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G^2 Pythagorean hodograph quintic transition between two circles

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Abstract

Walton & Meek [13] obtained fair G^2 Pythagorean hodograph (PH) quintic transition S- and C-shaped curves connecting two circles. It was shown that an S-shaped curve has no curvature extremum and a C-shaped curve has a single curvature extremum. We simplified and completed their analysis. A family of fair PH quintic transition curves between two circles has been derived. We presented an algorithm with less restrictive and most reasonable constraints.
Keywords: G^2 transition, quintic splines, curvature, spiral, *Mathematica*

1 Introduction

Parametric cubic curves are popular in CAD applications because they are the lowest degree polynomial curves that allow inflection points (where curvature is zero). They are suitable for blending, e.g. rounding corners, or for smooth transition between two curves, e.g. two circular arcs. The Bézier form of a parametric cubic curve is usually used in CAD and CAGD applications because of its geometric and numerical properties. Many authors have advocated their use in different applications like data fitting and font designing. The importance of using fair curves in the design process is well documented in the literature [1, 5, 3, 6, 9, 10]. Consumer products such as ping-pong paddles can be designed by blending circles. To be visually pleasing it is desirable that the blend be fair. For applications such as the design of highways or railways it is desirable that transitions be fair. Sudden changes between curves of widely different radii or between long tangents and sharp curves should be avoided by the use of curves of *gradually increasing or decreasing radii* without at the same time introducing an appearance of forced alignment. The importance of this design feature is discussed in [2].

Cubic curves, although smoother, are not always helpful since they might have unwanted inflection points and singularities (See [7, 8]). A cubic segment has the following undesirable features:

- Its arc-length is the integral part of the square root of a polynomial of its parameter.
- Its offset is neither polynomial, nor a rational algebraic function of its parameter.
- It may have more curavture extrema than necessary.

Pythagorean Hodograph (PH) curves do not suffer from the first two of the aforementioned undesirable features. A quintic is the lowest degree PH curve that may have an inflection point, as required for an S-shaped transition curve. Spirals have several advantages of containing neither inflection points, singularities nor curvature extrema (See [4, 12]). Such curves are useful for transition between two circles (See [11]). Walton & Meek [13] considered planar G^2 quintic transition between two circles. They showed there is no curvature extremum in the case of an S-shaped transition, and there is a curvature extremum in the case of C-transition.

The objectives of this paper are:

- To simplify and complete the analysis of Walton & Meek [13].
- To obtain a family of fair G^2 PH quintic transition curves between two non-enclosing circles.

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- To achieve more flexible and less restrictive constraints.
- To discuss and prove all the shape features of transition curve.
- To find the locus of the center of smaller circle.

Our compact algorithm also guarantee the absence of interior curvature extremum for an S-shaped transition curve and one curvature extremum for C-shaped transition curve. The organization of our paper is as follows. Next section gives a brief discussion of the notation and conventions used in this paper with some theoretical background and description of method. Results for S- and C-shaped G^2 cubic transitions are then presented followed by illustrative examples and concluding remarks.

2 Background and Description of Method

We denote the two circles by Ω_0, Ω_1, with centers C_0, C_1 and radii r_0, r_1, respectively. With $r(=\|C_1 - C_0\|)$ and $r_1 = \lambda^3 r_0, 0 < \lambda \leq 1$, we consider the following problems:
(i) For $r_0 + r_1 < r$ (the circles Ω_i, $i = 0, 1$ do not intersect), find an S- shaped transition cubic curve from Ω_0 to Ω_1.
(ii) For $r_0 - r_1 < r$ (the smaller circle Ω_1 is not enclosed in the larger circle Ω_0), find a C- shaped transition cubic curve from Ω_0 to Ω_1.
In each case, this paper treats the curve whose initial curvature is positive.
Consider PH quintic curve $\boldsymbol{z}(t) \left(= (x(t), y(t))\right), 0 \leq t \leq 1$ of the form:

$$\boldsymbol{z}'(t) = (u(t)^2 - v(t)^2, 2u(t)v(t)) \tag{2.1}$$

where

$$u(t) = u_0(1-t)^2 + 2u_1 t(1-t) + u_2 t^2, \quad v(t) = v_0(1-t)^2 + 2v_1 t(1-t) + v_2 t^2 \tag{2.2}$$

Then,

$$\boldsymbol{z}(t) = \sum_{i=0}^{5} \binom{5}{i} P_i (1-t)^{5-i} t^i \tag{2.3}$$

The Bézier points are easily obtained:

$$\begin{aligned}
&P_1 = P_0 + \frac{1}{5}\left((u_0^2 - v_0^2), 2u_0 v_0\right), \quad P_2 = P_1 + \frac{1}{5}\left((u_0u_1 - v_0v_1), (u_1v_0 + u_0v_1)\right) \\
&P_3 = P_2 + \frac{1}{15}\left((2u_1^2 + u_0u_2 - 2v_1^2 - v_0v_2), (u_2v_0 + 4u_1v_1 + u_0v_2)\right) \\
&P_4 = P_3 + \frac{1}{5}\left((u_1u_2 - v_1v_2), (u_2v_1 + u_1v_2)\right), \quad P_5 = P_4 + \frac{1}{5}\left((u_2^2 - v_2^2), 2u_2v_2\right)
\end{aligned} \tag{2.4}$$

Its signed curvature $\kappa(t)$ and $\kappa'(t)$ are given by

$$\kappa(t) = \frac{\boldsymbol{z}'(t) \times \boldsymbol{z}''(t)}{\|\boldsymbol{z}'(t)\|^3} \tag{2.5}$$

$$\begin{aligned}
\left\{u^2(t) + v^2(t)\right\}^3 \kappa'(t) = 2\,&[\{u(t)v''(t) - u''(t)v(t)\}\{u^2(t) + v^2(t)\} \\
&-4\{u(t)v'(t) - u'(t)v(t)\}\{u(t)u'(t) + v(t)v'(t)\}]\,(= 2w(t))
\end{aligned} \tag{2.6}$$

where $\times$ stands for the two-dimensional cross product, $(x_0, y_0)^T \times (x_1, y_1)^T = x_0y_1 - x_1y_0$. Letting $T_i = P_{i+1} - P_i$, define θ to be the angle from T_0 to T_1 and γ to be the angle from T_3 to T_4. As in [13], $T_0 \parallel T_4$ (note $\gamma = -\theta$; refer to Lemma 2.1) and $\gamma = \theta$ are to be taken for the $S-$ and $C-$curves, respectively. It will be easily checked that $\|\boldsymbol{z}'(0)\| / \|\boldsymbol{z}'(1)\| = (r_0/r_1)^{2/3}$; refer to (2.9) and (2.24). Without loss of generality, a shift and rotation enables us to assume that $C_0 = (0, r_0)$ and $P_0 = (0, 0)$; refer to Figures 1-2 (left). Denote $\boldsymbol{z}_i = (u_i, v_i), 0 \leq i \leq 2$. Then, to ensure fairness, we restrict $v_0 = 0$ since the initial curvature is positive. Two cases of an S-shaped and a C-shaped transition curves are now considered separately.

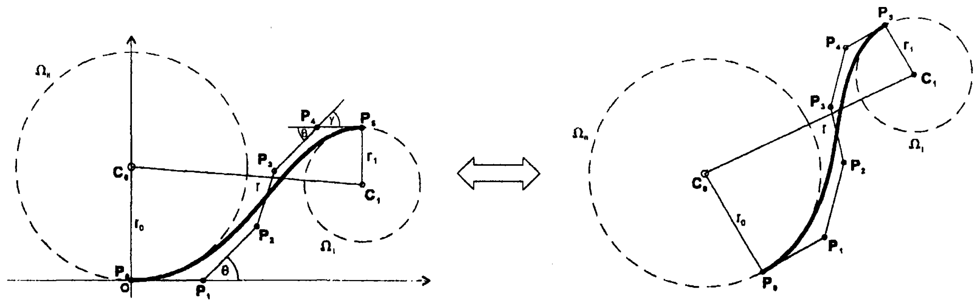

Figure 1: An S-shaped quintic Bézier transition curve.

2.1 S-Shaped Transition Curve

Here we consider an S-shaped transition curve $\boldsymbol{z}(t)$ of the form (2.1). Then, G^2 transition requires

Lemma 2.1

$$v_0 = 0, \quad v_2 = 0, \quad v_1 = \frac{u_0^3}{4r_0}, \quad u_2 = \lambda u_0 \tag{2.7}$$

where $v_0 = v_2 = 0$ *gives* $\theta = -\gamma$.

Proof. First, note $v_2 = 0$. Next, we only to note

$$\kappa(0)\left(= \frac{4v_1}{u_0^3}\right) = \frac{1}{r_0}, \quad \kappa(1)\left(= -\frac{4v_1}{u_2^3}\right) = -\frac{1}{r_1} \tag{2.8}$$

Letting $u_1 = mu_0$, Lemma 2.1 gives

$$\boldsymbol{z}_0 = u_0(1,0), \quad \boldsymbol{z}_1 = u_0\left(m, \frac{u_0^2}{4r_0}\right), \quad \boldsymbol{z}_2 = u_0\lambda(1,0) \tag{2.9}$$

Theorem 2.1 *($r > r_0 + r_1$): Assume that* $3/10 \le \lambda \le 1$. *Each value of* $m(\ge 3/4)$ *determines a* G^2 *quintic S-shaped transition curve of the form (2.1) with (2.9) between the two circles with no interior curvature extremum. It is free of loops and cusps and has a single inflection point, i.e., it is a pair of two spirals with monotone decreasing curvature which changes its sign from positive to negative.*

Proof. First, note $\boldsymbol{z}(1) = (p, q)$

$$p = \frac{u_0^2}{120}\left\{-\frac{u_0^4}{r_0^2} + 8(2m^2 + 3m(1+\lambda) + 3 + \lambda + 3\lambda^2)\right\}, \quad q = \frac{u_0^4}{60r_0}(4m + 3 + 3\lambda) \tag{2.10}$$

With $\rho = u_0^4/(25r_0^2)$, the center $C_1(=(c,d))(=\boldsymbol{z}(1) - r_1(0,1))$ is given by

$$c = \frac{r_0\sqrt{\rho}}{24}\left(16m^2 + 24(\lambda+1)m + 24\lambda^2 + 8\lambda + 24 - 25\rho\right), \quad d = \frac{r_0}{12}\left(20\rho m + 15\rho\lambda + 15\rho - 12\lambda^3\right) \tag{2.11}$$

$\|C_1 - C_0\| = r$ gives the cubic equation $f(\rho) = 0$, where

$$f(\rho) = \sum_{i=0}^{3} a_i \rho^i \tag{2.12}$$

with

$$\begin{aligned} a_3 &= 625r_0^2, \quad a_2 = 100r_0^2\left\{8m^2 + 12(1+\lambda)m - 3\lambda^2 + 14\lambda - 3\right\} \\ a_1 &= 32r_0^2\left\{8m^4 + 24(1+\lambda)m^3 + (42 + 44\lambda + 42\lambda^2)m^2\right. \\ &\quad \left. -24(1 - 2\lambda - 2\lambda^2 + \lambda^3)m - 27\lambda^4 - 33\lambda^3 + 38\lambda^2 - 33\lambda - 27\right\} \\ a_0 &= -576\left\{r^2 - r_0^2(1+\lambda^3)^2\right\} \left(= -576\left\{r^2 - (r_0 + r_1)^2\right\}\right) \end{aligned} \tag{2.13}$$

Since the signs of coefficients (a_3, a_2, a_1, a_0) are $(+, +, ?, -)$, combine Descartes' rule of signs and intermediate value of theorem to obtain that the above cubic equation has a unique positive root.

Now we examine the shape of the transition curve. First, the second component $y(t)$ of $\mathbf{z}(t)(=(x(t), y(t))$ satisfies

$$y'(t) = \frac{su_0^4(s^2 + 2ms + \lambda)}{r_0(1+s)^4} \ (> 0), \quad t = \frac{1}{1+s} \tag{2.14}$$

which implies that the curve is free of loops and cusps since "loop" means $\mathbf{z}(\alpha) = \mathbf{z}(\beta), 0 \le \alpha < \beta \le 1$, i.e., $y'(t)$ has at least one zero and "cusp" means $\mathbf{z}'(t) = \mathbf{0}$ for some $t \in (0,1)$, i.e., $y'(t)$ has at least one zero. Now we show that the transition curve has monotone decreasing curvature. A symbolic manipulator *Mathematica* gives with $t = 1/(1+s), 0 \le s < \infty$

$$w(t)\left(= \frac{\{u^2(t) + v^2(t)\}^3 \kappa'(t)}{2}\right) = -\frac{125 r_0^2 \rho^{3/2}}{4(1+s)^5}\left[4(s^2 + 2ms + \lambda)\{2m\mu(s) - (3s^3 - 5\lambda s^2 - 5\lambda s + 3\lambda^2)\} + 25\rho s \mu(s)\right] \tag{2.15}$$

where for $3/10 \le \lambda \le 1$,

$$\mu(s)\ (= 2s^3 - s^2 - \lambda s + 2\lambda) \ge 2s^3 - s^2 - s + \frac{3}{5} \ge \frac{7(16 - 5\sqrt{7})}{270} (\approx 0.0718) > 0 \tag{2.16}$$

For the term in the braces of (2.15), note $m \ge 3/4$ to obtain

$$2\{2m\mu(s) - (3s^3 - 5\lambda s^2 - 5\lambda s + 3\lambda^2)\} \ge (10\lambda - 3)s^2 + 7\lambda s + s + 6\lambda(1-\lambda) > 0 \tag{2.17}$$

Hence, $\kappa'(t) < 0$ for $3/10 \le \lambda \le 1$, i.e., the transition curve is a spiral whose curvature is monotone decreasing and has a single inflection point. This completes the proof of Theorem 2.1.

Here we note that the coefficient $4(4m-3)$ of s^5 (the highest term in the brackets of (2.15)) must be nonnegative for the spiral transition curve. Therefore, $m \ge 3/4$ is necessary and $m = 3/4$ means $\kappa'(0) = 0$ since then the numerator and the denominator is quartic and quintic in s, respectively.

2.2 *C*-Shaped Transition Curve

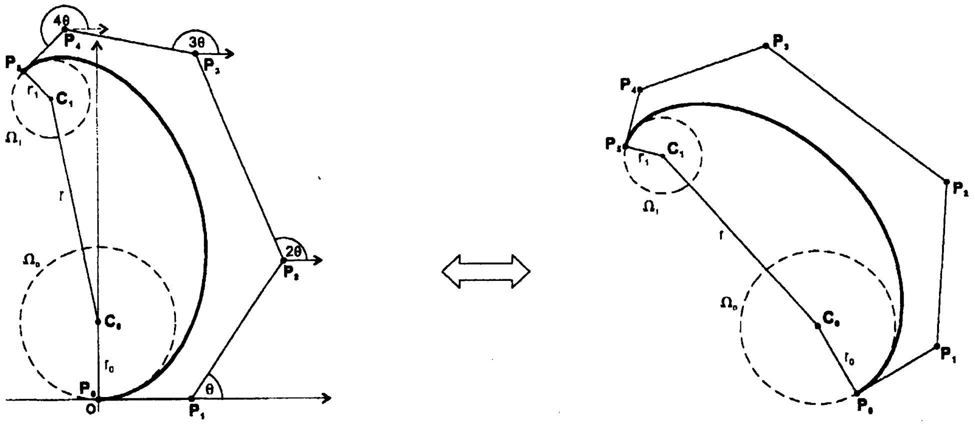

Figure 2: A *C*-shaped quintic Bézier transition curve.

Now we consider a *C*-shaped transition curve $\mathbf{z}(t)$ of the form (2.1). As in the *S*- shaped one, assume $C_0 = (0, r_0)$ and $P_0 = (0, 0)$. Then, G^2 transition requires

Lemma 2.2

$$v_0 = 0, \quad v_2 = u_2 \tan 2\theta, \quad u_1 = \frac{u_0^3}{4 r_0 \tan\theta}, \quad v_1 = u_1 \tan\theta, \quad u_2 = u_0 \lambda \cos 2\theta \tag{2.18}$$

Proof. Since the angle from T_0 to T_1 is θ, we obtain

$$v_1 = u_1 \tan\theta \tag{2.19}$$

Slopes of T_3 and T_4 are given by $(u_2v_1 + u_1v_2)/(u_1u_2 - v_1v_2)(= m_3)$ and $2u_2v_2/((u_2^2 - v_2^2)(= m_4)$, respectively. Therefore, note that the angle from T_3 to T_4 is θ to get

$$(m_4 - m_3)/(1 + m_3m_4) = \tan\theta, \text{ i.e., } -u_2v_1 + u_1v_2 = (u_1u_2 + v_1v_2)\tan\theta \tag{2.20}$$

from

$$v_1 = u_1 \tan\theta, \quad v_2 = u_2 \tan 2\theta \tag{2.21}$$

Next,

$$\kappa(0)\left(= \frac{4u_1 \tan\theta}{u_0^3}\right) = \frac{1}{r_0}, \quad \kappa(1)\left(= \frac{4u_1 \cos^2 2\theta \tan\theta}{u_2^3}\right) = \frac{1}{r_1} \tag{2.22}$$

from which follow

$$u_2 = u_0\lambda\cos 2\theta, \quad u_1 = \frac{u_0^3}{4r_0\tan\theta} \tag{2.23}$$

Then, note that the angles between $P_i - P_{i-1}$ and $P_{i+1} - P_i$, $1 \le i \le 4$ are all θ; refer to Figure 2. Letting $u_1 = mu_0$ and $\rho = \tan^2\theta$, Lemma 2.2 gives

$$\boldsymbol{z}_0 = 2\sqrt{mr_0}\rho^{1/4}(1,0), \boldsymbol{z}_1 = 2m\sqrt{mr_0}\rho^{1/4}(1,\sqrt{\rho}), \boldsymbol{z}_2 = \frac{2\sqrt{mr_0}\rho^{1/4}\lambda}{1+\rho}(1-\rho, 2\sqrt{\rho}) \tag{2.24}$$

Theorem 2.2 *($r > r_0 - r_1$): Assume $r \le 15.37r_0$; refer to Figures 3-4. Then, each value of $m(\in [1, 3.22])$ determines a G^2 quintic C-shaped transition curve of the form (2.1) with (2.24) between the two circles with a single interior curvature extremum. The curve is free of inflections, loops and cusps, i.e., it is a pair of two spirals with starting monotone decreasing curvature and ending monotone increasing curvature.*

Proof. Note

$$C_1 = P_5 + r_1\left(\frac{4(-1+\rho)\sqrt{\rho}}{(1+\rho)^2}, \frac{1-6\rho+\rho^2}{(1+\rho)^2}\right)(= P_5 + r_1(-\sin 4\theta, \cos 4\theta)) \tag{2.25}$$

A symbolic manipulator gives with $C_1 = (c, d)$

$$\begin{aligned} 15(1+\rho)^2c = {} & 4r_0\sqrt{\rho}\left[2(1-\rho)(1+\rho)^2m^3 + 3(1+\rho)\{1+\rho+\lambda(1-3\rho)\}m^2\right. \\ & \left.+\{3(1+\rho)^2 + \lambda(1-\rho^2) + 3\lambda^2(1-6\rho+\rho^2)\}m - 15\lambda^3(1-\rho)\right] \\ 15(1+\rho)^2d = {} & r_0\left[16\rho(1+\rho)^2m^3 + 12\rho(1+\rho)\{1+\rho+\lambda(3-\rho)\}m^2\right. \\ & \left.+8\lambda\rho\{1+\rho+6\lambda(1-\rho)\}m + 15\lambda^3(1-6\rho+6\rho^2)\right] \end{aligned} \tag{2.26}$$

Condition $\|C_1 - C_0\| = r$ gives the equation $f(\rho) = 0$ in ρ $(= \frac{u_0^4}{16r_0^2m^2})$ as

$$f(\rho) = \frac{r_0^2}{225(1+\rho)^2}\sum_{i=0}^{5} a_i\rho^i - r^2 \tag{2.27}$$

where the coefficients are given by

$$\begin{aligned} a_5 &= 64m^6, \quad a_4 = 16m^4\{16m^2 + 12(1+\lambda)m - 3\lambda^2 - 14\lambda - 3\} \\ a_3 &= 8m^2\{48m^4 + 72(1+\lambda)m^3 + 6(5 - 2\lambda + 5\lambda^2)m^2 \\ &\quad -24(1 + 4\lambda + 4\lambda^2 + \lambda^3)m - 27 + 33\lambda + 38\lambda^2 + 33\lambda^3 - 27\lambda^4\} \\ a_2 &= 256m^6 + 576(1+\lambda)m^5 + 48(13 + 10\lambda + 13\lambda^2)m^4 \end{aligned}$$

$$-348(1+\lambda+\lambda^2+\lambda^3)m^3-16(27+45\lambda+106\lambda^2+45\lambda^3+27\lambda^4)m^2$$
$$-240\lambda(1-6\lambda-6\lambda^2+\lambda^3)m+225(-1+\lambda^3)^2$$
$$a_1=64m^6+192(1+\lambda)m^5+16(21+22\lambda+21\lambda^2)m^4$$
$$-192(1-2\lambda-2\lambda^2+\lambda^3)m^3-8(27+123\lambda-38\lambda^2+123\lambda^3+27\lambda^4)m^2$$
$$-240\lambda(1+6\lambda+6\lambda^2+\lambda^3)m+450(1+6\lambda^3+\lambda^6)$$
$$a_0=225(1-\lambda^3)^2\left(=\frac{225}{r_0^2}(r_0-r_1)^2\right)$$

Intermediate value of theorem assures the existence of the positive root since

$$f(0)\left(=(r_0-r_1)^2-r^2\right)<0, \quad f(\infty)=\infty \tag{2.28}$$

To show the uniqueness of the positive root, note

$$f'(\rho)=\frac{8r_0^2}{225(1+\rho)^3}\sum_{i=0}^{5}b_i\rho^i \tag{2.29}$$

where to check the signs of the coefficients $b_i, 1\le i\le 5$ we use nonnegative $u(=m-1)$,

$$b_5=24m^6, \quad b_4=4m^4\left\{35-2\lambda-3\lambda^2+4(16+3\lambda)u+26u^2\right\}$$
$$b_3=m^2\left\{299-19\lambda-52\lambda^2+9\lambda^3-27\lambda^4+4(299+40\lambda-21\lambda^2-6\lambda^3)u\right.$$
$$\left.+2(783+190\lambda+3\lambda^2)u^2+8(109+21\lambda)u^3+176u^4\right\}$$
$$b_2=3m^2\left\{99-3\lambda-28\lambda^2+9\lambda^3-27\lambda^4+12(37+8\lambda-3\lambda^2-2\lambda^3)u\right.$$
$$\left.+6(89+34\lambda+5\lambda^2)u^2+24(11+3\lambda)u^3+48u^4\right\}$$
$$b_1=(1-\lambda)(137+102\lambda+150\lambda^2+111\lambda^3)+2(507+164\lambda-180\lambda^2+105\lambda^3-96\lambda^4)u$$
$$+3(809+389\lambda-70\lambda^2-91\lambda^3-27\lambda^4)u^2+8(338+170\lambda+39\lambda-9\lambda^3)u^3$$
$$+2(777+338\lambda+57\lambda^2)u^4+24(19+5\lambda)u^5+56u^6$$
$$b_0=8m^6+24(1+\lambda)m^5+2(21+22\lambda+21\lambda^2)m^4-24(1+\lambda)(1-3\lambda+\lambda^2)m^3$$
$$-(27+123\lambda-38\lambda^2+123\lambda^3+27\lambda^3)m^2-30\lambda(1+\lambda)(1+5\lambda+\lambda^2)m+450\lambda^3$$

Note $0<\lambda\le 1$ and $u>0$ to easily obtain

$$b_i>0, 1\le i\le 5 \tag{2.30}$$

Hence, if $b_0>0$, then the positive root of $f(\rho)=0$ is unique. If $b_0<0$, then $f'(\rho)$ has a single positive zero where $f'(\rho)$ changes its sign from $-$ to $+$. Therefore, $f(0)<0$ and $f(\infty)=\infty$ mean the unique positive root of $f(\rho)=0$.

Now we examine the shape of the transition curve. First, with $t=1/(1+s)$

$$u(t)v'(t)-u'(t)v(t)=\frac{16mr_0\rho\left\{2\lambda s+m(s^2+\lambda)(1+\rho)\right\}}{(1+s)^2(1+\rho)}\ (>0) \tag{2.31}$$

from which the curve is free of inflection points. Next, "cusps" require $z'(\alpha)=0, 0<\alpha<1$, i.e., $u(\alpha)=v(\alpha)=0$. On the other hand, with $t=1/(1+s)$

$$v(t)=\frac{4\sqrt{m}\sqrt{r_0}\rho^{3/4}\left\{\lambda+ms(1+\rho)\right\}}{(1+s)^2(1+\rho)}(>0) \tag{2.32}$$

from which the curve is free of cusps. Thirdly, For no loops, note

$$y(t)=\frac{4mr_0\rho}{15(1+s)^5(1+\rho)^2}\sum_{i=0}^{3}a_is^i\ (>0) \tag{2.33}$$

$$a_3 = 30m(1+\rho)^2, \quad a_2 = 10(1+\rho)\{2\lambda + m(3+4m)(1+\rho)\}$$
$$a_1 = 5(1+\rho)\left[4m^2 + 3m + 9m\lambda + 2\lambda + m\{4m + 3(1-\lambda)\}\rho\right]$$
$$a_0 = 4m^2 + 3m + 9m\lambda + 2\lambda + 12\lambda^2 + 2(4m^2 + 3m + 3m\lambda + \lambda - 6\lambda^2)\rho + m\{4m + 3(1-\lambda)\}\rho^2$$

If the curve had loops, there would exist at least two α and β $(0 < \alpha \neq \beta < 1)$ such that $z(\alpha) = z(\beta)$, i.e., $x(\alpha) = x(\beta), y(\alpha) = y(\beta)$ from which follows

$$\frac{x(\alpha)}{y(\alpha)} = \frac{x(\beta)}{y(\beta)}, \quad 0 < \alpha \neq \beta < 1 \tag{2.34}$$

Therefore, for no loops, it suffices to show that $\frac{x(t)}{y(t)}$ is monotone decreasing or $x'(t)y(t) - x(t)y'(t) < 0$. A symbolic manipulator gives

$$x'(t)y(t) - x(t)y'(t) = -\frac{16m^2 r_0^2 \rho^{3/2}}{15(1+s)^8(1+\rho)^2}\sum_{i=0}^{6} b_i s^i \ (< 0) \tag{2.35}$$

where all the following coefficients $b_i, 0 \leq i \leq 6$ are positive for $m \geq 1$

$$b_6 = 30m(1+\rho)^2, \quad b_5 = 20(1+\rho)\left\{4m^2 + 3m + 2\lambda + m(4m+3)\rho\right\}$$
$$b_4 = 5(1+\rho)\left\{8m^3 + 28m^2 + 9m + 31m\lambda + 14\lambda + m(16m^2 + 28m + 9 - 5\lambda)\rho + 8m^3\rho^2\right\}$$
$$b_3 = 4\left\{10m^3 + 6m^2(4+5\lambda) + m(3+59\lambda) + 12\lambda(1+\lambda)\right\} + 8\left\{15m^3 + 6m^2(4+5\lambda)\right.$$
$$\left. + m(3+23\lambda) + 6\lambda(1-\lambda)\right\}\rho + 4m\left\{30m^2 + 6m(4+5\lambda) + 3 - 13\lambda\right\}\rho^2 + 40m^3\rho^3$$
$$b_2 = 2\left\{2m^3(3+5\lambda) + 12m^2(1+4\lambda) + m\lambda(75+32\lambda) + 6\lambda(1+6\lambda)\right\}$$
$$+4\left\{3m^3(3+5\lambda) + 12m^2(1+4\lambda) + m\lambda(27+14\lambda) + 3\lambda(1-6\lambda)\right\}\rho$$
$$+2m\left\{6m^2(3+5\lambda) + 12m(1+4\lambda) - \lambda(21+4\lambda)\right\}\rho^2 + 4m^3(3+5\lambda)\rho^3$$
$$b_1 = 4\lambda\left\{2m^3 + m^2(6+4\lambda) + m(9+10\lambda) + 2\lambda(6+\lambda)\right\} + 8\lambda\left\{3m^3 + m^2(6+4\lambda)\right.$$
$$\left. + m(3+4\lambda) - \lambda(6-\lambda)\right\}\rho + 4m\lambda\left\{6m^2 + m(6+4\lambda) - 3 - 2\lambda\right\}\rho^2 + 8m^3\lambda\rho^3$$
$$b_0 = \lambda^2\left\{4m^2 + 3m(3+\lambda) + 2(6+\lambda)\right\} + 2\lambda^2\left\{4m^2 + 3m(1+\lambda) + \lambda - 6\right\}\rho$$
$$+m\lambda^2\left\{4m + 3(\lambda - 1)\right\}\rho^2$$

Therefore, the curve is free of loops. Finally we show that the curve has a single curvature extremum. A symbolic manipulator *Mathematica* gives with $t = 1/(1+s), 0 \leq s < \infty$

$$w(t)\left(= \frac{\{u^2(t) + v^2(t)\}^3 \kappa'(t)}{2}\right) = -\frac{64m^2 r_0^2 \rho^{3/2}}{(1+\rho)^2(1+s)^5}\left\{\sum_{i=0}^{5} c_i s^i\right\} \tag{2.36}$$

where to check the signs of the coefficients $c_i > 0, i = 5, 4, 3, 0$ we again use nonnegative $u(= m-1)$

$$c_5 = (1+\rho)\left\{1 - \lambda + \rho + 5(1+\rho)u + 4(1+\rho)u^2\right\} (> 0)$$
$$c_4 = (1+\rho)\left[\rho\{-5\lambda + 8(1+\rho)\} + (8 + 7\lambda + 32\rho - 5\lambda\rho + 24\rho^2)u\right.$$
$$\left. +8(2 + 5\rho + 3\rho^2)u^2 + 8(1+\rho)^2u^3\right] (> 0)$$
$$c_3 = 2\left[3\lambda^2(1-\rho) - \lambda(1 - 10\rho - 11\rho^2) - 2(1+\rho)^3\right.$$
$$\left. -(1+\rho)\left\{6(1+\rho)^2 - \lambda(11+23\rho)\right\}u - 6(1 - 2\lambda + \rho)(1+\rho)^2u^2 - 2(1+\rho)^3u^3\right]$$
$$c_2 = 4\lambda\left\{2m^3 - 12m^2 + \lambda(13m - 3) + (6m^3 - 24m^2 + 14m\lambda + 3\lambda)\rho\right.$$
$$\left. +m(6m^2 - 12m + \lambda)\rho^2 + 2m^3\rho^3\right\}$$
$$c_1 = \lambda(-8m^3 + 8m^2\lambda - 7m\lambda + 7\lambda^2) + \lambda(-24m^3 + 16m^2\lambda - 2m\lambda + 7\lambda^2)\rho$$
$$+\lambda(-24m^3 + 8m^2\lambda + 5m\lambda)\rho^2 - 8m^3\lambda\rho^3 (< 0)$$
$$c_0 = -\lambda^2(1+\rho)\left\{4(1-\lambda) + \rho(4 - 3\lambda) + (8 - 3\lambda)(1+\rho)u + 4(1+\rho)u^2\right\} (< 0)$$

Descartes' rule of signs implies that $c_2 \leq 0$ is a sufficient condition for the curvature to have a single extremum, i.e., a local minimum. From now on, we assume that $1 \leq m \leq 3.22$ where 3.22 is a necessary condition for the following ρ_0 to be positive for all $\lambda \in (0,1]$. Note that $2m^3 - 12m^2 + \lambda(13m - 3) \leq 2m^3 - 12m^2 + 13m - 3 \leq 0$ to obtain a sufficient condition for $c_2 \leq 0$:

$$0 < \rho \leq \frac{-6m^2 + 12m - \lambda + \phi(m,\lambda)}{4m^2} (= \rho_0) \tag{2.37}$$

with

$$\phi(m,\lambda) = \sqrt{-12m^4 + 48m^3 + 4(36 - 25\lambda)m^2 - 48\lambda m + \lambda^2} \tag{2.38}$$

Hence, $f(\rho_0) \geq 0$ gives a bound for r/r_0

$$r/r_0 \leq \sqrt{N(m,\lambda)/D(m,\lambda)}$$

where

$$\begin{aligned} N(m,\lambda) = &-768(11 + 3\lambda)m^9 + 64(1881 + 542\lambda + 81\lambda^2)m^8 - 384(1107 + 578\lambda + 160\lambda^2 + 6\lambda^3)m^7 \\ &-32(16065 - 23160\lambda - 6614\lambda^2 - 2172\lambda^3 + 81\lambda^4)m^6 - 96(-35442 + 2553\lambda + 2268\lambda^2 + 2453\lambda^3 \\ &+90\lambda^4)m^5 - 2(32043 + 309516\lambda + 34844\lambda^2 + 2874\lambda^3 - 16996\lambda^4 + 2268\lambda^5 + 75\lambda^6)m^4 \\ &+144(-1296 + 3207\lambda + 10822\lambda^2 + 4608\lambda^3 - 2270\lambda^4 + 597\lambda^5)m^3 - 72(-450 - 822\lambda + 1497\lambda^2 \\ &+4676\lambda^3 + 2098\lambda^4 - 1144\lambda^5 - 345\lambda^6 + 150\lambda^7)m^2 - 36\lambda(225 + 162\lambda - 218\lambda^2 - 862\lambda^3 - 274\lambda^4 \\ &+166\lambda^5 + 225\lambda^6)m + 9\lambda^2(25 + 12\lambda - 16\lambda^2 - 74\lambda^3 - 16\lambda^4 + 12\lambda^5 + 25\lambda^6) \\ &- \{256m^8 - 384(9 + \lambda)m^7 + 32(123 + 58\lambda - 21\lambda^2)m^6 + 192(482 + 67\lambda + 44\lambda^2 + 2\lambda^3)m^5 \\ &-48(5991 + 2195\lambda + 1008\lambda^2 + 95\lambda^3 - 9\lambda^4)m^4 + 144(204 + 1907\lambda + 770\lambda^2 + 267\lambda^3 - 56\lambda^4)m^3 \\ &+18(889 - 1560\lambda - 5232\lambda^2 - 3426\lambda^3 + 960\lambda^4 - 120\lambda^5 + 25\lambda^6)m^2 - 36(75 + 90\lambda - 122\lambda^2 \\ &-418\lambda^3 - 178\lambda^4 + 94\lambda^5 + 75\lambda^6)m + 9\lambda(25 + 12\lambda - 16\lambda^2 - 74\lambda^3 - 16\lambda^4 + 12\lambda^5 + 25\lambda^6)\}\, \phi(m,\lambda) \\ D(m,\lambda) = &\frac{225}{2}\{2m^2 - 12m + \lambda - \phi(m,\lambda)\}^2 \end{aligned}$$

This completes the proof of Theorem 2.2.

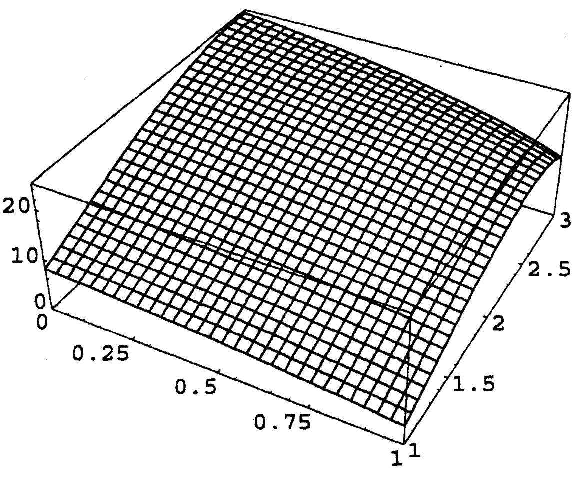

Figure 3: Graph of upper bound for r/r_0 for $0 < \lambda \leq 1, 1 \leq m \leq 3$.

Remark 1[Figure 4 (left)]: From above, a restriction is derived on the magnitude of the ratio r/r_0 independent of λ. $\phi(m,\lambda)$ is monotone decreasing with respect to λ, and so

$$\frac{-6m^2 + 12m - 1 + \phi(m,1)}{4m^2} (= \rho_1) \leq \rho_0 \tag{2.39}$$

$f(\rho_1) \geq 0$ gives $r/r_0 \leq \sqrt{N(m,1)/D(m,1)}$. *Mathematica* gives $r/r_0 \leq 15.37$ whose maximum occurs at about $m = 2.52$.

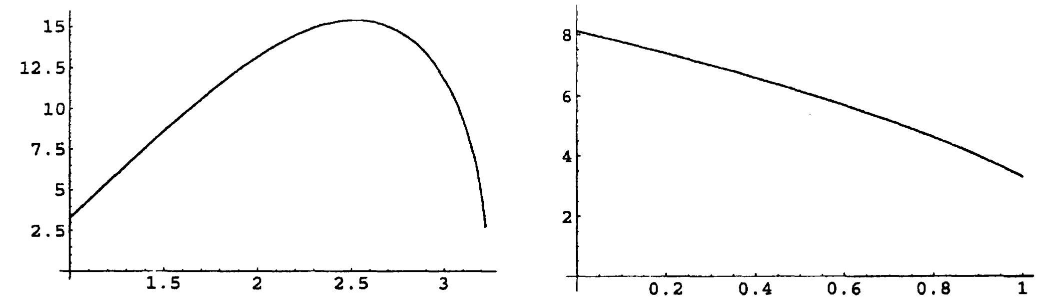

Figure 4: Graphs of upper bounds for r/r_0 for $1 \leq m \leq 3.22, \lambda = 1$(left) and for $0 < \lambda \leq 1, m = 1$ (right).

Remark 2[Figure 4 (right)]: For $m = 1$, the above inequality (2.37) is

$$0 < \rho \leq \frac{6 - \lambda + \phi(1,\lambda)}{4} (= \rho_2) \tag{2.40}$$

Hence, $f(\rho_2) \geq 0$ gives $r/r_0 \leq \sqrt{N(1,\lambda)/D(1,\lambda)}$.

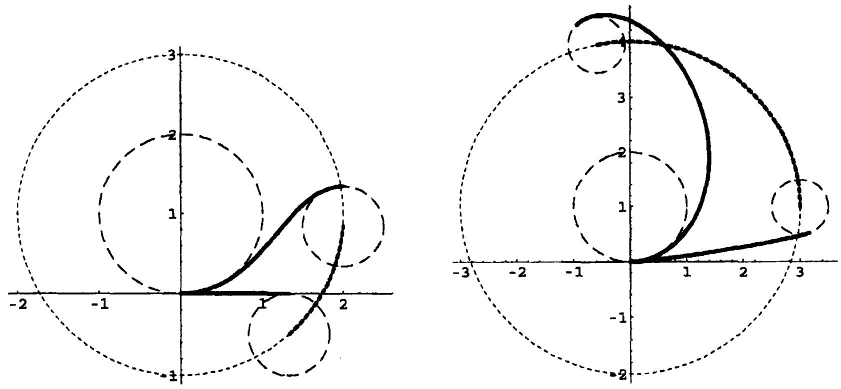

Figure 5: Locus of the center of smaller circle for S-case (left) and C-case (right).

Finally we give numerical results on the locus of the center of smaller circle when $(r_0, r_1, r) = (1, 0.5, 2)$ for $S-$transition curves in Figure 5 (left) and $(r_0, r_1, r) = (1, 0.5, 3)$ for $C-$transition curves in Figure 5 (right). Locus of the center is shown as *dotted thick* arc of the circle with the center $(0, r_0)$ and the radius r.

3 Numerical Examples

This section gives two numerical examples to assure our theoretical analysis. The figures with P_0 fixed on the larger circles show the effect of parameter m for $r_0 = 1$ where $(r_1, r) = r_0(0.5, 2)$ and $(r_1, r) = r_0(0.5, 3)$ for S and C, respectively. Figure 6 show S-shaped curves for $(m = 3/4, 1)$ and Figure 7 show C-shaped curves for $(m = 1, 4/3)$ with their curvature plots. Case $m = 1$ is shown with thick curves. Figure 8 is an exceptional case of [13] with $(r_0, r_1, m) = (1, 0.5, 4/3)$ for $r = 6$ (left) and $r = 10$ (right).

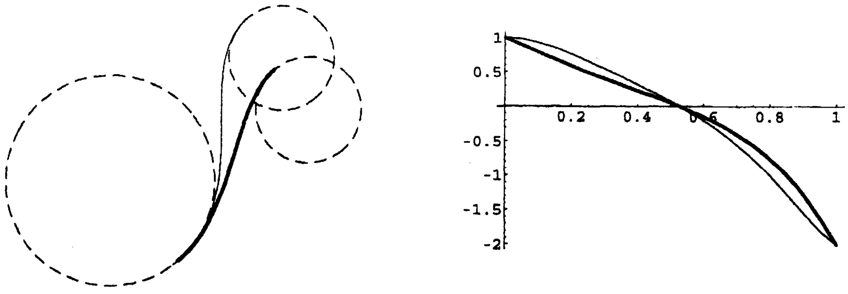

Figure 6: Graphs of $z(t)$ (left) with their curvature plots $\kappa(t)$ (right), $0 \le t \le 1$.

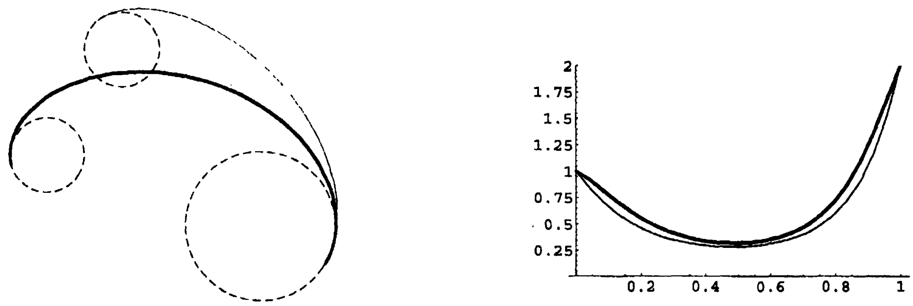

Figure 7: Graphs of $z(t)$ (left) with their curvature plots $\kappa(t)$ (right), $0 \le t \le 1$.

4 Conclusion

Use of a fair PH quintic curve for family of G^2 transition curves between two circles has been demonstrated. Such blending is often desirable in CAD, CAM and CAGD applications. We presented a very simple algorithm offering more flexible constraints than [13]. To guarantee the absence of interior curvature extremum (i.e., spiral segment) in S-shaped transition curve, the ratio of the larger to the smaller radii of the given circular arcs is constrained to less than or equal to $(10/3)^3 \approx 37$ $(\lambda \ge 3/10)$ for $m \ge 3/4$. On the other hand, the ratio must be less than 8 $(\lambda \ge 1/2)$ in [13] for $m = 3/4$. To guarantee a single curvature extremum (at which the curvature magnitude is a minimum) for a C-shaped transition curve, the distance between the centers of the circular arc is constrained to less than or equal to 15.37 times the larger radius $(r \le 15.37r_0)$ for $1 \le m \le 3.22$ where as in [13], it is only 3.3 times the larger radius $(r \le 3.3r_0)$ for $m = 1$. So our constraints are less restrictive, more reasonable and comfortable for practical applications.

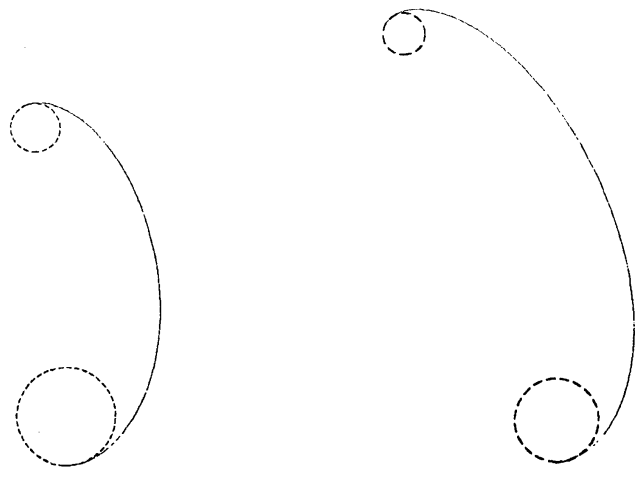

Figure 8: Graphs of $z(t)$ with $(r_1, r) = r_0(0.5, 6)$ (left) and $r_0(0.5, 10)$ (right).

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