup_{j=1}^{N} \sigma_j(K) = K. \tag{2.2}$$

In the case of essentially proper mappings σ_j, the definition is equivalent to the given in (2,3)([3]).
In this paper we assume that the following separation condition is satisfied:

$$\sigma_i(K^\circ) \cap \sigma_j(K^\circ) = \phi, \tag{2.3}$$

where E° implies the open kernel of E.
Next we proceed to the fractal sets of flower type and branch type.

(i) A fractal set of flower type
For a system of contractible mappings $\sigma_j : K_0 \mapsto K_0(j = 1, 2, .., N)$ with the separation condition(2,3), we put

$$K = \cap_{n=1}^{\infty} K_n, \text{where } K_i = \cup_{j=1}^{N} \sigma_j(K_{i-1}), \tag{2.4}$$

(ii)A fractal set of branch type

For a system of contractible mappings $\sigma_j : K_0 \mapsto K_0 (j = 1, 2, .., N)$, we choose a reference point p_0 in K_0 and define fundamental branches by

$$L_0 = \cup_{j=1}^{N} L_{0|j}, \quad L_{0|j} = \overline{p_0, \sigma_j(p_0)}, (j = 1, 2, ..N) \tag{2.5}$$

and we make a lattice L on K_0 in the following manner:

$$\begin{aligned} &(1) L \text{ is connected,} \\ &(2) L = \cup_{n=0}^{\infty} L_n, \text{ where } L_n = L_{n-1} \cup L_{n'}, L_{n'} = \cup_j \sigma_j(L_{n-1}), \\ &(3) L \text{ satisfies the separation conditions :} \\ &\mu(\sigma_j(L_{n-1}) \cap \sigma_k(L_{n-1})) = 0 (k \neq j), \quad \mu(L_{n-1} \cap L_{n'}) = 0 (n \neq n'), \end{aligned} \tag{2.6}$$

where μ is the Lebesgue measure on L, which is called a fractal set of *branch* type.

Then we see that they are connected through the following duality theorem:

Theorem I(Flower-Branch duality for self similar fractal sets)

The set of fractal sets of flower type(resp. branch type) in K_0 is denoted by $\mathcal{K}(K_0)$(resp. $\mathcal{L}(K_0)$. Then there exists a one to one mapping $\phi : \mathcal{K}(K_0) \mapsto \mathcal{L}(K_0)$ between fractal sets of lattice sets L and fractal sets of branch type L

Proof of Theorem I

Suppose that a fractal set of branch type is given. Then we can make the corresponding dual fractal set of flower type in the following manner:Putting

$$K_n = \text{the closure of } \{L - \cup_{k=1}^{n}(L_n)\}. \tag{2.7}$$

By the construction of L_n, we see that

$$K_{n+1} = \cup \sigma_j K_n \tag{2.8}$$

$$K_{n+1} \subset K_n. \tag{2.9}$$

Then we have the desired fractal set

$$K = \cap_{n=1}^{\infty} K_n, \tag{2.10}$$

which satisfies the invariant condition(2,2). The construction in the converse direction is given in (2,6). Hence we have proved the assertion.

In the case where the fractal set of flower type is a proper fractal set, we can find more direct correspondence. Taking the fact into account that each point x can be expressed as follows

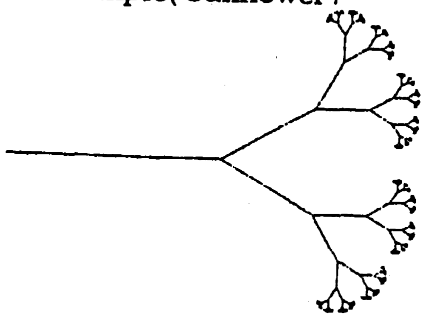
$$x = \lim_{n \to \infty} \sigma_{i_1} \sigma_{i_2} ... \sigma_{i_n}(x_0), \tag{2.11}$$

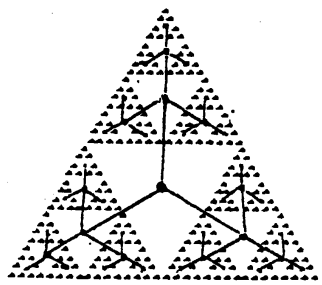
where x_0 is an arbitrary point of K, we can make a one to one correspondence $\Phi : K \times \mathbf{R}_+ \mapsto L$ which is defined by

$$\Phi(x)(t) = \cup_{n=1}^{\infty} \sigma_{i_2}\sigma_{i_3}...\sigma_{i_n}(L_{i_1})(t) \text{ where } t \in [n, n+1], \tag{2.12}$$

where we notice the $\mathbf{R}_+$ is the union of intervals $[n, n+1]$ which describe the parameters of pieces of lines. We give an example of califlower:

Example(Califlower)





3 Central extensions of Cuntz algebras and their representations on fractal sets of branch type

In this section we recall some basic facts on the Cuntz algebras and their central extensions at first and discuss representations on self similar fractal sets.
The Cuntz algebra $\mathcal{O}(N)$ is a C^*-algebra with generators $\{S_j\}(j = 1, 2, .., N)$ with the following commutation relations([2]):

$$(1) S^*{}_j S_j = 1 (j = 1, 2, .., N), \quad (2) \sum S_j S^*{}_j = 1. \tag{3.13}$$

These commutation relations give an algebraic description of the division of the total space into N-parts. Next we proceed to central extensions of the Cuntz algebras. A C^*-algebra $\mathcal{Z}(N)$ is called the Zunk algebra with generators $\{S_j\}(j = 1, 2, .., N)$ with the following commutation relations:

$$(1) S^*{}_j S_j = 1 (j = 1, 2, .., N), \quad (2) \sum S_j S^*{}_j + Q = 1, \text{ where } Q^* = Q, Q^2 = Q. \tag{3.14}$$

We see that the Zunk algebra is not simple and we see that it is obtained from the Cuntz algebra by the central extension. We can make representations of Cuntz algebras on fractal sets of flower type in a well known manner:

Proposition A(Hausdorff representations on fractal sets of flower type)([8],[9]) Let K be a fractal set of flower type which defined by $\{\sigma_j\}(j = 1, 2, ..N)$. Then we have the following representation $\pi : \mathcal{O}(N) \to B(L^2_D(K))$:

$$\pi(S_j)f(x) = \begin{cases} \Phi_j{}^{1/2}(\sigma_j^{-1}(x)) f(\sigma_j^{-1}(x)) & x \in \tau_j(K) \\ 0 & x \notin \tau_j(K) (j = 1, 2, .., N) \end{cases}$$

$$\pi(S^*{}_j)f(x) = \Phi_j^{-1/2}(x)f(\sigma_j(x))(j = 1, 2, ..N)$$

The representation defined in above is called a regular representation. We can prove the following proposition:

Proposition(The Kakutani's dichotomy theorem ([5]))
Let K and K' be two self similar fractal sets of flower type with the same number of generators N which are defined on compact sets K_0 and K_0' respectively. Then we see that the Hausdorff representations are equivalent, if and only if they satisfy the following conditions:

$$\lambda_{i,j}^D = \lambda'^{D'}_{i,j}(i, j = 1, 2). \tag{3.15}$$

Then, for two Hausdorff representations:$\pi_i : \mathcal{O}(N) \longmapsto B(L_D^2(K_i))(i = 1, 2)$, we can find a unitary operator $U : L_D^2(K_1) \longmapsto L_D^2(K_2)$ such that $\pi_1(S)U = U\pi_2(S)$ holds for any $S \in \mathcal{O}(N)$

Remark The equivalence does not imply that $D = D'$ holds.

Next we proceed to the construction of the representations of Zunk algebras on fractal sets of branch type. For this we prepare the so called *Haar basis* $e_{i_1,i_2,..,i_n}$:

$$e_{i_1,i_2,..,i_n} = \begin{cases} 1 & x \in \sigma_{i_2}\sigma_{i_3}...\sigma_{i_n}(L_{i_1}) \\ 0 & x \notin \sigma_{i_2}\sigma_{i_3}...\sigma_{i_n}(L_{i_1}) \end{cases}. \tag{3.16}$$

After the normalization with respect to the Borel measure $L^2(L, d\mu)$, we have a system of orthnormal basis and have the Hilbert space $\mathbf{H}$. Then we can prove the following theorem:

Proposition B(The Fock representation on a fractal set of branch type)
Let L be a fractal set of branch type. Then we have the following Fock representation $\rho_b : \mathcal{Z}(4) \mapsto B(\mathbf{H})$:Choosing the vaccum $|0>$ we define

$$\begin{aligned} Q|0> &= 0> && (3.17) \\ T_j|0> &= e_j && (3.18) \\ T_j(e_{i_1,i_2,..,i_n}) &= e_{j,i_1,i_2,..,i_n}, && (3.19) \\ Q^*|0> &= 0> && (3.20) \\ T_j^*(e_{i_1,i_2,..,i_n}) &= \begin{cases} e_{i_2,..,i_n} & j = i_1 \\ 0 & j \neq i_1 \end{cases} \end{aligned}$$

The proof is easy and may be omitted. Next we proceed to the duality theorem to representations which arise from the Theorem I.

Then we can prove the following theorem

Theorem II(Flower-Branch duality theorem for representations of Hausdorff type)
Let L be a fractal set of branch type and let K be the corresponding dual fractal set.

Then the following duality is induced between the representations:For a representation of Hausdorff type, $\pi_{flower} : \mathcal{O}(N) \mapsto L^2(K, d\mu_K^D)$, there exists an central extension $\mathcal{Z}(N)$ of the Cuntz algebra $\mathcal{O}(N)$ so that there is a representation $\pi_{branch} : \mathcal{Z}(N) \mapsto L^2(L, d\mu_L)$. The converse is also true.

Proof of Theorem II

We give a proof of Theorem II. At first we assume that the Fock representation is given on a fractal set of branch type. We make the fractal set of flower type. Then we can extend the representation to the representation of Hausdorff type in the following manner:Choosing the Hilbert space $\mathbf{H}(L)$ spanned by the basis

$$\{e_\alpha : \alpha = (\alpha_1, \alpha_2, ...) \in \prod_{n=1}^{\infty}\{1, 2, ..., N\}\}$$

of infinite paths. Then we see that the representation above defined can be written as follows:

$$\pi(S_j)e_\alpha = e_{(j,\alpha_1,\alpha_2,...)}, \quad \pi(S_j^*)e_\alpha = \begin{cases} e_{(\alpha_2,\alpha_3,...)} & j = \alpha_1 \\ 0 & j \neq \alpha_1 \end{cases}$$

The converse direction can be given in the similar manner and may be omitted.

4 Applications

In this section we demonstrate how we can apply the duality theorem to several topics in mathematics and physics. We treat the following three topics:(1)Infinite dimensional Clifford algebras, (2)Lattice models on fractal sets and (3)Complex anaysis. The applications can be performed as follows:We realize the complex systems as infinite dimensional objects and represent them by use of those of Cuntz algebras on fractal sets. We approach the systems from finite dimensional systems by approximation. This can be done by the representations of Zunk algebras. By use of the duality theorems, we can discuss the original complex systems.

(α)Infinite dimensional Clifford algebras ([6])

We define the infinite dimensional Clifford algebras by use of the inductive limit of finite dimensional Clifford algebras. For example, we can choose the exahusions in the following manner:At first we notice that the Clifford algebra $Cl_{2N+1}(\mathbf{C})$ can be realized on the matrice space $M(2^N : \mathbf{C})$ in the inductive manner. The $Cl_3(\mathbf{C})$ can be given by the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.21}$$

For the generators $A_j (j = 1, 2, .., 2p-1)$ of $C_{2p-1}(\mathbf{C})$, putting

$$\begin{pmatrix} A_j & 0 \\ 0 & -A_j \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & iI_2 \\ -iI_2 & 0 \end{pmatrix} (j = 1, 2, .., p), \tag{4.22}$$

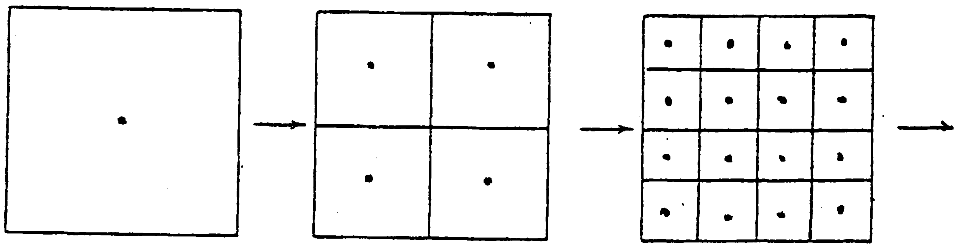
we have the generators of $C_{2p+1}(\mathbf{C})$. We notice that

$$Cl_{2N+1}(\mathbf{C}) \cong M(2^N, \mathbf{C}) \tag{4.23}$$

We can introduce the infinite dimensional Clifford algebra by use of the inductive limit:

$$Cl(\infty : \mathbf{C}) = \underrightarrow{lim} Cl_{2N+1}(\mathbf{C}). \tag{4.24}$$

By this construction we are temptated to introduce a fractal method to the infinite dimensional Clifford algebras. In fact we can realize the algebras in terms of fractal sets of Peano floer type which are defined by the four contractible mappings $\{\sigma_{i,j}|i,j = 1,2\}$ between the unit rectangle $K_0(=\{(x,y)|0 \leq x \leq 1, 0 \leq y \leq 1\}$ with the separate condition. Considering the dual fractal set of branch type, we can realize the representations of a sequence of finite dimensional Clifford algebras by use of the Zunk representations.



$M_1(\mathbf{C})$ $\qquad$ $M_2(\mathbf{C})$ $\qquad$ $M_4(\mathbf{C})$

In fact we can construct the representation in the following manner. At first we notice the following fact:

$$Cl(\infty : \mathbf{C}) \subset \mathcal{Z}(4). \tag{4.25}$$

Then restricting the representation given in Proposition, we have the following

Theorem III(Duality theorem for infinite dimensional Clifford algebras)
(1)**Existence)**:We have a representation $\pi_b : Cl(\infty : \mathbf{C}) \mapsto B(L^2(L))$.
(2)**(Equivalence)**:Two representations of (1) are unitary equivalent, if and only if the dual representations $\pi_f : \mathcal{O}(4) \mapsto B(L^2_D(K))$ satisfy the Kakutani's condition on K in the case where K is a proper fractal set.

By this theorem we may discuss the Clifford analysis for $Cl(\infty : \mathbf{C})$. The detail will be given in ([6]).

(β) **Lattice models on fractal sets**([1])
We can treat the interacting lattice models of fermionic N-spin particles in terms of fractal geometry and discuss their phase transitions by use of the duality theorem. We consider the following standard(i.e. free) lattice model on the lattice of positive integers $\mathbf{N}$:

$$\mathcal{H}_0 = \beta \sum a_n^{(i)} a_n^{(i)\dagger}, \tag{4.26}$$

where $a_n^{(j)}$ are the annihilation operators of frmionic type at the site $n(n = 1,2,..)$ for the spin $j(j = 1,2,..,N)$ and $a_m^{(i)\dagger}$ are the corresponing creation operator. The algebra is called the fermionic algebra with spin N and is denoted by $AF(N)$. We will treat interacting lattice models in terms of fractal geometry. This can be performed by use of the representation of the Zunk algebra.

Theorem IV(Duality theorem for lattice models on fractal sets)
Let L be a fractal set of branch type and let $\pi : \mathcal{Z}(N) \mapsto B(L^2(L))$ be a representation. Then we have
(1)(**Existence**):When $M = 2^N$, we have a subalgebra $AF(N)$ of $\mathcal{Z}(M)$ with generators $\{a_n^{(i)}, a_n^{(i)\dagger} | n, m = 1,2,3,..,i,j = 1,2,..N\}$. Hence we have the Hamiltonian

$$\mathcal{H}_L = \beta \sum \pi(a_n^{(i)})\pi(a_n^{(i)\dagger}), \tag{4.27}$$

which is called the *standard Hamiltonian* on L.
(2)(**Equivalence**):Let $\pi' : \mathcal{Z}(N) \mapsto B(L^2(L'))$ be another representation. Then the dynamical systems define by the Hamiltonian are unitary equivalent if and only if the Kakutani's conditions are satisfied on the corresponding fractal sets of flower type.

Here the dynamical system is defined by

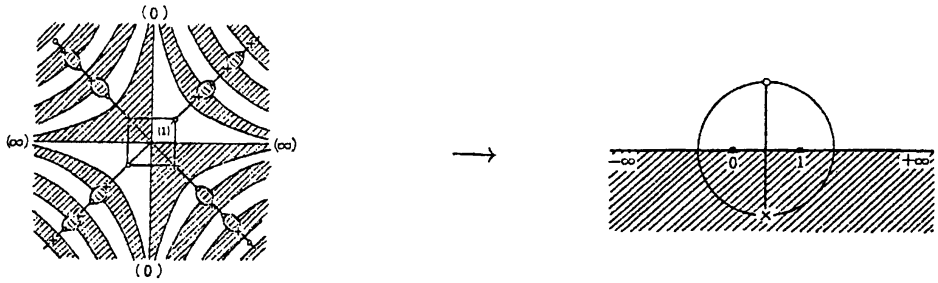
$$i\frac{dx}{dt} = [x, H_L]. \tag{4.28}$$

On the base of this theorem, we can treat the phase transitions by considering deformations of the corresponding fractal sets of flower type.

(γ)**Complex anaysis**([7])
Finally we shall show a possibility of treating complex analysis by use of the fractal geometry. In this paragraph, we will be concerned with the following two topics.
(1)The boundary behavior of a holomorphic function
The one of the important subjects in complex anaysis is to consider the behavior of holomorphic functions on the natural boundaries. This can be done in the following manner. At first we take a holomorphic function and consider its "Streck Komplex". This is defined in the following manner:Let f be a holomorphic(or meromorphic) function on D with the ∂D as a natural boundary of f. We choose a reference point z_0 in D and we put $f(z_0) = c$. We consider the points set

$$\{z_n\}, \text{ where } f(z_n) = c. \tag{4.29}$$

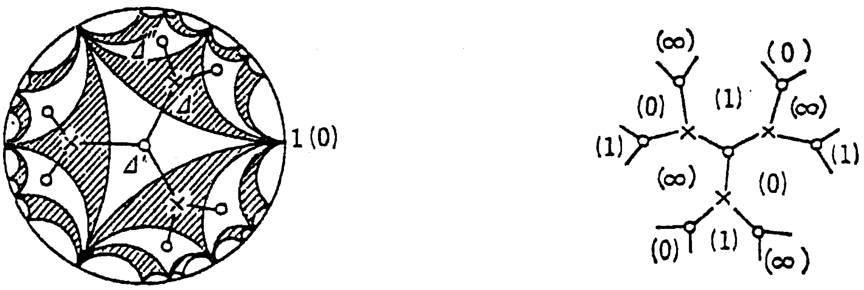
Following the analytic continuation of f from z_0, we trace the point z_1 and continuating it further, we have the sequence. Here we give an example for the holomorphic function $exp \quad z^2$:



Then following the construction rule of the fractal set of branch type, we can define the fractal set $L(f:c)$. We consider the dual fractal set $K(f:c)$. Then we may identify this set as the cluster set of the value c. Hence, considering the dual fractal set, we may treat the behavior of holomorphic functions on the boundary. We may expect to give a proof of the Picard theorem or Nevannlinna theory in terms of the fractal geometry. For example we can formulate the following problem:

Problem
"If the Hausdorff dimension of $K(f:c)$ is positive for some c, does the Picard theorem not hold ?"

(2)The moduli of Riemann surfaces
The second application is to the moduli structure of Riemann surfaces. Here we assume that the universal covering of the Riemann surface $R_g(g>1)$ is the unit disck D. Hence we can represent it by the Decktransformationen group $\Gamma(R_g)$ as $R_g = D/\Gamma(R_g)$. We denote the generators of $\Gamma(R_g)$ by $\{g_j | j = 1,2,..,2g)\}$. Taking a reference point p_0 of a fundamental region and making the fundamental branch $L_j = \overline{p_0, g_j(p_0)}(j = 1,2,..,2g)$, we can make a fractal set $L(R_g)$ of branch type by the construction method.



Then we can introduce a representation:

$$\pi_b : \mathcal{Z}(2g) \mapsto B(L^2(L(R_g)). \tag{4.30}$$

By use of the duality theorem we may discuss the moduli spaces through the anaysis on the fractal set of flower type. For example, we can formulate the following problem:

Problem
(1)Let R_g and R'_g be two Riemann surfaces. Then we can we show the biholomorphic equivalence thorough the unitary equivalence of the dual representation

$$\pi_f : \mathcal{O}(2g) \mapsto B(L^2(K(R_g))? \tag{4.31}$$

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