数理解析研究所講究録 1336 巻 2003 年 152-157

CONSTRUCTION OF THE SOLUTIONS FOR MICRODIFFERENTIAL EQUATIONS WITH FRACTIONAL POWER SINGULARITIES

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1. Introduction

As is seen in the Stokes phenomenon, differential equations with irregular singularities are more complex than ones with regular singularities. In spite of the difficulties, we treat them as if they had regular singularities by using a suitable transformation on a Riemannian sphere.

In this article, we introduce how to construct solutions for some partial differential equations of irregular type. We actually consider an ordinary differential equation because we regard a differential operator appear in that partial differential equation as a large parameter according to the microlocal analysis and the WKB method. In addition, our solutions are composed of amplitude and phase functions similar to the WKB solutions.

By the several transformations, the partial differential equation we first consider is reduced to a microdifferential equation with a fractional filtration. For the equation, we do not make microfunction solutions but microdifferential operators which act on microfunctions. Successive approximation is well performed to make such solutions.

Our construction does not give a concrete expression of the WKB solutions, but it gives an algorithm of making them. In the similar way of usual microdifferential operators, formal norms are used to show the convergence of the asymptotic expansion of the microdifferential operator solutions with a fractional power.

2. Problem

We are going to consider the next partial differential equation:

$$P(x,\partial_x,\partial_z)u(x,z)$$
$$= \left(\partial_x^m + a_1(x)\partial_z\partial_x^{m-1} + a_2(x)\partial_z^2\partial_x^{m-2} + \cdots + a_m(x)\partial_z^m\right)u(z,x) = 0,$$

where ∂_* stands for $\partial/\partial *$ $(* = x, z)$. For the sake of brevity, let x and z be one dimensional complexes.

We assume some conditions on the partial differential operator above.
(Assumption 1.) The principal symbol $\sigma(P)(x,\xi,\zeta)$ of the operator $P(x,\partial_x,\partial_z)$ has the following form in a neighborhood of $(0,0;0,i\eta)$:

$$\sigma(P)(x,\xi,\zeta) = \prod_{j=1}^{m}(\xi - x^\lambda\alpha_j(x)\zeta),$$

where λ is a positive number, $\alpha_j(x)$ $(j = 1, 2, \cdots, m)$ are analytic in a neighborhood of the origin. Moreover, we assume that each $\alpha_j(0)$ are mutually distinct.
(Assumption 2.) We assume that we have

$$a_k(x) = x^{l_k} b_k(x), \quad l_k \geq k\lambda,$$

where each $b_k(x)$ $(k = 1, 2, \cdots, m)$ is holomorphic in a neighborhood of the origin.
Furthermore, we regard ∂_z as a large parameter ζ because of the commutativity of operators.

On the hypothesis above, setting $y = x^{\lambda+1}/(\lambda+1)$ for the operator $x^m P(x, D_x, \zeta)$ we have

$$(2.1) \quad x^m P(x, D_x, \zeta) = \sum_{k=0}^{m} \{(\lambda+1)y\}^{\frac{k}{\lambda+1}} a_k\left(((\lambda+1)y)^{\frac{1}{\lambda+1}}\right) \zeta^k \times \prod_{l=0}^{m-k-1} \{(\lambda+1)yD_y - l\},$$

where D_* stands for an ordinary differential operator $(* = x, y)$.
It is WKB solutions of type

$$(2.2) \qquad \int S(w, \zeta) \exp(-\zeta y w) dw$$

that we need to obtain. Here S is an amplitude and $-\zeta y w$ is a phase. For this aim, we use the quantized Legendre transformation with respect to y:

$$(2.3) \qquad \beta_k : \begin{cases} \dfrac{d}{dy} = -\zeta w, \\ y = \dfrac{1}{\zeta}\dfrac{d}{dw}, \end{cases}$$

which keeps the sheaf isomorphism of microfunction.
We can finally reduce the partial differential operator P to the microdifferential operator $\widehat{P}$ as follows:

$$\widehat{P}(w, D_w, \zeta) = \left\{ \sum_{k=0}^{m} \widetilde{b_k}\left((\zeta^{-1}D_w)^{\frac{1}{\lambda+1}}\right)(-w)^{m-k} \right\} D_w^m + c_1\left(w, (\zeta^{-1}D_w)^{\frac{1}{\lambda+1}}\right) D_w^{m-1} + \cdots + c_m\left(w, (\zeta^{-1}D_w)^{\frac{1}{\lambda+1}}\right),$$

where c_0(=the coefficient of D_w^m), $c_1, \cdots, c_m$ are 0-th order operators with respect to $\zeta^{-1}D_w$.
By the division theorem of Späth type about microdifferential operators, the coefficient c_0 is split as $c_0(w, D_w, \zeta) = d(w, D_w, \zeta)(w - \Phi(D_w, \zeta))$ (ref. [KtS]). Therefore we consider the operator

$$(2.4) \qquad L = wD_w^m + \gamma_1(w, D_w, \zeta)D_w^{m-1} + \cdots + \gamma_m(w, D_w, \zeta)$$

for the sake of brevity.
A successive approximation plays an important role in our construction of the solutions for the equation

$$L(w, D_w, \zeta)U(w, \zeta) = 0,$$

where the microdifferential operator $U(w,\zeta)$ has a form

$$U(w,\zeta)=\sum_{j=-\infty}^{0} w^{-1+\frac{j}{\lambda+1}}u_j(\zeta)$$

at $w=\infty$.

Our successive approximation is the following:

$$\begin{cases} L_0 = L|_{\zeta^{-1}D_w=0}, \\ L_1 = L - L_0 = \sum_{j=0}^{m} \gamma_j(w,\zeta^{-1}D_w)D_w^{m-j}, \end{cases}$$

where

$$L_1=\sum_{j=0}^{m}\tilde{\gamma}_j(w,\zeta^{-1}D_w)\zeta^{-\frac{1}{\lambda+1}}D_w^{m-j+\frac{1}{\lambda+1}}.$$

Using the terminologies of physics, we call L_0 a classical term and L_1 a quantum term with a small parameter $\zeta^{-1/(\lambda+1)}$.

Set

$$D_\nu=\{(w,z;\tau,\zeta)\in T^*(\mathbb{C}\times\mathbb{C}); |w|\le 1+\nu, |z-\dot{x}|\le\nu, \left|\frac{\tau}{|\zeta|}-\frac{i\dot{\eta}}{|\dot{\eta}|}\right|\le\nu\}$$

and

$$V_\nu=\{(z;\zeta)\in\mathbb{C}\times\mathbb{C}; |z-\dot{x}|\le\nu, \left|\frac{\zeta}{|\zeta|}-\frac{i\dot{\eta}}{|\dot{\eta}|}\right|\le\nu\}.$$

Theorem. *Let $U_0\equiv U_{00}$ be an arbitrary holomorphic solution of $L_0U_0=0$ in a domain $D_\nu(\supset\{w\in\mathbb{C};|w|\le 1\}\times\{(\dot{x};i\dot{\eta})\})$ with homogeneous degree 0 with respect to ζ. We have a series U_k $(k=1,2,\cdots)$ of a successive process so that each $U_k=\sum_{j=-\infty}^{0}U_{jk}$ is a formal symbol defined in a neighborhood of $D_{\nu/2}$ satisfying*

$$\partial_w^l U_k|_{w=0}=0 \quad (l=0,1,2,\cdots,m-2; k\ge 1).$$

Then $U=\sum_{k=0}^{\infty}U_k$ converges in a suitable formal norm $N_m(\cdot;T)$ uniformly on a certain domain and it becomes a solution of the microdifferential equation

$$L(w,\partial_w,\partial_z)U(w,\partial_z)=0$$

to which is reduced the partial differential equation

$$P(x,\partial_x,\partial_z)u(x,z)=0.$$

3. Convergence of microdifferential operator solutions

In this section, we estimates the term of the successive approximation introduced in section2.

To begin with, we introduce a fractional derivation as the Riemann-Liouville integral.

A derivation of fractional order $1/q$ $(1 < q < \infty)$ as the Riemann-Liouville integral for a function $f(w)$ holomorphic in a neighborhood of γ including the origin $w = 0$ can be defined as

$$\left(\frac{d}{dw}\right)^{1/q} f(w) := \frac{1}{\Gamma(1-1/q)}\frac{d}{dw}\int_\gamma \frac{f(t)}{(w-t)^{1/q}}dt \ ,$$

where $\Gamma(\cdot)$ is a gamma function and γ is a proper integral contour which begins and ends at $t = w$ enclosing $t = \infty$ on $\mathbb{CP}$ once in the positive sense. Here the value of a many-valued function $(w-t)^{1/q}$ is defined by

$$(w-t)^{1/q} = |w-t|^{1/q} \exp\left(\sqrt{-1}\frac{1}{q}\arg(w-t)\right).$$

Lemma 3.1. *We have the following estimation around the origin for a function $f(w)$ holomorphic in a neighborhood of γ:*

$$\left|\left(\frac{d}{dw}\right)^{1/q} f(w)\right| \\ \leq \frac{1}{\Gamma(1-1/q)}\left\{C_1|w|^{-1/q}\sup_\gamma|f(w)| + C_2|w|^{1-1/q}\sup_\gamma|f'(w)|\right\} ,$$

with some constants $C_1, C_2 > 0$.

By means of this lemma, we have the estimation around the origin by the following:

$$\left|\left(\frac{d}{dw}\right)^{1/q} f(w)\right| \\ \leq \frac{1}{\Gamma(1-1/q)}\left\{C_1\varepsilon^{-1/q}\sup_\gamma|f(w)| + C_2\left(1-\frac{1}{q}\right)\varepsilon^{1-1/q}\sup_\gamma|f'(w)|\right\} ,$$

where $|w| < \varepsilon$. Taking suitable constants, the inequality

$$\left|\left(\frac{d}{dw}\right)^{1/q} f(w)\right| \leq \frac{C'}{\Gamma(1-1/q)}\varepsilon^{-1/q}\sup_\gamma|f(w)| \tag{3.1}$$

holds.

Next, we introduce the definition of microdifferential operators with a fractional filtration.

Definition 3.2. Let $a_j(z,\zeta)$ $(j = 0, \pm1, \pm2, \cdots)$ be holomorphic functions on Γ. When each $a_j(z,\zeta)$ satisfies the following conditions, we call the sum $\sum_j a_j(z,\zeta)$ a formal symbol of microdifferential operator with fractional order $1/q$:

(1) for any $\delta > 0$, there exists a constant $C_\delta > 0$ such that we have

$$\sup_\Gamma |a_j(z,\zeta)| \leq C_\delta \delta^{j/q}|\zeta|^{j/q}/[j/q]! \quad \text{for } j = 1, 2, \cdots , \tag{3.2}$$

(2) there exists a positive constant C such that

(3.3) $$\sup_{\Gamma} |a_j(z,\zeta)| \le C^{j/q+1}|\zeta|^{j/q}[-j/q]! \text{ for } j = 0, -1, -2, \cdots .$$

We will find the solution operator of type

$$\begin{aligned} U(w,\zeta) &= \sum_{j=-\infty}^{0} U_j(w,\zeta) \\ &= \sum_{j=-\infty}^{0} w^{-1+\frac{j}{\lambda+1}} u_j(\zeta) \end{aligned}$$

for the microdifferential equation

(3.4) $$\begin{aligned} L(w,D_w,\zeta)U(w,\zeta) &= \left(\sum_{k=0}^{m} \gamma_k(w,D_w,\zeta)D_w^{m-k}\right) U(w,\zeta) \\ &= \sum_{k=0}^{m} \left\{ \sum_{j=-\infty}^{0} \gamma_{jk}(w,D_w,\zeta) \right\} D_w^{m-k} \cdot U(w,\zeta) , \end{aligned}$$

where γ_k is defined as (2.4).

For this aim, we construct the solution operator by the aid of an iteration scheme.

To begin with, we take a part of $D_w = 0$ in $\gamma_k(w,D_w,\zeta)$ out of the operator $L(w,D_w,\zeta)$, that is, we set

(3.5) $$L_0 = \sum_{k=0}^{m} \gamma_k(w,0,\zeta)D_w^{m-k} .$$

We evaluate the rest of the equation defined above. Set

$$\begin{aligned} R(w,D_w,\zeta) &= \sum_{k=0}^{m} R_k(w,D_w,\zeta)D_w^{m-k} \\ &= \sum_{k=0}^{m} \sum_{j=-\infty}^{-1} R_{jk}(w,D_w,\zeta)D_w^{m-k}. \end{aligned}$$

Then the operator $R\circ$ can be defined as

$$R \circ U = \sum_{j=-\infty}^{-1} \left(\sum_{\substack{-s+l+p=j \\ 0\le k\le m}} \frac{1}{s!}(\partial_\tau^s R_{lk})(w,0,\zeta)\partial_w^{m-k+s}U_p(w,\zeta) \right) ,$$

where τ is the symbol of D_w. This $R \circ U$ becomes a formal symbol of order $-1/(\lambda+1)$. Hence we can show the convergence of the microdifferential operator $R \circ U$ by using a suitable formal norm.

We have an algorithm of the iteration scheme of successive approximation process by the following:

(3.6) $$\begin{cases} L_0U_0 = 0, \\ L_0U_{-p-1} = -R \circ U_{-p} \qquad (p = 0, 1, 2, \cdots). \end{cases}$$

This process is reduced to the equation

$$\left(\sum_{k=0}^{m}\gamma_k^0(w,\partial_z)\partial_w^{m-k}+R(w,\partial_w,\partial_z)\right)U(w,\partial_z)=0,$$

where $\gamma_k^0(w,\zeta)=\gamma_k(w,0,\zeta)$.

It is possible to show the convergence of the operator by formal norms as in [KtS].

Example 3.3. The Schrödinger equation with a large parameter of Airy type

$$(D_x^2-\zeta^2 x)u(x,\zeta)=0$$

becomes the hypergeometric differential equation by the transformations considered in section 2.

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