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On exponential time decay solutions to Schrödinger equations

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0 Introduction

Let us consider the Cauchy problem for the free Schrödinger equation:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = i\triangle u(t,x) \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where u is a complex-valued unknown function of $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $i = \sqrt{-1}$ and $\triangle$ is the Laplacian in $\mathbb{R}^n$ defined by

$$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{2}$$

In this Cauchy problem, it is interesting how the solution u behaves as $t \to \infty$ in virtue of given initial data u_0. In general, the initial data u_0 provide all of properties of the solution u to the Cauchy problem.

For example, if $u_0 \in L^1(\mathbb{R}^n)$ then we can solve the Cauchy problem explicitly by the expression

$$u(t,x) = \frac{1}{(4\pi it)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y)\, dy. \tag{3}$$

This implies the estimate

$$|u(t,x)| \leqq c\|u_0\|_{L^1} t^{-\frac{n}{2}} \tag{4}$$

with positive constant $c = (4\pi)^{-\frac{n}{2}}$. This example shows that u decays at most polynomial order in respect of t, but there are no information at which order u decays as $t \to \infty$.

To investigate the decay order more concretely, we see the next example. If

$$u_0(x) = e^{-|x|^2}, \tag{5}$$

then we can solve the Cauchy problem more explicitly by the expression

$$u(t,x) = (1+4it)^{-\frac{n}{2}} e^{-\frac{|x|^2}{1+4it}}. \tag{6}$$

This concrete solution has our desired information at which order u decays as $t \to \infty$. From the expression, we can immediately estimate the solution u by

$$c_0 t^{-\frac{n}{2}} \leqq |u(t,x)| \leqq c_1 t^{-\frac{n}{2}} \tag{7}$$

for some positive constants c_0 and c_1 which may depend on x, but independent of t. The estimate from both and lower shows that the solution u decays at the rate of $t^{-\frac{n}{2}}$ as $t \to \infty$, at the x to be fixed.

The second example contains much more information. It seems that solutions cannot decay more rapidly than polynomial order in t, even if initial value u_0 is smooth, or u_0 decays rapidly in the x direction. In fact, $u_0(x) = e^{-|x|^2}$ is one of the rapidly decreasing functions with good character.

In spite of our examples in Schrödinger equation, we can easily find a lot of examples of the solution decaying at the rate of exponential order in wave equation. For example in $n = 1$,

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = \dfrac{\partial^2}{\partial x^2} u(t,x) \\ u(0,x) = e^{-x^2}, \quad \dfrac{\partial}{\partial t} u(0,x) = 0 \end{cases} \tag{8}$$

has the solution

$$u(t,x) = \frac{1}{2}\left(e^{-(x+t)^2} + e^{-(x-t)^2}\right) \tag{9}$$

which decays at the rate of e^{-t} as $t \to \infty$ at every fixed point x.

Are there solutions in Schrödinger equation which decay at the rate of exponential order in t? If there are, to what function space do initial data belong? These problems are not so trivial.

In the present paper, we will construct the example of the solution which decay at the rate of exponential order in t, and then discuss the function space to which the solutions belong.

1 Exponential Time Decay

First of all, we give a definition of terminology "exponential time decay". The solution u is said to decay exponentially in time, or have exponential time decay property, if for any compact set K in $\mathbb{R}^n$ there are positive constants C and ε independent of t such that

$$\sup_{x\in K}|u(t,x)| \leqq Ce^{-\varepsilon t} \tag{10}$$

for any $t \in [0, \infty)$.

The purpose of this section is to construct the solutions with exponential time decay property.

The function space to which the solution belongs for each time t, is defined by

$$H^s_\delta = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n);\ e^{\delta\langle x\rangle}u(x) \in H^s\} \tag{11}$$

for a negative constant δ, where $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ denotes the set of all locally square-integrable functions, and H^s denotes the usual Sobolev space with regularity s. It is trivial from the definition that H^s_δ contains the usual Sobolev space H^s as a proper subset when δ is negative. We call H^s_δ weighted Sobolev spaces.

Let ψ be a function in $H^{[\frac{n}{2}]+1}$. Denote a phase function by

$$\varphi(x,\xi) = x\cdot\left(\xi - \frac{i\mu\xi}{|\xi|}\right) \tag{12}$$

for a complex number $\mu \in \mathbb{C}$, and define

$$I_\varphi(x,D)\psi(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}\hat{\psi}(\xi)\ d\xi \tag{13}$$

where $\hat{\psi}$ stands for the Fourier transform of ψ. Then, $u_0 = I_\varphi(x, D)\psi$ belongs to $H^{[\frac{n}{2}]+1}_{-(|\mathrm{Re}\mu|+\delta)}$ for any positive δ, and the solution u to the Cauchy problem (1) with the initial data u_0 has the estimate

$$|u(t,x)| \leqq Ce^{-2\mathrm{Re}\mu\mathrm{Im}\mu t+\delta_0\langle x\rangle} \tag{14}$$

for some $C > 0$ and $0 < \mathrm{Re}\mu < \delta_0 = \mathrm{Re}\mu + \delta$. This estimate shows that the solution decays exponentially with respect to time t, when both $\mathrm{Re}\mu$ and $\mathrm{Im}\mu$ are positive. See [1, 2] for the proof in detail.

THEOREM 1. *Let $\psi \in H^{[\frac{n}{2}]+1}$ and $\varphi(x,\xi) = x \cdot \left(\xi - \frac{i\mu\xi}{|\xi|}\right)$ for $\mu \in \mathbb{C}$. If $Re\mu > 0$, $Im\mu > 0$ and $u_0(x) = I_\varphi(x, D)\psi(x)$, then the solution u to the Cauchy problem (1) is estimated by*

$$|u(t,x)| \leqq Ce^{-2Re\mu Im\mu t+(Re\mu+\delta)\langle x\rangle} \tag{15}$$

for any $\delta > 0$ and $(t,x) \in [0,\infty) \times \mathbb{R}^n$.

As an example, we can take a function $\psi(x) = e^{-|x|^2}$ when $n = 1$. After the calculation of $I_\varphi(x, D)\psi(x)$, we have

$$u_0(x) = \frac{1}{2}e^{-x^2}\left\{e^{+\mu x} + e^{-\mu x} + \frac{i}{\sqrt{2\pi}}(e^{+\mu x} - e^{-\mu x})\int_0^x e^{y^2}\,dy\right\} \tag{16}$$

which increase exponentially in x, but decrease exponentially in t!

The proof of the THEOREM 1 is based on the following lemma.

LEMMA 1.1. *Let s be a non-negative integer. If φ is as in THEOREM 1, then $I_\varphi(x,D)$ operates continuously from H^s to $H^s_{-(|Re\mu|+\delta)}$ for any $\delta > 0$.*

LEMMA 1.1 claims that there is $C > 0$ such that

$$\|I_\varphi(x,D)\psi\|_{H^s_{-(|\mathrm{Re}\mu|+\delta)}} \leqq C\|\psi\|_{H^s} \tag{17}$$

for any $\psi \in H^s$. According to the definition of $I_\varphi(x, D)$, the estimate of

$$e^{-(|\mathrm{Re}\mu|+\delta)\langle x\rangle}I_\varphi(x,D)\psi(x) \tag{18}$$

proves LEMMA 1.1. In the estimate, we have to face the pseudo-differential operator

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi}p(x,\xi)\hat{\psi}(\xi)\,d\xi \tag{19}$$

where $p(x,\xi) = \exp\left[\mu \frac{x\cdot\xi}{|\xi|}\right]$. Because $p(x,\xi)$ has singularity at $\xi = 0$, it is delicate to treat near the origin.

So, we make partition of integral region, such as $|\xi| \leqq 1$ and $|\xi| \geqq 1$. More precise, taking $\chi \in C_0^\infty(\mathbb{R}^n)$ such that supp$\chi \subset S^{n-1} = \{\xi \in \mathbb{R}^n; |\xi| \leqq 1\}$, $0 \leqq \chi(\xi) \leqq 1$ for all $\xi \in \mathbb{R}^n$ and $\chi(\xi) = 1$ in some neighborhood of $\xi = 0$, then we decompose $p(x,\xi)$ into $p(x,\xi)\chi(\xi)$ and $p(x,\xi)(1-\chi(\xi))$. Consideration on the compact set $|\xi| \leqq 1$ near $\xi = 0$ becomes our discussion much easier, because ξ can only move in the compact set.

On the other hand, when $|\xi| \geqq 1$, we conclude the estimate from the behavoir of $p(x,\xi)$. That is, $p(x,\xi)(1-\chi(\xi))$ belongs to the symbol class $S^0_{1,0}$ (Hörmander's notation) Therefore, we conclude LEMMA 1.1 by the L^2 boundedness theorem for pseudo-differential operators.

To prove THEOREM 1, we use the solution to the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) = -i(|D| - i\mu)^2 v(t,x) \\ v(0,x) = v_0(x), \end{cases} \tag{20}$$

when we give the initial data v_0 in the function space $H^{[\frac{n}{2}]+1}$. The solution is expressed by

$$v(t,x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-i(|\xi|-i\mu)^2 t} \hat{v}_0(\xi)\, d\xi, \tag{21}$$

so that if we give u by the relation

$$u(t,x) = I_\varphi(x,D) v(t,x), \tag{22}$$

then it is concluded that u solves the Cauchy problem (1), and that u has exponential time decay property. The regularity of u, that is $\left[\frac{n}{2}\right] + 1$, is neccesary for the estimate when we use Sobolev's lemma;

$$\begin{aligned} e^{-(|\mathrm{Re}\mu|+\delta)\langle x\rangle} |u(t,x)| &\leqq \text{Const.} \| e^{-(|\mathrm{Re}\mu|+\delta)\langle x\rangle} u(t,x) \|_{H^s} \\ &= \text{Const.} \| u(t,x) \|_{H^s_{-(|\mathrm{Re}\mu|+\delta)}} \\ &= \text{Const.} \| I_\varphi(x,D) v(t,x) \|_{H^s_{-(|\mathrm{Re}\mu|+\delta)}} \\ &\leqq \text{Const.} \| v(t,x) \|_{H^s} \end{aligned} \tag{23}$$

Thus, we have just obtained exponential time decay solutions u, by virtue of v which is constructed so as to have exponential time decay property.

2 Uniqueness Theorem

By the way, it is well known that the Cauchy problem (1) is wellposed in the Sobolev space H^s. The estimate (14) makes sense when the uniqueness of the solution is assured, because the estimate (14) should be valid for all of the solutions. That is, the discussion in the preceding section is essentially based on the unique solution to the Cauchy problem (1). Consequently, we are confronted with require to prove the uniqueness theorem in the Sobolev space with the exponential weights.

THEOREM 2. *Let s be a non-negative integer. If $u_0 = 0$ and a solution u to Cauchy problem (1) belongs to $C^0([0,\infty); H^s_{-\sigma}) \cap C^1([0,\infty); H^{s-2}_{-\sigma})$ for a fixed $\sigma > 0$, then $u = 0$.*

The key idea in the proof is that existence of a solution to the adjoint problem implies the uniqueness of a solution to the original Cauchy problem (1). The uniqueness statement maybe seems to be almost trivial, but we need precise discussion because wellposedness is not always valid in the weighted Sobolev spaces.

REMARK. THEOREM 2 is not devoted to existence, but to uniqueness of the solution. Although it is not clear whether the existence does hold or not, THEOREM 1 have constructed a solution by giving a certain condition to initial data.

In the proof of THEOREM 2 below, we use the estimate for $e^{\lambda\langle D\rangle}a(x)e^{-\lambda\langle D\rangle}$ where a is analytic in the following sense.

DEFINITION. Let $a(x)$ be a smooth function which belongs to $C^\infty(\mathbb{R}^n)$. We say that $a(x)$ is analytic in $\mathbb{R}^n$ if there exist $\rho_0 > 0$ and $C_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} |\partial_x^\alpha a(x)| \leqq C_0|\alpha|!\rho_0^{-|\alpha|} \tag{24}$$

for any $\alpha \in \mathbb{N}^n$.

The exact estimate for $e^{\lambda\langle D\rangle}a(x)e^{-\lambda\langle D\rangle}$ is given by the next lemma. See [3] for the proof in detail.

LEMMA 2.1. *Let $\lambda > 0$ and $a(x)$ be analytic in $\mathbb{R}^n$. Put*

$$\tilde{a}(x,D) = e^{\lambda\langle D\rangle}a(x)e^{-\lambda\langle D\rangle}. \tag{25}$$

If λ is sufficiently small, ($\lambda < \rho$ for some ρ) then there exists a polynomial $C(\lambda)$ on $[0, \infty)$ such that

$$\|\tilde{a}(x, D)u\|_{L^2} \leqq C(\lambda)\|u\|_{L^2} \tag{26}$$

and that

$$C(\lambda) = \sum_{j=0}^{N_0} c_j \lambda^j \tag{27}$$

for some $N_0 \geqq 0$ and $c_j > 0$ for $j = 0, 1, \ldots, N_0$.

The proof of THEOREM 2 is given here.

Proof. When we would like to prove the uniqueness, it is sufficient to prove that if $u \in C^0([0,T]; H^s_{-\sigma}) \cap C^1([0,T]; H^{s-2}_{-\sigma})$ is a solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = i\Delta u(t,x) \\ u(0,x) = 0, \end{cases} \tag{28}$$

then $u = 0$ in the function space of $C^0([0,T]; H^s_{-\sigma}) \cap C^1([0,T]; H^{s-2}_{-\sigma})$. Change unknown functions by putting $v = e^{-\sigma\langle x\rangle} u$, we have

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,x) = i \displaystyle\sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} + \sigma \frac{x_j}{\langle x\rangle} \right)^2 v(t,x) \\ v(0,x) = 0 \end{cases} \tag{29}$$

for $v \in C^0([0,T]; H^s) \cap C^1([0,T]; H^{s-2})$. This Cauchy problem cannot be solved generally, because the imaginary part of the coefficients of the first-order terms, that is $x_j\langle x\rangle^{-1}$, do not satisfy the necessary condition to be well-posed in H^∞. Let

$$L = \frac{\partial}{\partial t} - i \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} + \sigma \frac{x_j}{\langle x\rangle} \right)^2, \tag{30}$$

then the Cauchy problem (29) is equivalent to

$$\begin{cases} Lv = 0 & \text{in } (0,T) \times \mathbb{R}^n \\ v(0) = 0 & \text{on } \mathbb{R}^n. \end{cases} \tag{31}$$

Putting

$$L^* = -\frac{\partial}{\partial t} + i\Delta - \frac{2i\sigma}{\langle x\rangle} x \cdot \nabla + ic(x) \tag{32}$$

where $c(x) = \sigma^2 - \frac{(n-1)\sigma}{\langle x\rangle} - \frac{\sigma^2}{\langle x\rangle^2} - \frac{\sigma}{\langle x\rangle^3}$, and we will consider the adjoint problem:

$$\begin{cases} L^* w = 0 & \text{in } (0,T) \times \mathbb{R}^n \\ w(T) = g & \text{on } \mathbb{R}^n. \end{cases} \tag{33}$$

On the other hand, if v is a solution to the Cauchy problem (31), we have

$$\begin{aligned} 0 &= \int_0^T (Lv(t), w(t))\, dt & (34)\\ &= \int_0^T (v(t), L^* w(t))\, dt + [(v(t), w(t))]_0^T & (35)\\ &= \int_0^T (v(t), L^* w(t))\, dt + (v(T), w(T)). & (36) \end{aligned}$$

Hence, in order that we obtain the uniqueness statement of the solution, it is sufficient that the Cauchy problem (33) can be solved for a certain function class, that is, there exists a solution to (33) for any $g \in L_1^2(\mathbb{R}^n)$, where

$$L_1^2(\mathbb{R}^n) = \{g \in L^2(\mathbb{R}^n); e^{\delta_0 \langle \xi\rangle} \hat{g}(\xi) \in L^2(\mathbb{R}_\xi^n)\} \tag{37}$$

is dense in $L^2(\mathbb{R}^n)$ for a positive constant δ_0 with $\delta_0 < \frac{1}{n\sqrt{6n}}$. In fact, if there is w satisfying (33) for any $g \in L_1^2(\mathbb{R}^n)$, then $(v(T), g) = 0$ for any $g \in L_1^2(\mathbb{R}^n)$, which implies $v(T) = 0$.

Here, we change variables t to τ, such as $\tau = T - t$, and we put $\tilde{w}(\tau) = e^{\nu(T-\tau)\langle D\rangle} w(T-\tau)$ for some $\nu > 0$, in order to make our later discussion simpler. Hence, we obtain

$$\begin{cases} \dfrac{\partial}{\partial \tau}\tilde{w}(\tau) \\ \quad = i\Delta \tilde{w}(\tau) + \left(2\sigma \displaystyle\sum_{j=1}^n \tilde{b}_j(\tau; x, D) D_j - \nu\langle D\rangle\right) \tilde{w}(\tau) + i\tilde{c}(\tau; x, D)\tilde{w}(\tau) \\ \tilde{w}(0) = e^{\nu T\langle D\rangle} g \end{cases}$$

where

$$\begin{cases} \tilde{b}_j(\tau; x, D) = e^{\nu(T-\tau)\langle D\rangle} \dfrac{x_j}{\langle x\rangle} e^{-\nu(T-\tau)\langle D\rangle} \\ \tilde{c}(\tau; x, D) = e^{\nu(T-\tau)\langle D\rangle} c(x) e^{-\nu(T-\tau)\langle D\rangle}. \end{cases} \tag{39}$$

It should be remarked that both $\frac{x_j}{\langle x\rangle}$ and $c(x)$ in the above operators are analytic in $\mathbb{R}^n$. We can find the solution $\tilde{w} \in C^0([0,T]; H^s) \cap C^1([0,T]; H^{s-2})$ of this equation (38) if we give initial data $e^{\nu T\langle D\rangle} g \in H^s$.

For $j = 1, \ldots, n$

$$\mathrm{Re}(\tilde{b}_j(\tau; x, D) D_j \tilde{w}, \tilde{w}) \leqq B_j(\nu, T) \|\langle D\rangle^{\frac{1}{2}} \tilde{w}\|_{L^2}^2, \tag{40}$$

where $B_j(\nu, T)$ is a positive constant which depends on ν and T, but is independent of τ. Then, we obtain the energy estimate from the equation (38) such as

$$\|\tilde{w}(\tau)\|_{L^2} \leqq e^{C\tau} \|e^{\nu T\langle D\rangle} g\|_{L^2} \quad (0 \leqq \tau \leqq T) \tag{41}$$

for some positive constant C which depends on ν, T and σ, when the condition that $\nu(T-\tau) < \rho$ and $2\sigma \sum_{j=1}^{n} B_j(\nu, T) \leqq \nu$ hold. From LEMMA 2.1, where we make precise estimate of constant B_j,

$$\sum_{j=1}^{n} B_j(\nu, T) \leqq C'(1 + \nu T)^{N_0} \tag{42}$$

holds for a sufficiently large constant $C' > 0$, therefore when we choose ν and T such that $\nu T = \delta_0$ and $2\sigma C'(1 + \nu T)^{N_0} \leqq \nu$ at the beginning of our discussion, then the proof has just completed. □

3 Vision for the future

The equation may be generalized to Schrödinger type equations

$$\begin{cases} \dfrac{1}{i}\dfrac{\partial}{\partial t} u(t,x) \\ \quad = \displaystyle\sum_{j,k=1}^{n} D_j\left(a_{jk}(x) D_k u\right)(t,x) + \sum_{j=1}^{n} b_j(x) D_j u(t,x) + c(x) u(t,x) \\ u(0,x) = u_0(x) \end{cases} \tag{43}$$

with some assumptions on a_{jk}, b_j and c. It is an issue whether or not we can obtain similar result even in the generalized form.

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