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On the Fundamental Solutions of Linear Fuchsian Partial Differential Equations

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Abstract

Without any assumption on the characteristic exponents, we give fundamental solutions of linear Fuchsian partial differential equations.

1. Introduction and Main result.

Let $\mathbb{C}$ be the set of complex numbers, $t \in \mathbb{C}$, $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $\mathbb{N} = \{0, 1, \ldots\}$, $m \in \mathbb{N}^* = \mathbb{N} - \{0\}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Let Δ be a polydisc centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x^n$ and set $\Delta_0 = \Delta \cap \{t = 0\}$. Let $a_{j,\alpha}(t,x)$ $(j + |\alpha| \le m,\ j < m)$ be holomorphic functions defined on Δ satisfying the following

$$a_{j,\alpha}(0,x) \equiv 0 \quad \text{on } \Delta_0 \text{ if } |\alpha| > 0. \tag{1.1}$$

We consider a Fuchsian partial differential operator

$$P = \left(t\frac{\partial}{\partial t}\right)^m + \sum_{\substack{j+|\alpha|\le m \\ j<m}} a_{j,\alpha}(t,x) \left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha \tag{1.2}$$

and the following linear partial differential equation

$$Pu = 0. \tag{1.3}$$

This operator P in (1.2) was introduced by M. S. Baouendi and C. Goulaouic [1], they proved a Cauchy-Kowalevsky type theorem and a Holmgren type theorem. Also, H. Tahara [2] has investigated the structure of singular solutions of $Pu = 0$.

Now, let us introduce

i) $\mathfrak{R}(\mathbb{C} \setminus \{0\})$ the universal covering space of $\mathbb{C} \setminus \{0\}$,

ii) $S(\epsilon) = \{t \in \mathfrak{R}(\mathbb{C} \setminus \{0\});\ 0 < |t| < \epsilon\}$,

iii) $D_L = \{x \in \mathbb{C}^n;\ |x_i| < L,\ i = 1, \ldots, n\}$,

iv) $\tilde{\mathcal{O}}$ the set of functions $u(t,x)$ satisfying the following:
there are $\epsilon > 0$ and $L > 0$ such that $u(t,x)$ is holomorphic on $S(\epsilon) \times D_L$,

v) $\mathcal{O}_0$ the set of germs of holomorphic functions at $x = 0$ and it is the same as $\mathbb{C}\{x\}$ the ring of convergent power series in x,

vi) The polynomial algebra in φ with the coefficients in a ring K is denoted by $K[\varphi]$,

vii) $\mathcal{O}(D)$ the set of holomorphic functions on D.

We set
$$C(\lambda, x) = \lambda^m + \sum_{j<m} a_{j,0}(0,x)\lambda^j.$$
This polynomial in λ is called the characteristic polynomial of P. We denote by $\lambda_1(x), \dots, \lambda_m(x)$ the roots of the equation
$$C(\lambda, x) = 0.$$
These $\lambda_1(x), \dots, \lambda_m(x)$ are called the characteristic exponent functions of P. Now, let us recall the result in [2].

Theorem 1.1 ([H. Tahara (1979)]). *If the condition*

$$\lambda_i(0) - \lambda_j(0) \notin \mathbb{Z} - \{0\} \quad \text{for } 1 \le i \ne j \le m \tag{1.4}$$

is satisfied, there are holomorphic functions $E_i(t,x,y)$ $(i = 1, \dots, m)$ *on*

$$\Omega = \{(t,x,y) \in S(\epsilon) \times D_L \times D_L;\ |t| < M|x_i - y_i|^m,\ i = 1, \dots, n\}$$

for some $\epsilon > 0$, $L > 0$, *and* $M > 0$ *which satisfy the following properties:*
(I) *For any* $\varphi_i(x) \in \mathcal{O}_0$ $(i = 1, \dots, m)$ *the function* $u(t,x)$ *defined by*

$$u(t,x) = \sum_{i=1}^m \oint E_i(t,x,y)\varphi_i(y)dy \tag{1.5}$$

is an $\tilde{\mathcal{O}}$*-solution of* $Pu = 0$.
(II) *Conversely, if* $u(t,x)$ *is an* $\tilde{\mathcal{O}}$*-solution of* $Pu = 0$*, then* $u(t,x)$ *is expressed in the form* (1.5) *for some* $\varphi_i(x) \in \mathcal{O}_0$ $(i = 1, \dots, m)$.

Here, the meaning of the integration in (1.5) is as follows:

$$\oint E_i(t,x,y)\varphi_i(y)dy = \int_{\Gamma_1} \cdots \int_{\Gamma_n} E_i(t,x,y)\varphi_i(y)dy_1 \cdots dy_n$$

and for $i = 1, \dots, n$, Γ_i denotes the circle

$$\{y_i \in \mathbb{C};\ |y_i - x_i| = s_i\}$$

with an orientation of counter clock-wise in the y_i-plane. Let $\varphi_i(x)$ be a holomorphic function on D_L. Since $E_i(t,x,y)$ is holomorphic with respect to y_i-variable on

$$\left\{ y_i \in \mathbb{C};\ \left(\frac{|t|}{M}\right)^{\frac{1}{m}} < |x_i - y_i|,\ |y_i| < L \right\},$$

we take the radius s_i so that

$$\left(\frac{|t|}{M}\right)^{\frac{1}{m}} < s_i < L.$$

H. Tahara called such functions $E_i(t,x,y)$ $(i = 1, \dots, m)$ a fundamental system of solutions (or fundamental solutions) of (1.3) in $\tilde{\mathcal{O}}$. It should be noted that if we denote by S the set of all $\tilde{\mathcal{O}}$-solutions of (1.3) the map defined by

$$\begin{array}{cccc} \Phi\colon & (\mathcal{O}_0)^m & \longrightarrow & S \\ & \Psi & & \Psi \\ & (\varphi_1, \dots, \varphi_m) & \longmapsto & \displaystyle\sum_{i=1}^m \oint E_i(t,x,y)\varphi_i(y)dy \end{array} \tag{1.6}$$

is an isomorphism. In the case where the condition (1.4) is not satisfied, the construction of fundamental solutions of (1.3) in $\tilde{\mathcal{O}}$ seemed to be very complicated and it remained as unsolved problem. About two decades later, T. Mandai [3] proved the following theorem without any assumption on the characteristic exponents of P. The following theorem is his result:

Theorem 1.2 **([T. Mandai (2000)]).** *Without any assumption on the characteristic exponents of P, we can construct an isomorphism*

$$\Psi\colon (\mathcal{O}_0)^m \longrightarrow S, \qquad (\varphi_1, \ldots, \varphi_m) \longmapsto \sum_{i=1}^{m} K_i[\varphi_i]. \tag{1.7}$$

T. Mandai called this map a solution map of (1.3) in $\tilde{\mathcal{O}}$. The construction of $K_i[\varphi_i]$ is very elegant, but still the construction of fundamental solutions as in (1.6) has remained as unsolved problem. In this paper we will solve this problem. The following is the main theorem of this paper.

Theorem 1.3 (Main result). *Without any assumption on the characteristic exponents, we can construct holomorphic functions $E_i(t,x,y)$ $(i = 1,\ldots,m)$ on*

$$\Omega = \{(t,x,y) \in S(\epsilon) \times D_L \times D_L;\ |t| < M|x_i - y_i|^m,\ i = 1,\ldots,n\}$$

for some $\epsilon > 0$, $L > 0$, and $M > 0$ such that the $K_i[\varphi_i]$ $(i = 1,\ldots,m)$ in Theorem 1.2 are expressed in the form

$$K_i[\varphi_i] = \int_{\Gamma_1} \cdots \int_{\Gamma_n} E_i(t,x,y)\varphi_i(y)dy_1 \cdots dy_n$$

for any $\varphi_i(x) \in \mathcal{O}_0$.

2. A key proposition to the proof of the main theorem.

We begin by introducing some notation and definition that will be used throughout this work. We define the indicial polynomial of P is

$$C(\mu) = \mu^m + \sum_{j<m} a_{j,0}(0,0)\mu^j$$

and a characteristic exponent of P is a root of the equation $C(\mu) = 0$. Let $\mu_1,\ldots,\mu_d$ be the distinct characteristic exponents, and let r_j $(j = 1,\ldots,d)$ be the multiplicity of μ_j. Then, for each $j = 1,\ldots,d$, we can take a domain S_j in $\mathbb{C}$ enclosed by a simple closed curve γ_j such that

$$\mu_j \in S_j \ (1 \le j \le d)$$

and

$$\bar{S}_i \cap \bar{S}_j = \emptyset \text{ if } i \neq j$$

and

$$C(\lambda + \nu, 0) \neq 0 \text{ for every } \lambda \in \left(\cup_{j=1}^{d}(\bar{S}_j \setminus \{\mu_j\})\right) \text{ and every } \nu \in \mathbb{N}$$

where $\bar{S}$ denote the closure of S. Thus, if we take $L > 0$ sufficiently small, then we have

$$C(\lambda + \nu, x) \neq 0 \text{ for every } x \in D_L, \text{ every } \lambda \in \left(\cup_{j=1}^{d}\gamma_j\right), \text{ and every } \nu \in \mathbb{N}.$$

For every $x \in D_L$, above condition implies that the number of the roots of $C(\lambda,x) = 0$ in S_j is r_j. Then, there exists monic polynomials $B_j(\lambda,x)$ such that

$$C(\lambda,x) = \prod_{j=1}^{d} B_j(\lambda,x)$$

where $B_1(\lambda,x) = (\lambda - \lambda_1(x))\cdots(\lambda - \lambda_{r_1}(x))$, $B_2(\lambda,x) = (\lambda - \lambda_{r_1+1}(x))\cdots(\lambda - \lambda_{r_1+r_2}(x))$, ..., $B_j(\lambda,x) = (\lambda - \lambda_{r_1+\cdots+r_{j-1}+1}(x))\cdots(\lambda - \lambda_{r_1+\cdots+r_j}(x))$ and $B_j(\lambda,x) \in \mathcal{O}(D_L)[\lambda]$ $(1 \le j \le d)$. For $0 < L < 1$ we set

$$\Omega_L = \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^n;\ |x_i| < L,\ |y_i| < L,\ x_i \neq y_i,\ i = 1,\ldots,n\}$$

and for $(x,y) \in \mathbb{C}^n \times \mathbb{C}^n$ we write

$$\psi_L(x,y) = \min\{L - |x_i|,\ |x_i - y_i|,\ i = 1,\dots,n\}.$$

Then, we see that

$$0 < \psi_L(x,y) < 1 \quad \text{for any } (x,y) \in \Omega_L.$$

Here, we will have a review of T. Mandai [3]. The following theorem is his result:

Theorem 2.1. *For any $\varphi_{j,k}(x) \in \mathcal{O}_0$ and for $1 \le j \le d$ and $1 \le k \le r_j$, there exists a unique solution $K_{j,k}(t,x,\lambda) \in \mathcal{O}(\{t=0\} \times D_L \times (\cup_{j=1}^d \gamma_j))$ of the equation*

$$P(K_{j,k}(t,x,\lambda)t^\lambda) = \frac{C(\lambda,x)\cdot \partial_\lambda^k B_j(\lambda,x)\cdot \varphi_{j,k}(x)}{B_j(\lambda,x)} t^\lambda.$$

And the function

$$K_{j,k}[\varphi_{j,k}] = \frac{1}{2\pi i}\int_{\gamma_j} K_{j,k}(t,x,\lambda)t^\lambda d\lambda$$

is an $\tilde{\mathcal{O}}$-solution of $Pu = 0$.
Moreover, we have a linear isomorphism

$$\begin{array}{cccc} \Psi\colon & (\mathcal{O}_0)^m & \longrightarrow & S \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & (\varphi_{j,k})_{\substack{1\le j\le d\\ 1\le k\le r_j}} & \longmapsto & \displaystyle\sum_{j=1}^{d}\sum_{k=1}^{r_j} K_{j,k}[\varphi_{j,k}]. \end{array}$$

This result will be useful later. We now consider the following partial differential equation:

$$P(F_{j,k}(t,x,y,\lambda)t^\lambda) = \frac{\partial_\lambda^k B_j(\lambda,y)\cdot C(\lambda,x)t^\lambda}{(2\pi i)^n B_j(\lambda,y)(y_1-x_1)\cdots(y_n-x_n)}. \tag{2.1}$$

The above equation is essence of our construction and we will prepare the following proposition:

Proposition 2.2. *For $1 \le j \le d$ and $1 \le k \le r_j$, the equation* (2.1) *has a unique holomorphic solution $F_{j,k}(t,x,y,\lambda)$ on*

$$\Omega' = \left\{(t,x,y,\lambda);\ (x,y)\in\Omega_L,\ \lambda \in (\cup_{j=1}^d \gamma_j) \text{ and } \frac{|t|}{\psi_L(x,y)^m} < M\right\}$$

for some $L > 0$ and $M > 0$.

By using this proposition, we can prove the Theorem 1.3.

3. Sketch of the proof of Proposition 2.2.

We still need to show the Proposition 2.2. To avoid confusion, we write β instead of j in (1.2). By expanding $a_{\beta,\alpha}(t,x)$ into Taylor series in t and using (1.1), the equation (2.1) is reduced to the form

$$\begin{aligned} &C\left(t\frac{\partial}{\partial t},x\right)(F_{j,k}(t,x,y,\lambda)t^\lambda) \\ &= -\sum_{\substack{\beta+|\alpha|\le m\\ \beta<m}}\sum_{p\ge 1} a_{\beta,\alpha,p}(x)t^p\left(t\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\alpha F_{j,k}(t,x,y,\lambda)t^\lambda \\ &\quad + \frac{\partial_\lambda^k B_j(\lambda,y)\cdot C(\lambda,x)t^\lambda}{(2\pi i)^n B_j(\lambda,y)(y_1-x_1)\cdots(y_n-x_n)}, \end{aligned} \tag{3.1}$$

where $a_{\beta,\alpha,p}(x) \in \mathcal{O}(D_L)$ for some $L > 0$. Let us find a formal solution of (3.1) of the form

$$F_{j,k}(t,x,y,\lambda) = \sum_{\nu=0}^{\infty} F_{j,k,\nu}(x,y,\lambda)t^{\nu}.$$

Then (3.1) is reduced to the following recursive formula:

$$\begin{aligned} &C(\lambda+\nu,x)F_{j,k,\nu}(x,y,\lambda) \\ &\quad = -\sum_{\substack{\beta+|\alpha|\le m \\ \beta<m}} \sum_{\substack{p+q=\nu \\ p\ge 1}} a_{\beta,\alpha,p}(x)(\lambda+q)^{\beta}\left(\frac{\partial}{\partial x}\right)^{\alpha} F_{j,k,q}(x,y,\lambda) \end{aligned} \tag{3.2}$$

for $\nu = 1, 2, \ldots,$

$$F_{j,k,0}(x,y,\lambda) = \frac{\partial_{\lambda}^{k} B_j(\lambda,y)}{(2\pi i)^n B_j(\lambda,y)(y_1-x_1)\cdots(y_n-x_n)}. \tag{3.3}$$

It follows from (3.2) and (3.3) that the equation (3.1) has a unique formal solution $F_{j,k}(t,x,y,\lambda) = \sum_{\nu=0}^{\infty} F_{j,k,\nu}(x,y,\lambda)t^{\lambda}$. From now on, we will investigate the domain of convergence of $F_{j,k}(t,x,y,\lambda)$. Now, we may assume:

(a) $|a_{\beta,\alpha,p}(x)| \le b_{\beta,\alpha,p}$ on D_L for any (β,α,p);

(b) $\sum_{p\ge 1} b_{\beta,\alpha,p}t^p \in \mathbb{C}\{t\}$ for any (β,α);

(c) There is a positive constant k_0 such that

$$|C(\lambda+\nu,x)| \ge k_0(\nu+1)^m \text{ on } (\cup_{j=1}^{d}\gamma_j)\times D_L \text{ for } \nu = 0,1,2,\ldots.$$

Moreover, we write

$$J = \max_{\lambda\in(\cup_{j=1}^{d}\gamma_j)} |\lambda|.$$

The following lemma will play an important role later.

Lemma 3.1. *If $F(x,y)$ is holomorphic on Ω_L and the following estimate holds:*

$$|F(x,y)| \le \frac{A}{\psi_L(x,y)^{\zeta}} \quad \text{on } \Omega_L$$

for some $A \ge 0$ and $\zeta > 0$, then we have

$$\left|\frac{\partial F}{\partial x_i}(x,y)\right| \le \frac{A(1+\zeta)e}{\psi_L(x,y)^{\zeta+1}} \quad \text{on } \Omega_L$$

for $i = 1,\ldots,n$.

By using this lemma and (3.3), we have

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} F_{j,k,0}(x,y,\lambda)\right| \le \frac{B}{\psi_L(x,y)^{n+m}} \tag{3.4}$$

on $\Omega_L \times \left(\bigcup_{j=1}^{d}\gamma_j\right)$ for any $|\alpha| \le m$, for some $B > 0$.

For any fixed $(x,y) \in \Omega_L$, we consider the following linear equation with respect to $G = G(t,x,y)$:

$$\begin{aligned} k_0 G &= \frac{k_0 B}{\psi_L(x,y)^{n+m}} \\ &\quad + \frac{1}{\psi_L(x,y)^m} \sum_{\substack{\beta+|\alpha|\le m \\ \beta<m}} \sum_{p\ge 1} \frac{b_{\beta,\alpha,p}}{\psi_L(x,y)^{m(p-1)}}(J+1)^m t^p \big(e(n+m)\big)^m G. \end{aligned} \tag{3.5}$$

It is obvious that the above equation has a unique holomorphic solution

$$G=\sum_{l=0}^{\infty}G_l(x,y)t^l\in\mathbb{C}\{t\}.$$

By using (3.5), we have

$$G_l(x,y)=\frac{\epsilon_l}{\psi_L(x,y)^{n+(l+1)m}} \tag{3.6}$$

for some $\epsilon_l\geq 0$. Here, we note the following proposition.

Proposition 3.2. *For any $|\alpha|\leq m,\ 1\leq j\leq d$, and $1\leq k\leq r_j$, the following inequality holds:*

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}F_{j,k,\nu}(x,y,\lambda)\right|\leq(\nu+1)^{|\alpha|}\big(e(n+m)\big)^{m}G_\nu(x,y)$$

on $\Omega_L\times\left(\bigcup_{j=1}^{d}\gamma_j\right)$ for $\nu=0,1,2,\ldots$.

By applying (3.4) and (3.6) we then obtain this proposition. This proposition implies that $\big(e(n+m)\big)^{m}G$ is a majorant series of $F_{j,k}(t,x,y,\lambda)$. From (3.5) and (3.6), we see that the domain of convergence of G includes Ω'. Consequently, $F_{j,k}(t,x,y,\lambda)$ is holomorphic on Ω'. This proves the Proposition 2.2. Now, we remark in [2], if $\lambda_i(0)-\lambda_j(0)\notin\mathbb{Z}$ for $1\leq i\neq j\leq m$ holds, then the author has constructed fundamental solutions $E_j(t,x,y)=K_j(t,x,y)t^{\lambda_j(y)}$ $(1\leq j\leq m)$ by using partial differential equations

$$P\big(K_j(t,x,y)t^{\lambda_j(y)}\big)=\frac{C(\lambda_j(y),x)t^{\lambda_j(y)}}{(2\pi i)^n(y_1-x_1)\cdots(y_n-x_n)} \tag{3.7}$$

for $1\leq j\leq m$. First we note the following result in [2].

Lemma 3.3. *If the characteristic exponents of P do not differ by integer, the equation* (3.7) *has a unique holomorphic solution $K_j(t,x,y)$ on*

$$\{(t,x,y)\in\mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^n;\ |t|<\epsilon,\ |x_i|<L,\ |y_i|<L,\ |t|<M|x_i-y_i|^m,\ i=1,\ldots,n\}$$

for some $\epsilon>0,\ L>0$, and $M>0$.

If we admit this lemma, the following proposition is proved immediately.

Proposition 3.4. *Under the situation in Lemma* 3.3, *then our fundamental solutions $E_j(t,x,y)$ $(1\leq j\leq m)$ in Theorem* 1.3 *coincide with the ones in Theorem* 1.1.

References

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