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On the Wellposedness of the Cauchy Problem for Weakly Hyperbolic Operators

By Piero D'Ancona and 木下 保 (Tamotu Kinoshita)

Dipartimento di Matematica
Università "La Sapienza" di Roma

Institute of Mathematics
University of Tsukuba
(筑波大, 数学系)

§1. Introduction

We first consider the Cauchy problem on $[0,T]\times\mathbf{R}^n_x$

$$(P_1)\qquad \begin{cases} D_t^m u = \displaystyle\sum_{j+|\alpha|=m} A_{j,\alpha}(t)D_t^j D_x^\alpha u + \sum_{j+|\alpha|\le d} B_{j,\alpha}(t)D_t^j D_x^\alpha u + F(t,x) \\ D_t^j u(0,x) = u_j(x) \quad (j=0,\cdots,m-1), \end{cases}$$

where $D_t=-i\partial_t$, $D_x=-i(\partial_{x_1},\cdots,\partial_{x_n})$ and $0\le d\le m-1$. The coefficients of the principal part depend only on t. We shall write for the principal part

$$p(t,\tau,\xi)=\prod_{k=1}^m(\tau-\lambda_k(t,\xi))=\tau^m-\sum_{j+|\alpha|=m}A_{j,\alpha}(t)\tau^j\xi^\alpha,$$

and for the lower order terms

$$p_d(t,\tau,\xi)=\sum_{j+|\alpha|\le d}B_{j,\alpha}(t)\tau^j\xi^\alpha.$$

We assume that the principal part p is hyperbolic with respect to τ, that is, the characteristic roots are all real-valued and named $\lambda_k(t,\xi)$ according to the rule

$$(1)\qquad \lambda_1(t,\xi)\ge\lambda_2(t,\xi)\ge\cdots\ge\lambda_m(t,\xi).$$

We recall that the functions $\lambda_k(t,\xi)$ are homogeneous of degree 1 in ξ.

In this paper we use the following notations:

$$\Omega_\sigma^k(\xi)=\{t\in[0,T]\ :\ |\lambda_k(t,\xi)-\lambda_{k+1}(t,\xi)|\le\sigma\}\ \text{for}\ 0<\sigma<1\ \text{and}\ 1\le k\le m-1,$$

$$\Omega_\sigma^*(\xi)=\{t\in[0,T]\ :\ |\lambda_1(t,\xi)-\lambda_m(t,\xi)|\le\sigma\},\quad \Omega_\sigma(\xi)=\bigcup_{k=1}^{m-1}\Omega_\sigma^k(\xi)\ \text{for}\ 0<\sigma<1$$

$\mu(S)$ is the Lebesgue measure in $\mathbf{R}_t$ of the set $S \subseteq [0,T]$.
$AC([0,T])$ is the space of absolutely continuous functions on $[0,T]$.
$G^s(\mathbf{R}^n)(s \geq 1)$ is the space of Gevrey functions $g(x)$ satisfying,

$$\sup_{x\in K} |D_x^\alpha g(x)| \leq C_K \rho_K^{|\alpha|} |\alpha|!^s \text{ for any compact set } K \subset \mathbf{R}^n \text{ and } \alpha \in \mathbf{N}^n.$$

Our previous result is the following:

Theorem 0. *([DK]). Assume that $B_{j,\alpha}$ belong to $C^0([0,T])$ and $\lambda_1, \cdots, \lambda_m$ belong to $AC([0,T])$ and that there exist $C > 0$, $a \geq 0$ and $b > 0$ such that for any $0 < \sigma < 1$, $|\xi| = 1$,*

(2) $$\mu(\Omega_\sigma(\xi)) \leq C\sigma^a,$$

(3) $$\int_{[0,T]\setminus\Omega_\sigma^k(\xi)} \frac{|\lambda_k'(t,\xi)| + |\lambda_{k+1}'(t,\xi)|}{|\lambda_k(t,\xi) - \lambda_{k+1}(t,\xi)|} dt \leq C\sigma^{-b} \text{ for } 1 \leq k \leq m-1,$$

Then the Cauchy problem (P_1) is wellposed in G^s if

(4) $$1 \leq s < \begin{cases} 1 + \dfrac{a+1}{b} & \text{when } d \leq \dfrac{m(a+b)}{a+b+1}, \\ \dfrac{m}{d + a(d-m)} & \text{when } d > \dfrac{m(a+b)}{a+b+1}, \end{cases}$$

i.e., for any data $u_j \in G^s(\mathbf{R}^n)$ and $f \in C^0([0,T]; G^s(\mathbf{R}^n))$ the Cauchy problem (P_1) has a unique solution $u \in C^m([0,T]; G^s(\mathbf{R}^n))$.

Remark 1. If $\lambda_1, \cdots, \lambda_m$ and $\psi_1, \cdots, \psi_{m-1}$ are analytic in t and vanish at $t = 0$ and there exist $C > 0$, $c > 0$ and $0 < \alpha < \beta$ such that for any $t \in [0,T]$, $|\xi| = 1$

(5) $$|\lambda_k(t,\xi)| \leq Ct^\alpha \quad \text{for} \quad 1 \leq k \leq m,$$

(6) $$|\lambda_{k+1}(t,\xi) - \lambda_k(t,\xi)| \geq ct^\beta \quad \text{for} \quad 1 \leq k \leq m-1,$$

we can take $a = 1/\beta$ in (2) and $b = 1 - \alpha/\beta$ in (3). Then the Cauchy problem (P_1) when $B_{j,\alpha} \equiv 0$ is wellposed in G^s if

(7) $$1 \leq s < 1 + \frac{\beta+1}{\beta-\alpha}.$$

Remark 2. When $\lambda_1, \cdots, \lambda_m$ vanish of infinite order, (2) can be dropped (one is forced to choose $a = 0$). Then the Cauchy problem (P_1) is wellposed in G^s if

$$1 \leq s < \min\left\{1 + \frac{1}{b}, \frac{m}{d}\right\}.$$

In case of constant multiplicity, we can take $a = 0$ in (2), but (3) does not make a sense, since $[0,T]\backslash\Omega_\sigma^k(\xi) = \phi$. From Remark 8 in §3, we may take $b = 1$. Then the Cauchy problem (P_1) is wellposed in G^s if $1 \le s < \min\{2, m/d\}$.

Remark 3. When $\lambda_1, \cdots, \lambda_m$ belong to $C^1([0,T])$, we can take $0 \le a < 1$ and $0 < b \le 1 - a$ (see Lemma 3 and Remark 7). Then the Cauchy problem (P_1) is wellposed in G^s if

$$1 \le s < \min\left\{\frac{2}{1-a}, \frac{m}{d + a(d-m)}\right\}.$$

We next consider the Cauchy problem on $[0,T] \times \mathbf{R}_x^n$

$$(P_2) \qquad \begin{cases} D_t U = \sum_{|\alpha|=1} A_\alpha(t) D_x^\alpha U + B(t)U + F(t,x) \\ U(0,x) = U_0(x), \end{cases}$$

where $A_\alpha(t)$ and $B(t)$ are $m \times m$ matrices.

By the Cauchy-Kowalevski Theorem any type of systems can be solvable locally in the analytic class. Weakly hyperbolic systems are solvable globally in the Gevrey classes of order $1 \le s < m/(m-1)$. In this paper the main result is the following:

THEOREM 1. *Assume that A_α and B belong to $C^{m-1}([0,T])$ and $\lambda_1, \cdots, \lambda_m$ belong to $AC([0,T])$ and satisfy (2) and (3). Then the Cauchy problem (P_2) is wellposed in G^s if*

$$(8) \qquad 1 \le s < \min\left\{1 + \frac{a+1}{b}, \frac{m}{m-1-a}\right\},$$

i.e., for any data $U_0 \in G^s(\mathbf{R}^n)$ and $F \in C^{m-1}([0,T]; G^s(\mathbf{R}^n))$ the Cauchy problem (P_2) has a unique solution $U \in C^1([0,T]; G^s(\mathbf{R}^n))$.

Remark 4. When $\lambda_1, \cdots, \lambda_m$ belong to $C^1([0,T])$, we can take $0 \le a < 1$ and $0 \le b \le 1 - a$ (see Lemma 3 and Remark 7). Then the Cauchy problem (P_2) is wellposed in G^s if

$$1 \le s < \min\left\{\frac{2}{1-a}, \frac{m}{m-1-a}\right\} = \frac{m}{m-1-a}.$$

We also consider the special case when

$$\sum_{|\alpha|=1} A_\alpha(t) D_x^\alpha \equiv \begin{pmatrix} \lambda_1(t,D_x) & \psi_1(t,D_x) & 0 & \cdots & 0 \\ 0 & \lambda_2(t,D_x) & \psi_2(t,D_x) & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \lambda_{m-1}(t,D_x) & \psi_{m-1}(t,D_x) \\ 0 & \cdots & & 0 & \lambda_m(t,D_x) \end{pmatrix},$$

where $\lambda_k(t,\xi)$ $(1 \le k \le m)$ and $\psi_k(t,\xi)$ $(1 \le k \le m-1)$ are homogeneous of degree 1 in ξ.

THEOREM **2.** *Assume that B belongs to $C^{m-1}([0,T])$ and $\lambda_1,\cdots,\lambda_m$ and $\psi_1,\cdots,\psi_{m-1}$ belong to $C^{m-1}([0,T])$ and satisfy (2) and that there exists $C>0$ and $b>0$ such that for any $0<\sigma<1$, $|\xi|=1$*

(9) $$\int_{[0,T]\setminus\Omega^*_\sigma(\xi)} \frac{|\lambda'_k(t,\xi)|}{|\lambda_1(t,\xi)-\lambda_m(t,\xi)|}dt \le C\sigma^{-b} \text{ for } 1\le k\le m,$$

(10) $$\int_{[0,T]\setminus\Omega^*_\sigma(\xi)} \frac{|\psi'_k(t,\xi)|}{|\lambda_1(t,\xi)-\lambda_m(t,\xi)|}dt \le C\sigma^{-b} \text{ for } 1\le k\le m-1.$$

Then the Cauchy problem (P_2) is wellposed in G^s if

(11) $$1 \le s < 1 + \frac{a+1}{b+m-2}.$$

Remark 5. (9) is a more relaxed condition than (3) (when $m=2$, (9) is the same as (3)). Since $\lambda_1,\cdots,\lambda_m$ belong to $C^{m-1}([0,T]) \subset C^1([0,T])$, we find that $b \le 1-a$ and by Theorem 1 we get the wellposedness in G^s if

$$1 \le s < \frac{m}{m-1-a}\left(=1+\frac{a+1}{1-a+m-2}\right) \le 1+\frac{a+1}{b+m-2}$$

(see Remark 4). The Gevrey index (11) for this special case is larger than (8).

Example A. Theorem 2 can be also applied when the principal part has the form of the Jordan matrix:

$$\sum_{|\alpha|=1} A_\alpha(t)D_x^\alpha \equiv \sum_{|\alpha|=1} \begin{pmatrix} \lambda_1(t) & 1 & 0 & \cdots & 0 \\ 0 & \lambda_2(t) & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \lambda_{m-1}(t) & 1 \\ 0 & \cdots & & 0 & \lambda_m(t) \end{pmatrix} D_x^\alpha.$$

Since (10) is valid for any $0<b(\le 1-a)$, with only (9) we obtain (11).

Example B. Yamahara [Y] studied 4 by 4 systems with the principal part

$$\sum_{|\alpha|=1} A_\alpha(t)D_x^\alpha \equiv \begin{pmatrix} \lambda t^\beta & 1 & 0 & 0 \\ 0 & \lambda t^\beta & \psi t^\alpha & 0 \\ 0 & 0 & \mu t^\beta & 1 \\ 0 & 0 & 0 & \mu t^\beta \end{pmatrix} D_x, \text{ where } \alpha,\beta>0, \lambda\ne\mu \text{ and } \psi\ne 0$$

and proved that the Cauchy problem (P_2) is wellposed in G^s if

$$1 \leq s < \begin{cases} 2 & \text{for } \alpha \geq 2\beta, \\ 1 + \dfrac{\beta}{3\beta - \alpha} & \text{for } 0 < \alpha < 2\beta, \end{cases} \tag{12}$$

and the Cauchy problem (P_2) is not wellposed in G^s otherewise. In particular, let us consider the case when $0 < \alpha < \beta$. Taking $m = 4$, $a = 0$ and $b = 1 - \alpha/\beta$ which is determind by (10), by Theorem 2 we can also get the optimal Gevrey index (12) if $0 < \alpha < \beta$.

§2. Sketch of Proof of Theorem 0

When $s = 1$, the Cauchy problem (P_1) is wellposed in the class of real analytic functions. Therefore we can suppose that $s > 1$ for the proof. By Fourier transform with respect to x, the Cauchy problem (P_1) turns into

$$\begin{cases} p(t, D_t, \xi)\hat{u} = \hat{f}(t, \xi) + p_d(t, D_t, \xi)\hat{u} \\ D_t^j \hat{u}(0, \xi) = \hat{u}_j(\xi) \quad (j = 0, \cdots, m-1). \end{cases}$$

Let $0 < \sigma < 1$ and $\varphi(r)$ be a non-negative function such that $\varphi \in C_0^\infty(\mathbf{R})$, $\varphi(r) \equiv 0$ for $|r| \geq 2$ and $\varphi(r) \equiv 1$ for $|r| \leq 1$. We define

$$\omega(t, \xi) = \sigma|\xi| \sum_{l=1}^{m-1} \varphi\Big(\sigma^{-1}\Big\{\lambda_l\Big(t, \frac{\xi}{|\xi|}\Big) - \lambda_{l+1}\Big(t, \frac{\xi}{|\xi|}\Big)\Big\}\Big),$$

$$\mu_k(t, \xi) = \lambda_k(t, \xi) + ik\omega(t, \xi) \quad \text{for} \quad k = 1, \cdots, m.$$

Moreover we denote by $q(t, \tau, \xi)$ the polynomial of degree m in τ

$$q(t, \tau, \xi) = \prod_{k=1}^{m} (\tau - \mu_k(t, \xi)).$$

Now we set the energy density

$$E(t, \xi) = \frac{1}{2} \sum_{l=1}^{m} |q_l(t, D_t, \xi)\hat{u}|^2,$$

where $q_l(t, \tau, \xi)$ is the polynomial of degree $m - 1$ in τ defined by

$$q_l(t, \tau, \xi) = \frac{q(t, \tau, \xi)}{\tau - \mu_l(t, \xi)} \Big(= \prod_{k=1, k\neq l}^{m} (\tau - \mu_k(t, \xi))\Big).$$

Hence we can derive the energy estimate

$$\sqrt{E(t,\xi)} \le \exp\Big\{C\int_0^T \Big(\max_{1\le k\le m-1} \frac{|\lambda_k'|+|\lambda_{k+1}'|+|\omega'|}{|\lambda_k-\lambda_{k+1}|+\omega} + \omega + \frac{|\xi|^d}{\prod_{k=1}^{m-1}|\lambda_k-\lambda_{k+1}|+\omega^{m-}}$$

$$\times \Big\{\sqrt{E(0,\xi)} + \int_0^T |\hat f(t,\xi)|dt\Big\}.$$

By (2)-(4), there exist $C>0$ and $\rho>0$ such that for any $(t,\xi)\in[0,T]\times\mathbf{R}^n_\xi\backslash 0$

$$\sqrt{E(t,\xi)} \le C\exp\{\rho|\xi|^{\frac{1}{s}}\}\Big\{\sqrt{E(0,\xi)} + \int_0^T |\hat f(t,\xi)|dt\Big\}.$$

§3. Sketch of Proof of Theorem 1

In Theorem 1 the coefficients A_α, B and F belong to $C^{m-1}([0,T])$ and the assumptions for the characteristic roots of the principal part are the same as Theorem 0. So, we shall use the result of Theorem 0. Let $C(t,\tau,\xi)$ be the cofactor matrix of $\tau I - \sum_{|\alpha|=1} A_\alpha(t)\xi^\alpha$, i.e.,

$$\begin{cases} C(t,\tau,\xi)\cdot\Big\{\tau I - \sum\limits_{|\alpha|=1} A_\alpha(t)\xi^\alpha\Big\} = p(t,\tau,\xi)I, \\ \Big\{\tau I - \sum\limits_{|\alpha|=1} A_\alpha(t)\xi^\alpha\Big\}\cdot C(t,\tau,\xi) = p(t,\tau,\xi)I, \end{cases}$$

where I is the $m\times m$ identity matrix and the polynomial p has degree m:

$$p(t,\tau,\xi) = \det\Big\{\tau I - \sum_{|\alpha|=1} A_\alpha(t)\xi^\alpha\Big\}.$$

Mutiplying $\Big\{D_t - \sum_{|\alpha|=1} A_\alpha(t)D_x^\alpha\Big\}$ by $C(t,D_t,D_x)$, we have the following operator:

$$p(t,D_t,D_x)I + G(t,D_t,D_x), \tag{13}$$

where G contains all the coefficients A_α, B and their derivatives up to order $m-1$ in each row. Applying Theorem 0 into (13), we get the existence and the uniqueness.

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