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Microlocal WKB method applied to a simple well eigen-value asymptotics

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0 Introduction

The subject of this report is the semiclassical distribution of eigenvalues for the Schrödinger equation

$$-h^2\Delta u+V(x)u=Eu.$$

"Semiclassical distribution" means the asymptotics with respect to h as h tends to 0, while the energy E is restrained in a neighborhood of a fixed real energy E_0.

In this report, we restrict ourselves to the most fundamental problem of a **simple well** potential in one dimension:

$$-h^2\frac{d^2u}{dx^2}+V(x)u=Eu, \tag{0.1}$$

where the potential $V(x)$ is a real-valued analytic function on $\mathbb{R}$ and the classically allowed region $\{x\in\mathbb{R}; V(x)\leq E_0\}$ is a connected interval $[\alpha,\beta]$ $(-\infty<\alpha<\beta<+\infty)$. We assume moreover that $V'(\alpha)<0, V'(\beta)>0$. For $E\in(E_0-\epsilon,E_0+\epsilon)$ with sufficiently small ϵ, the classically allowed region is still connected interval $[\alpha(E),\beta(E)]$.

It is well known that the eigenvalues near E_0 are given by the so-called **Bohr-Sommerfeld quantization condition**:

$$C(E)=(2n+1)\pi h+O(h^2),\quad n\in\mathbb{N}=\{0,1,2,...\}, \tag{0.2}$$

where the function $C(E)$ is the **action** defined by

$$C(E) = 2\int_{\alpha(E)}^{\beta(E)} \sqrt{E - V(x)}dx. \tag{0.3}$$

In the case of the harmonic oscillator $V(x) = x^2$, $C(E) = \pi E$.

In the following, we shall show how to derive the Bohr-Sommerfeld quantization condition (0.2) by using the **WKB method** in a microlocal way. This technique was used in [Gé-Sj] in multi-dimensional case for the quantization condition of resonances created by a hyperbolic closed trajectory.

The microlocal way is based on the **FBI transformation**. Roughly speaking, the FBI transformation is a Fourier integral operator with complex phase, and the associated canonical transformation maps the phase space $\mathbb{R}^2_{x,\xi}$ to an I-Lagrangian manifold $\Lambda \subset \mathbb{C}^2$ whose projection on $\mathbb{C}_x$ is bijective. This enables us to have the phase space geometry on the complex base space and to avoid the problem of the caustics (or equivalently the connection problem at turning points in the one-dimensional case).

1 FBI transformation

In this section, we review some elements of the microlocal and semiclassical analysis. For proofs and more details, see [Ma].

For $u \in L^2(\mathbb{R}^n)$, we define the *FBI transform* by

$$\begin{aligned}(Tu)(z,h) &= \int_{\mathbb{R}^n} e^{-(z-y)^2/2h} u(y)dy \\ &= e^{\xi^2/2h}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi/h-(x-y)^2/2h} u(y)dy,\end{aligned}$$

where $z = x - i\xi$. Define also

$$\tilde{T}(x,\xi;h) = c_{n,h}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi/h-(x-y)^2/2h} u(y)dy, \quad c_{n,h} = 2^{-n/2}(\pi h)^{-3n/4}.$$

We easily see the following properties:

Proposition 1.1

(1) *$(Tu)(z;h)$ is an entire function with respect to z.*

(2) *$\tilde{T}$ is unitary from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$, that is,*

$$\|\tilde{T}u\|_{L^2(\mathbb{R}^{2n}_{x,\xi})} = \|u\|_{L^2(\mathbb{R}^n_x)}.$$

(3) *The image of $L^2(\mathbb{R}^n)$ by $\tilde{T}$ is $e^{-\xi^2/2h}\mathcal{H}(\mathbb{C}^n_z) \cap L^2(\mathbb{R}^{2n}_{x,\xi})$ and the adjoint is given by*

$$(\tilde{T}^* v)(y) = c_{n,h} \iint e^{-i(x-y)\cdot\xi/h-(x-y)^2/2h} v(x,\xi)dxd\xi.$$

(4) *Let P and Q be the pseudo-differential operators whose Weyl symbols are $p(x,\xi)$ and $q(z,\zeta)$ respectively. Then*

$$T \circ P = Q \circ T$$

if and only if

$$q(z,\zeta) = p(z + i\zeta, \zeta).$$

An advantage of the FBI transformation is that it enables us to localise the functions in x and ξ simultaneously. We define the notion of *microsupport*:

Definition 1.2 For $u \in \mathcal{S}'(\mathbb{R}^n)$ (h-dependent) and $(x_0,\xi_0) \in \mathbb{R}^{2n}$, one says that u is microlocally exponentially small near (x_0,ξ_0) if and only if there exists $\delta > 0$ such that

$$\tilde{T}u(x,\xi;h) = O(e^{-\delta/h})$$

uniformly for (x,ξ) in a neighbourhood of (x_0,ξ_0) and sufficiently small $h > 0$. The complement of such points (x_0,ξ_0) is called microsupport of u and denoted by $MS(u)$.

Proposition 1.3 *If $Pu = 0$ and $\|u\| = 1$, where $\|\cdot\|$ is the L^2 norm, then $MS(u) \subset \mathrm{Char}\,(P)$ where $\mathrm{Char}\,(P) = \{(x,\xi); p(x,\xi) = 0\}$ and p is the principal symbol of P.*

Proposition 1.4 *If $u(x;h) = a(x,h)\exp(i\phi(x)/h)$ and $\|u\| = 1$, where a is an analytic symbol, then $MS(u) \subset \{(x,\xi); \xi = \partial_x\phi(x)\}$.*

2 Derivation of the Bohr-Sommerfeld quantization condition

Let $p(x,\xi) = \xi^2 + V(x)$ be the Weyl symbol of the Schrödinger operator. By the change of the dependent variable $v(z,\zeta) = Tu$, the equation (0.1) is reduced to the equation

$$Qv = Ev, \tag{2.1}$$

where Q is the pseudo-differential operator whose Weyl symbol is

$$q(z,\zeta) = p(z + i\zeta, \zeta).$$

(see Proposition 1.1 (4)). This can be written as $q = p \circ \kappa^{-1}$ with the canonical transformation

$$\kappa\colon (x,\xi) \mapsto (z,\zeta) = (x - i\xi, \xi).$$

The new symbol $q(z,\zeta)$ is defined on the I-Lagrangian manifold $\Lambda = \{(z,\zeta) \in \mathbb{C}^{2n}; \operatorname{Re}\zeta = -\operatorname{Im} z, \operatorname{Im}\zeta = 0\}$. The point is that the projection π of Λ on $\mathbb{C}_z$ is bijective.

The Hamiltonian flow $(z(t), \zeta(t))$ of q defined on Λ by the canonical system

$$\begin{cases} \dot{z} = \partial_\zeta q(z,\zeta), \\ \dot{\zeta} = -\partial_z q(z,\zeta), \end{cases} \tag{2.2}$$

is the image by κ of the Hamiltonian flow $(x(t), \xi(t))$ of p:

$$(z(t), \zeta(t)) = \kappa(x(t), \xi(t)).$$

It is a curve on the energy plane $q^{-1}(E) = \{(z,\zeta) \in \Lambda; q(z,\zeta) = E\}$ for a fixed energy E.

By the simple well assumption on the potential $V(x)$ (see Introduction), the Hamiltonian flow of p on $p^{-1}(E)$, $E \in (E_0 - \epsilon, E_0 + \epsilon)$ is a simple periodic curve $\gamma(E)$, and so is the Hamiltonian flow $\kappa \circ \gamma(E)$ of q on $q^{-1}(E)$. The action $C(E) = \int \xi dx$ (see (0.3)) and the period $T(E)$ are also invariant by κ:

$$C(E) = \int_{\pi\circ\kappa\circ\gamma(E)} \zeta dz, \quad T(E) = C'(E) = \int_{\alpha(E)}^{\beta(E)} \frac{dx}{\sqrt{E - V(x)}}.$$

It is important to remark that the true solution $v(z;E,h)$ of (2.1) is not necessarily single-valued on $\mathbb{C}_z$. By Proposition 1.1 (3), we know that the quantization condition of the original equation (0.1) is equivalent to the condition

$$v(z;E,h) \in \mathcal{H}(\mathbb{C}_z) \cap e^{\xi^2/2h} L^2(\mathbb{R}^2_{x,\xi}). \tag{2.3}$$

On the other hand, we also know that the microsupport of u is included in $\gamma(E)$ by Propositions 1.3 or 1.4. From this point of view, it is natural to modify the condition (2.3) as follows:

(Q_2) The solution $v(z;E,h)$ of (2.1) is single-valued on $\pi \circ \kappa \circ \gamma(E)$.

Let us study the equation (2.1) by the WKB method. Put

$$v(z;E,h) = a(z;E,h)e^{i\psi(z;E)/h}, \quad a(z;E,h) \sim \sum_{j=0}^{n} a_j(z;E)h^j. \tag{2.4}$$

We then obtain the eikonal and the first transport equations:

$$q(z;\psi') = E, \tag{2.5}$$

$$\partial_\zeta q(z,\psi')\frac{da_0}{dz} + \frac{1}{2}\{\partial_\zeta^2 q(z,\psi')\psi'' + \partial_z\partial_\zeta q(z,\psi')\}a_0 = 0, \tag{2.6}$$

where $' = d/dz$. Note that

$$\frac{d}{dz}\{\partial_\zeta q(z,\psi'(z))\} = \partial_\zeta^2 q(z,\psi')\psi'' + \partial_z\partial_\zeta q(z,\psi').$$

So one can solve the first transport equation (2.6) and gets

$$a_0(z) = \text{const.}\{\partial_\zeta q(z,\psi')\}^{-1/2}. \tag{2.7}$$

Now to understand the condition (Q_2), we continue the WKB solution (2.4) along the closed trajectory $\pi \circ \kappa \circ \gamma(E)$.

First we have

$$\begin{aligned}(Q_2) &\iff a(z(t),h)\exp(i\psi(z(t))/h)|_{t=0}^{T} = 0,\\ &\iff \frac{a(z(T),h)}{a(z(0),h)}\exp\{i(\psi(z(T)) - \psi(z(0)))/h\} = 1.\end{aligned}$$

On $\kappa\circ\gamma(E)$, we have $\zeta = \psi'(z)$ by the eikonal equation (2.5), and so $\psi(z(T)) - \psi(z(0)) = \int_{z(0)}^{z(T)} \psi' dz = C(E)$. Hence

$$(Q_2) \Longleftrightarrow C(E) - ih \log M(E,h) = 2n\pi h \quad (n \in \mathbb{Z}),$$

where

$$M(E,h) = \frac{a(z(T),h)}{a(z(0),h)}.$$

Next we replace a by its principal term a_0:

$$M(E,h) = \frac{a_0(z(T))}{a_0(z(0))}(1 + O(h)).$$

The solution a_0 of the first transport equation (2.6) is given by (2.7), and if moreover $z = z(t)$ is on $\pi \circ \kappa \circ \gamma(E)$, then $\partial_\zeta q(z, \psi') = \dot{z}$ by (2.2). On the other hand, by the simple-well assumption, we have

$$\dot{z}(T) = e^{-2\pi i}\dot{z}(0), \quad \text{i.e.} \left\{\frac{\dot{z}(T)}{\dot{z}(0)}\right\}^{-1/2} = e^{-\pi i}.$$

Hence we have

$$M(E,h) = e^{-\pi i}(1 + C(h)).$$

Thus we obtain the Bohr-Sommerfeld condition from the condition (Q_2);

$$(Q_2) \Longleftrightarrow C(E) = (2n+1)\pi h + O(h^2).$$

References

[Gé-Sj] C. Gérard, J. Sjöstrand: *Semiclassical resonances generated by a closed trajectory of hyperbolic type*, Commun. Math. Phys., **108**, (1987), 391-421.

[Ma] A. Martinez: *Introduction to Semiclassical and Microlocal Analysis*, Springer (2002).