特異一階半線型偏微分方程式に対する形式解の収束及び発散
(Convergence and Divergence of formal solutions for solutions for singular 1st order semi-linear PDE)

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1   Introduction and Main Result.

In this paper we are concerned with formal power series solutions of the following first order semi-linear partial differential equation:

(1.1)
$$P(x,D) \equiv \sum_{i=1}^{d} a_i(x) D_i u(x) = f(x, u(x)), \quad u(0) = 0,$$

$$x = (x_1, \ldots, x_d) \in \mathbf{C}^d, \quad D_i = \frac{\partial}{\partial x_i},$$

where coefficients $a_i(x)$ ($i = 1, \ldots, d$) and $f(x, u)$ are holomorphic in a neighborhood of $x = 0$ and $(x, u) = (0, 0)$, respectively.

Our problems in this paper are the existence, the uniqueness, convergence and divergence of formal power series solutions $u(x) = \sum_{|\alpha| \geq 1} u_\alpha x^\alpha$ ($\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{N}^d$, $\mathbf{N} = \{0, 1, 2, \ldots\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$) centered at the origin for the equation (1.1). If $a_i(0) \neq 0$ for some i, the solvability is well known by Cauchy-Kowalevsky's theorem. Therefore we shall study the case where

(1.2)
$$a_i(0) = 0 \quad \text{for all} \quad i = 1, \ldots, d,$$

which is called a singular or degenerate case. In the following we always assume (1.2).

Furthermore, as a compatibility condition, we always assume the following:

(1.3)
$$f(0, 0) = 0.$$

As we will see later, we can prove the existence and the uniqueness of the formal solution of (1.1) under some condition on the principal part $P(x, D)$. However, this

formal solution $u(x)$ does not necessarily converge. Our main purpose in this paper is to obtain the rate of divergence, which is called the Gevrey order, of the formal solution (cf. Definition 1.1).

Now let us state the main result. Firstly, we state the assumptions.

Let $D_x a(0) := (D_i a_j(0))_{i,j=1,\ldots,d}$ be the Jacobi matrix at the origin of the mapping $a = (a_1, \ldots, a_d)$ and let its Jordan canonical form be

$$\begin{pmatrix} A & & & & \\ & B_1 & & & \\ & & \ddots & & \\ & & & B_k & \\ & & & & O_p \end{pmatrix},$$

where

$$A = \begin{pmatrix} \lambda_1 & \delta_1 & & \\ & \lambda_2 & \ddots & \\ & & \ddots & \delta_{m-1} \\ & & & \lambda_m \end{pmatrix}, \quad B_h = \underbrace{\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}}_{n_h}, \quad \begin{array}{l} \lambda_i \neq 0 \ (i = 1, \ldots, m), \\ \delta_i = 0 \text{ or } 1 \\ (i = 1, \ldots, m-1), \\ h = 1, \ldots, k, \end{array}$$

and O_p is a zero-matrix of order p $(m, k, p \geq 0; n_h \geq 2; m + n_1 + \cdots + n_k + p = d)$.

Let us assume the following condition (Po) according to the value of m ("Po" derives from Poincaré):

$$(\textbf{Po}) \quad \begin{cases} \left| \sum_{i=1}^{m} \lambda_i \alpha_i - f_u(0,0) \right| > \delta|\alpha| \quad \text{for all} \ \alpha \in \mathbf{N}^m \quad (\text{if } m \geq 1), \\ f_u(0,0) \neq 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (\text{if } m = 0), \end{cases}$$

where δ is a positive constant independent of $\alpha \in \mathbf{N}^m$, and $f_u(0,0) = (\partial f / \partial u)(0,0)$.

Before stating the main result, let us give the definition of the Gevrey order, which gives the rate of divergence of formal power series.

Definition 1.1   Let $u(x) = \sum_{\alpha \in \mathbf{N}^d} u_\alpha x^\alpha$ be a formal power series centered at the origin. We say that $u(x)$ *belongs to* $G^{\{s\}}$ $(s = (s_1, \ldots, s_d) \in \mathbf{R}^d)$, if the power series

$$v(\xi) = \sum_{\alpha \in \mathbf{N}^d} u_\alpha \frac{\xi^\alpha}{(\alpha!)^{s-1^{(d)}}}$$

converges in a neighborhood of $\xi = 0$, where $1^{(d)} = (\overbrace{1, \ldots, 1}^{d})$, $s - 1^{(d)} = (s_1 - 1, \ldots, s_d - 1)$ and $(\alpha!)^{s-1^{(d)}} = (\alpha_1!)^{s_1 - 1} \cdots (\alpha_d!)^{s_d - 1}$. Especially, $u(x) \in G^{\{1^{(d)}\}}$ if and only if $u(x)$ is a convergent power series near $x = 0$. ————

The main result in this paper is stated as follows:

Theorem 1.1   *Under the condition* (Po), *the equation* (1.1) *has a unique formal power series solution* $u(x) = \sum_{|\alpha| \geq 1} u_\alpha x^\alpha$. *Furthermore the formal solution* $u(x)$ *belongs to* $G^{\{2N,\dots,2N\}}$, *where*

$$
N = \begin{cases}
\max\{n_1, \dots, n_k\} & (if \ k \geq 1), \\
1 & (if \ k = 0 \ and \ p \geq 1), \\
\dfrac{1}{2} & (if \ k = p = 0).
\end{cases}
$$

Therefore in the case $k = p = 0$ *the formal solution converges, but in other cases it diverges in general.* ———

We will start the proof of Theorem 1.1 from the next section. For simplicity, we consider the two dimensional case, and consider the case $m = 1$ and $k = 0$ in this paper.

In order to prove Theorem 1.1, we shall transform the equation (1.1) in §2. For that transformed equation we can obtain the precise Gevrey order in individual variables of the formal solution (cf. Theorem 2.1). We shall prove the unique existence of the formal solution and its Gevrey order separately. Admitting the unique existence of the formal solution, we will prove its Gevrey order in §4 by using the contraction mapping principle in a Banach space which consists of formal power series. The Banach spaces employed in the proof will be introduced in §3. The unique existence of the formal solution will be proved in §5.

2   Reduction of Equation and Newton Polyhedron.

As mentioned in the previous section, we consider the two dimensional case from this section, and we will prove Theorem 1.1 in the case $m = 1$ and $k = 0$. Firstly let us rewrite the equation (1.1) in the two dimensional case:

$$
a(x,y)D_x u(x,y) + b(x,y)D_y u(x,y) = f(x,y,u(x,y)), \quad u(0,0) = 0,
$$

(2.1)
$$
x, y \in \mathbf{C}, \quad D_x = \frac{\partial}{\partial x}, \quad D_y = \frac{\partial}{\partial y},
$$

where $a(x,y)$, $b(x,y)$ and $f(x,y,u)$ are holomorphic in a neighborhood of the origin such that $a(0,0) = b(0,0) = 0$ and $f(0,0,0) = 0$. Our assumptions imply that

$$
\begin{pmatrix} D_x a(0,0) & D_y a(0,0) \\ D_x b(0,0) & D_y b(0,0) \end{pmatrix} \sim \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(Jordan canonical form)},
$$

where λ is a nonzero eigenvalue satisfying (Po). That is, there exists some positive number $\delta > 0$ such that

(2.2)
$$
|\lambda \alpha - f_u(0,0,0)| > \delta \alpha \quad \text{for all} \quad \alpha = 0, 1, 2, \dots.
$$

Our purpose is to prove the following fact under the above conditions:

(∗) The equation (2.1) has a unique formal solution $u(x,y) = \sum_{\alpha+\beta\geq 1} u_{\alpha\beta}x^\alpha y^\beta$ and the formal solution belongs to $G^{\{2,2\}}$.

In order to do that, let us transform the equation (2.1) by a linear transform of independent variables which reduces the Jacobi matrix to its Jordan canonical form. A reduced equation is written as follows:

$$(2.3) \qquad P_1 u = g_0(x,y) + g(x,y,u(x,y)), \quad u(0,0) = 0,$$

where g_0 and g are holomorphic at the origin with $g_0(0,0) = 0$ and $g(x,y,0) \equiv g_u(x,y,0) \equiv 0$, respectively. Furthermore P_1 is a linear partial differential operator which has the following form:

$$(2.4) \qquad P_1 = \lambda x D_x - f_u(0,0,0) + P_1{}' + P_1{}'' + P_1{}''' + P_1{}'''' + h,$$

where

$$P_1{}' = \left(\sum_{\substack{\alpha+\beta\geq 2 \\ \alpha\geq 1}}^{finite} c_{\alpha\beta}(x,y)x^\alpha y^\beta \right) D_x, \quad P_1{}'' = \left(\sum_{\substack{\alpha+\beta\geq 2 \\ \alpha\geq 1}}^{finite} e_{\alpha\beta}(x,y)x^\alpha y^\beta \right) D_y,$$

$$P_1{}''' = \left(\sum_{\beta\geq 2}^{finite} e_\beta(x,y)y^\beta \right) D_y, \qquad P_1{}'''' = \left(\sum_{\beta\geq 2}^{finite} c_\beta(x,y)y^\beta \right) D_x,$$

$$h = h(x,y) = \sum_{\alpha+\beta\geq 1}^{finite} h_{\alpha\beta}(x,y)x^\alpha y^\beta.$$

Here all coefficients $c_{\alpha\beta}(x,y)$, $e_{\alpha\beta}(x,y)$, $e_\beta(x,y)$, $c_\beta(x,y)$ and $h_{\alpha\beta}(x,y)$ are holomorphic at the origin, and none of them vanish at the origin unless they vanish identically.

Now we shall study the equation (2.3).

In order to give the Gevrey order in an individual variable for formal solutions of the equation (2.4), we study the Newton polyhedron of linear partial differential operators.

Newton Polyhedron. Let

$$P(x,y,D_x,D_y) = \sum_{\alpha,\beta,\alpha',\beta'\geq 0}^{finite} a_{\alpha\beta\alpha'\beta'}(x,y)x^\alpha y^\beta D_x{}^{\alpha'} D_y{}^{\beta'}$$

be a linear partial differential operator, where all coefficients are holomorphic at the origin and do not vanish at the origin unless they vanish identically.

Let us define $Q(\alpha,\beta,\alpha',\beta') \subset \mathbf{R}^3$ by

$$Q(\alpha,\beta,\alpha',\beta') = \{(\mathcal{X},\mathcal{Y},\mathcal{Z}) \in \mathbf{R}^3; \ \mathcal{X} \geq \alpha - \alpha', \ \mathcal{Y} \geq \beta - \beta', \ \mathcal{Z} \leq \alpha' + \beta'\}$$

and let us define the Newton polyhedron $N(P)$ of the operator P by

$$N(P) = \begin{cases} \mathrm{Ch}\left\{ \displaystyle\bigcup_{(\alpha,\beta,\alpha',\beta') \text{ with } a_{\alpha\beta\alpha'\beta'}\neq 0} Q(\alpha,\beta,\alpha',\beta') \right\} & (\text{if } P \neq 0), \\ Q(0,0) & (\text{if } P = 0), \end{cases}$$

where $\mathrm{Ch}\, A$ denotes the convex hull of a set $A \subset \mathbf{R}^3$.

Now we shall apply the above general definition to our first order linear partial differential operator P_1. In order to state the main theorem in this section, we shall define the sets $\widetilde{S}$, $\widetilde{S}'$, $\widetilde{S}''$, S, S', S'', whose elements give the Gevrey orders of formal solutions, as follows: We define $\widetilde{\Pi}(\rho, \sigma)$ and $\Pi(\rho, \sigma)$ $((\rho, \sigma) \in [1, +\infty)^2)$ by

$$(2.5) \qquad \widetilde{\Pi}(\rho, \sigma) = \{(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathbf{R}^3;\ (\rho - 1)\mathcal{X} + (\sigma - 1)\mathcal{Y} - \mathcal{Z} \geq -1\}$$

and

$$(2.6) \qquad \Pi(\rho, \sigma) = \{(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathbf{R}^3;\ (\rho - 1)\mathcal{X} + (\sigma - 1)\mathcal{Y} - \mathcal{Z} \geq 0\},$$

respectively, and define $\widetilde{S}$, $\widetilde{S}'$, $\widetilde{S}''$, S, S', S'' by

$$
\begin{aligned}
\widetilde{S} &= \{(\rho, \sigma) \in [1, +\infty)^2;\ N(P_1') \subset \widetilde{\Pi}(\rho, \sigma)\}, \\
\widetilde{S}' &= \{(\rho, \sigma) \in [1, +\infty)^2;\ N(P_1'') \subset \widetilde{\Pi}(\rho, \sigma)\}, \\
\widetilde{S}'' &= \{(\rho, \sigma) \in [1, +\infty)^2;\ N(P_1'''') \subset \widetilde{\Pi}(\rho, \sigma)\}, \\
S &= \{(\rho, \sigma) \in [1, +\infty)^2;\ N(P_1''') \subset \Pi(\rho, \sigma)\}, \\
S' &= \{(\rho, \sigma) \in [1, +\infty)^2;\ N(P_1'') \subset \Pi(\rho, \sigma)\}, \\
S'' &= \{(\rho, \sigma) \in [1, +\infty)^2;\ N(P_1'''') \subset \Pi(\rho, \sigma)\}.
\end{aligned}
$$

Then we obtain the following theorem.

Theorem 2.1 *Under the contidion* (2.2) *the equation* (2.3) *has a unique formal power series solution. Furthermore the formal solution belongs to* $G^{\{\rho, \sigma\}}$ *if* (ρ, σ) *satisfies the following condition:*

$$
\begin{aligned}
&P_1'''' = 0 \Rightarrow (\rho, \sigma) \in \widetilde{S} \cap S \cap \widetilde{S}', \\
&P_1'' = 0 \Rightarrow (\rho, \sigma) \in \widetilde{S} \cap S \cap \widetilde{S}'', \\
&P_1'',\ P_1'''' \neq 0 \Rightarrow (\rho, \sigma) \in \widetilde{S} \cap S \cap \{(\widetilde{S}' \cap S'') \cup (S' \cap \widetilde{S}'')\}.
\end{aligned}
$$

Remark 2.1 We can easily see that the following s_0 always satisfies the condition in Theorem 2.1:

$s_0 = (3/2, 2)$ (if $P_1'' \neq 0$), $= (1, 2)$ (if $P_1'' = 0$).

Therefore by a linear transform of independent variables again we obtain (∗) from Theorem 2.1 and the next Lemma 2.1. Thus the proof of (∗) is reduced to that of Theorem 2.1. ————

Lemma 2.1 *Let* $u(x, y) = \sum_{(\alpha, \beta) \in \mathbf{N}^2} u_{\alpha\beta} x^\alpha y^\beta \in G^{\{s,s\}}$ $(s \geq 1)$. *Then for any linear transform* $L : \mathbf{C}^2 \to \mathbf{C}^2$, *it holds that* $v(x', y') := u(L(x', y')) \in G^{\{s,s\}}$. ————

We omit the proof of Lemma 2.1 (cf. Hibino[1]).

3 Banach Spaces $G^{\{\rho,\sigma\}}(X,Y)$ and $\widetilde{G}^{\{\rho,\sigma\}}(X,Y)$.

Theorem 2.1 is proved by contraction mapping principle in Banach spaces which consist of formal power series. For this purpose we shall define two types of Banach spaces necessary in the proof, and we shall give some lemmas needed later. We omit the proof.

Definition 3.1 (1) Let $(\rho,\sigma) \in \mathbf{R}_+{}^2$ ($\mathbf{R}_+ = [0,+\infty)$) and $(X,Y) \in (\mathbf{R}_+ \setminus \{0\})^2$. The spaces of formal power series $G^{\{\rho,\sigma\}}(X,Y)$ and $\widetilde{G}^{\{\rho,\sigma\}}(X,Y)$ are defined as follows:
We say that $u(x,y) = \sum_{(\alpha,\beta)\in\mathbf{N}^2} u_{\alpha\beta}x^\alpha y^\beta$ belongs to $G^{\{\rho,\sigma\}}(X,Y)$ if

$$\|u\|_{X,Y}^{\{\rho,\sigma\}} := \sum_{(\alpha,\beta)\in\mathbf{N}^2} |u_{\alpha\beta}|\frac{(\alpha+\beta)!}{(\rho\alpha+\sigma\beta)!}X^\alpha Y^\beta < +\infty.$$

We say that $u(x,y) = \sum_{(\alpha,\beta)\in\mathbf{N}^2} u_{\alpha\beta}x^\alpha y^\beta$ belongs to $\widetilde{G}^{\{\rho,\sigma\}}(X,Y)$ if

$$\||u\||_{X,Y}^{\{\rho,\sigma\}} := \sum_{(\alpha,\beta)\in\mathbf{N}^2} |u_{\alpha\beta}|\frac{\alpha!\beta!}{(\rho\alpha+\sigma\beta)!}X^\alpha Y^\beta < +\infty.$$

Here $k! = \Gamma(k+1)$, $k \geq 0$. Then $G^{\{\rho,\sigma\}}(X,Y)$ and $\widetilde{G}^{\{\rho,\sigma\}}(X,Y)$ are Banach spaces equipped with the norms $\| \cdot \|_{X,Y}^{\{\rho,\sigma\}}$ and $\|| \cdot \||_{X,Y}^{\{\rho,\sigma\}}$, respectively.

(2) We define the subspace $G_0^{\{\rho,\sigma\}}(X,Y)$ (resp. $\widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$) of the Banach space $G^{\{\rho,\sigma\}}(X,Y)$ (resp. $\widetilde{G}^{\{\rho,\sigma\}}(X,Y)$) by

$$G_0^{\{\rho,\sigma\}}(X,Y) := \left\{u(x,y) = \sum_{(\alpha,\beta)\in\mathbf{N}^2} u_{\alpha\beta}x^\alpha y^\beta \in G^{\{\rho,\sigma\}}(X,Y);\ u_{00}(= u(0,0)) = 0\right\}$$

$$\left(\text{resp. } \widetilde{G}_0^{\{\rho,\sigma\}}(X,Y) := \left\{u(x,y) = \sum_{(\alpha,\beta)\in\mathbf{N}^2} u_{\alpha\beta}x^\alpha y^\beta \in \widetilde{G}^{\{\rho,\sigma\}}(X,Y);\ u_{00} = 0\right\}\right).$$

Then $G_0^{\{\rho,\sigma\}}(X,Y)$ (resp. $\widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$) is also a Banach space as a closed linear subspace of $G^{\{\rho,\sigma\}}(X,Y)$ (resp. $\widetilde{G}^{\{\rho,\sigma\}}(X,Y)$). ————

Lemma 3.1 *If $\rho,\ \sigma \geq 1$, then*

$$G^{\{\rho,\sigma\}} = \bigcup_{(X,Y)\in(\mathbf{R}_+\setminus\{0\})^2} G^{\{\rho,\sigma\}}(X,Y) = \bigcup_{(X,Y)\in(\mathbf{R}_+\setminus\{0\})^2} \widetilde{G}^{\{\rho,\sigma\}}(X,Y).$$

Lemma 3.2 *Let us fix $(K,L) \in (\mathbf{R}_+ \setminus \{0\})^2$ and let us assume that $a(x,y) = \sum_{(\alpha,\beta)\in\mathbf{N}^2} a_{\alpha\beta}x^\alpha y^\beta$ are holomorphic on $\{x \in \mathbf{C};\ |x| \leq K\} \times \{y \in \mathbf{C};\ |y| \leq L\}$. If $0 < X \leq K$ and $0 < Y \leq L$, then the multiplication operator $a(x,y)\cdot$ is bounded on $G^{\{\rho,\sigma\}}(X,Y)$, $G_0^{\{\rho,\sigma\}}(X,Y)$, $\widetilde{G}^{\{\rho,\sigma\}}(X,Y)$ and $\widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$ for all $(\rho,\sigma) \in [1,+\infty)^2$ with the norm bounded by $|a|(X,Y)$, where $|a|(X,Y) := \sum_{(\alpha,\beta)\in\mathbf{N}^2} |a_{\alpha\beta}|X^\alpha Y^\beta$. Especially in each space the operator norm is bounded by $|a|(K,L)$.* ————

The following lemma will play a very important role when we deal with nonlinear terms.

Lemma 3.3 (1) *Let* $(\rho, \sigma) \in [1, +\infty)^2$ *and assume that* $u(x, y)$ *and* $v(x, y)$ *belongs to* $G^{\{\rho,\sigma\}}(X, Y)$ *(resp.* $G_0^{\{\rho,\sigma\}}(X, Y)$*). Then* $u(x, y) \cdot v(x, y)$ *also belongs to* $G^{\{\rho,\sigma\}}(X, Y)$ *(resp.* $G_0^{\{\rho,\sigma\}}(X, Y)$*). Furthermore for all* $u(x, y)$ *and* $v(x, y)$ *as the above it holds that*

$$(3.1) \qquad \|u \cdot v\|_{X,Y}^{\{\rho,\sigma\}} \leq \mathsf{M} \|u\|_{X,Y}^{\{\rho,\sigma\}} \cdot \|v\|_{X,Y}^{\{\rho,\sigma\}},$$

where $\mathsf{M} = \max\{\rho, \sigma\}$.

(2) *Let* $(\rho, \sigma) \in [1, +\infty)^2$ *and assume that* $u(x, y)$ *and* $v(x, y)$ *belongs to* $\widetilde{G}^{\{\rho,\sigma\}}(X, Y)$ *(resp.* $\widetilde{G}_0^{\{\rho,\sigma\}}(X, Y)$*). Then* $u(x, y) \cdot v(x, y)$ *also belongs to* $\widetilde{G}^{\{\rho,\sigma\}}(X, Y)$ *(resp.* $\widetilde{G}_0^{\{\rho,\sigma\}}(X, Y)$*). Furthermore for all* $u(x, y)$ *and* $v(x, y)$ *as the above it holds that*

$$(3.2) \qquad \||u \cdot v\||_{X,Y}^{\{\rho,\sigma\}} \leq \mathsf{M} \||u\||_{X,Y}^{\{\rho,\sigma\}} \cdot \||v\||_{X,Y}^{\{\rho,\sigma\}},$$

where M *is same as in* (1). ————

4   Proof of Theorem 2.1.

Let us start the proof of Theorem 2.1. We shall prove the unique existence of the formal solution in §5. So in this section, admitting the unique existence of the formal solution, we will prove its Gevrey order.

We assume that (ρ, σ) satisfies the condition in Theorem 2.1, and prove that the formal solution of the equation (2.3) belongs to $G^{\{\rho,\sigma\}}$.

Proof of Theorem 2.1.   First we define the operator $\Lambda : G^{\{\rho,\sigma\}} \to G^{\{\rho,\sigma\}}$ by

$$\Lambda = \lambda x D_x - f_u(0, 0, 0).$$

The condition (2.2) implies that $\lambda\alpha - f_u(0, 0, 0) \neq 0$ for all $\alpha \in \mathbf{N}$. Hence the operator Λ is bijective and Λ^{-1} is given by

$$\Lambda^{-1} \left(\sum_{(\alpha,\beta)\in\mathbf{N}^2} U_{\alpha\beta} x^\alpha y^\beta \right) = \sum_{(\alpha,\beta)\in\mathbf{N}^2} \frac{U_{\alpha\beta}}{\lambda\alpha - f_u(0, 0, 0)} x^\alpha y^\beta.$$

Now we introduce a new unknown function $U(x, y)$ by

$$U(x, y) = \Lambda u(x, y), \quad \text{that is,} \quad u(x, y) = \Lambda^{-1} U(x, y).$$

Then the equation (2.3) is equivalent to the following one:

$$(4.1) \qquad P_2 U = g_0(x, y) + g(x, y, \Lambda^{-1} U(x, y)), \quad U(0, 0) = 0,$$

where

$$P_2 = I + (P_1{}' + P_1{}'' + P_1{}''' + P_1{}'''' + h)\Lambda^{-1}$$

$$(I : \text{identity mapping}).$$

Let us define the operator T by

(4.2)
$$TU = -(P_1{}' + P_1{}'' + P_1{}''' + P_1{}'''' + h)\Lambda^{-1}U + g_0(x,y) + g(x,y,\Lambda^{-1}U(x,y)),$$

and let us write the ε-closed ball $\widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$ and $G_0^{\{\rho,\sigma\}}(X,Y)$ as

$$\widetilde{G}_0^{\{\rho,\sigma\}}(X,Y;\varepsilon) := \left\{ U(x,y) = \sum_{\alpha+\beta\geq 1} U_{\alpha\beta}x^\alpha y^\beta \in \widetilde{G}_0^{\{\rho,\sigma\}}(X,Y);\ |||U|||_{X,Y}^{\{\rho,\sigma\}} \leq \varepsilon \right\}$$

and

$$G_0^{\{\rho,\sigma\}}(X,Y;\varepsilon) := \left\{ U(x,y) = \sum_{\alpha+\beta\geq 1} U_{\alpha\beta}x^\alpha y^\beta \in G_0^{\{\rho,\sigma\}}(X,Y);\ \|U\|_{X,Y}^{\{\rho,\sigma\}} \leq \varepsilon \right\},$$

respectively.

We shall prove that T is well-defined as a mapping from G to itself by choosing X, Y and ε suitably and that it becomes a contraction mapping there, where

$$G = \begin{cases} \widetilde{G}_0^{\{\rho,\sigma\}}(X,Y;\varepsilon) & \text{(when } P_1{}'''' = 0 \text{ or ``}P_1{}'', P_1{}'''' \neq 0 \text{ and } (\rho,\sigma) \in \widetilde{S}\cap S\cap \widetilde{S}'\cap S''')\text{,} \\ G_0^{\{\rho,\sigma\}}(X,Y;\varepsilon) & \text{(when } P_1{}'' = 0 \text{ or ``}P_1{}'', P_1{}'''' \neq 0 \text{ and } (\rho,\sigma) \in \widetilde{S}\cap S\cap S'\cap \widetilde{S}''')\text{.} \end{cases}$$

Let us estimate the operator norm of $(P_1{}' + P_1{}'' + P_1{}''' + P_1{}'''' + h)\Lambda^{-1}$ on the spaces $\widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$ and $G_0^{\{\rho,\sigma\}}(X,Y)$.

By the condition (2.2) there is some constant C such that $|1/(\lambda\alpha - f_u(0,0,0))| \leq C$ for all $\alpha \in \mathbf{N}$. Hence the operator $\Lambda^{-1} : \widetilde{G}_0^{\{\rho,\sigma\}}(X,Y) \to \widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$ (resp. $G_0^{\{\rho,\sigma\}}(X,Y) \to G^{\{\rho,\sigma\}}(X,Y)$) is bounded and we have

(4.3)
$$|||\Lambda^{-1}U|||_{X,Y}^{\{\rho,\sigma\}} \leq C|||U|||_{X,Y}^{\{\rho,\sigma\}} \quad \left(\text{resp. } \|\Lambda^{-1}U\|_{X,Y}^{\{\rho,\sigma\}} \leq C\|U\|_{X,Y}^{\{\rho,\sigma\}}\right).$$

Therefore it follows from Lemma 3.2 that the operator $h\cdot\Lambda^{-1} : \widetilde{G}_0^{\{\rho,\sigma\}}(X,Y) \to \widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$ (resp. $G_0^{\{\rho,\sigma\}}(X,Y) \to G_0^{\{\rho,\sigma\}}(X,Y)$) is bounded and we have

(4.4)
$$|||h\cdot\Lambda^{-1}U||| \leq A_1(X,Y)|||U|||_{X,Y}^{\{\rho,\sigma\}}$$
$$\left(\text{resp. } \|h\cdot\Lambda^{-1}U\|_{X,Y}^{\{\rho,\sigma\}} \leq A_1(X,Y)\|U\|_{X,Y}^{\{\rho,\sigma\}}\right),$$

where

$$A_1(X,Y) = C_1\left(\sum_{\alpha+\beta\geq 1}^{finite} X^\alpha Y^\beta\right)$$

for some constant C_1. Here and hereafter X and Y are taken so small that the coefficients in the equation (4.1) are holomorphic on $\{x \in \mathbf{C};\ |x| \leq X\} \times \{y \in \mathbf{C};\ |y| \leq Y\}$. In order to estimate the operator norm of $(P_1{}' + P_1{}'' + P_1{}''' + P_1{}'''')\Lambda^{-1}$ we need the following

Lemma 4.1 (1) *Let ρ, σ, μ, ν, μ', ν' satisfy*

(4.5) $$\rho, \sigma \geq 1 \quad and \quad \rho(\mu - \mu') + \sigma(\nu - \nu') \geq \mu + \nu.$$

Then the operator $x^\mu y^\nu D_x^{\mu'} D_y^{\nu'} \Lambda^{-1}$ is bounded both on $\widetilde{G}^{\{\rho,\sigma\}}(X,Y)$ and on $G^{\{\rho,\sigma\}}(X,Y)$, and the operator norm is bounded by $C(X^\mu Y^\nu)/(X^{\mu'} Y^{\nu'})$, where C is the same constant as in (4.3). Futhermore if $\mu + \nu \geq 1$, the operator $x^\mu y^\nu D_x^{\mu'} D_y^{\nu'} \Lambda^{-1}$ is bounded both on $\widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$ and on $G_0^{\{\rho,\sigma\}}(X,Y)$, and the operator norm has the same estimate.

(2) *If $\mu \geq 1$,*

(4.6) $$\rho, \sigma \geq 1 \quad and \quad \rho(\mu - \mu') + \sigma(\nu - \nu') \geq \mu + \nu - 1,$$

then the operator $x^\mu y^\nu D_x^{\mu'} D_y^{\nu'} \Lambda^{-1}$ is bounded both on $\widetilde{G}^{\{\rho,\sigma\}}(X,Y)$ and on $\widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$, and the operator norm is bounded by $C_{\mu\mu'}(X^\mu Y^\nu)/(X^{\mu'} Y^{\nu'})$ for some constant $C_{\mu\mu'}$.

(3) *If $\mu' \geq 1$ and (4.6) hold, then the operator $x^\mu y^\nu D_x^{\mu'} D_y^{\nu'} \Lambda^{-1}$ is bounded on $G^{\{\rho,\sigma\}}(X,Y)$, and the operator norm is bounded by $C_{\mu\nu\mu'\nu'}(X^\mu Y^\nu)/(X^{\mu'} Y^{\nu'})$ for some constant $C_{\mu\nu\mu'\nu'}$. Furthermore if $\mu + \nu \geq 1$, then $x^\mu y^\nu D_x^{\mu'} D_y^{\nu'} \Lambda^{-1}$ is bounded on $G_0^{\{\rho,\sigma\}}(X,Y)$ and the operator norm has the same estimate.* ———

Remark 4.1 Let us write the Newton polyhedron of the operator $x^\mu y^\nu D_x^{\mu'} D_y^{\nu'}$ as

$$N(x^\mu y^\nu D_x^{\mu'} D_y^{\nu'}) = \left\{ (\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in \mathbf{R}^3; \; \mathcal{X} \geq \mu - \mu', \; \mathcal{Y} \geq \nu - \nu', \; \mathcal{Z} \leq \mu + \nu \right\}.$$

Furthermore we define $\widetilde{\Pi}(\rho, \sigma)$ and $\Pi(\rho, \sigma)$ by (2.5) and (2.6), respectively, and define $\widetilde{S}$ and S as follows

$$\begin{aligned}
\widetilde{S} &= \{ (\rho, \sigma) \in [1, +\infty)^2; \; N(x^\mu y^\nu D_x^{\mu'} D_y^{\nu'}) \subset \widetilde{\Pi}(\rho, \sigma) \}, \\
S &= \{ (\rho, \sigma) \in [1, +\infty)^2; \; N(x^\mu y^\nu D_x^{\mu'} D_y^{\nu'}) \subset \Pi(\rho, \sigma) \}.
\end{aligned}$$

Then the condition $(\rho, \sigma) \in \widetilde{S}$ and $(\rho, \sigma) \in S$ are equivalent to (4.6) and (4.5), respectively.

———

We omit the proof of Lemma 4.1 (cf. [1]). We remark that the condition (2.2) plays an important role in the proof.

Proof of Theorem 2.1 (continued). When $P_1'''' = 0$, it follows from the assumption $(\rho, \sigma) \in \widetilde{S} \cap S \cap \widetilde{S}'$, Lemma 3.2, Lemma 4.1, (1) and (2) that the operator $(P_1' + P_1'' + P_1''')\Lambda^{-1} : \widetilde{G}_0^{\{\rho,\sigma\}}(X,Y) \to \widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$ is bounded for sufficiently small X and Y. Moreover we have

(4.7) $$\||(P_1' + P_1'' + P_1''')\Lambda^{-1} U\||_{X,Y}^{\{\rho,\sigma\}} \leq A_2(X,Y)\||U\||_{X,Y}^{\{\rho,\sigma\}},$$

where

$$A_2(X,Y) = C_2 \left\{ \left(\sum_{\substack{\alpha+\beta \geq 2 \\ \alpha \geq 1}}^{finite} X^\alpha Y^\beta \right) \frac{1}{X} + \left(\sum_{\substack{\alpha+\beta \geq 2 \\ \alpha \geq 1}}^{finite} X^\alpha Y^\beta \right) \frac{1}{Y} + \left(\sum_{\beta \geq 2}^{finite} Y^\beta \right) \frac{1}{Y} \right\}$$

for some constant C_2.

When $P_1'' = 0$, it follows from the assumption $(\rho, \sigma) \in \widetilde{S} \cap S \cap \widetilde{S}''$, Lemma 3.2, Lemma 4.1, (1) and (3) that the operator $(P_1' + P_1''' + P_1'''')\Lambda^{-1} : G_0^{\{\rho,\sigma\}}(X, Y) \to G_0^{\{\rho,\sigma\}}(X, Y)$ is bounded. Moreover we have

$$(4.8) \qquad \|(P_1' + P_1''' + P_1'''')\Lambda^{-1}U\|_{X,Y}^{\{\rho,\sigma\}} \leq A_3(X,Y)\|U\|_{X,Y}^{\{\rho,\sigma\}},$$

where

$$A_3(X, Y) = C_3\left\{ \left(\sum_{\substack{\alpha+\beta\geq 2 \\ \alpha\geq 1}}^{finite} X^\alpha Y^\beta \right)\frac{1}{X} + \left(\sum_{\beta\geq 2}^{finite} Y^\beta \right)\frac{1}{Y} + \left(\sum_{\beta\geq 2}^{finite} Y^\beta \right)\frac{1}{X} \right\}$$

for some constant C_3.

When P_1'', $P_1'''' \neq 0$ and $(\rho, \sigma) \in \widetilde{S} \cap S \cap \widetilde{S}' \cap S'''$, it follows from Lemma 3.2, Lemma 4.1, (1) and (2) that the operator $(P_1' + P_1'' + P_1''' + P_1'''')\Lambda^{-1} : \widetilde{G}_0^{\{\rho,\sigma\}}(X, Y) \to \widetilde{G}_0^{\{\rho,\sigma\}}(X, Y)$ is bounded. Moreover we have

$$(4.9) \qquad |||(P_1' + P_1'' + P_1''' + P_1'''')\Lambda^{-1}U|||_{X,Y}^{\{\rho,\sigma\}} \leq A_4(X,Y)|||U|||_{X,Y}^{\{\rho,\sigma\}},$$

where

$$A_4(X, Y) = C_4\left\{ \left(\sum_{\substack{\alpha+\beta\geq 2 \\ \alpha\geq 1}}^{finite} X^\alpha Y^\beta \right)\frac{1}{X} + \left(\sum_{\substack{\alpha+\beta\geq 2 \\ \alpha\geq 1}}^{finite} X^\alpha Y^\beta \right)\frac{1}{Y} + \left(\sum_{\beta\geq 2}^{finite} Y^\beta \right)\frac{1}{Y} + \left(\sum_{\beta\geq 2}^{finite} Y^\beta \right)\frac{1}{X} \right\}$$

for some constant C_4. When P_1'', $P_1'''' \neq 0$ and $(\rho, \sigma) \in \widetilde{S} \cap S \cap S' \cap \widetilde{S}''$, it follows from Lemma 3.2, Lemma 4.1, (1) and (3) that the operator $(P_1' + P_1'' + P_1''' + P_1'''')\Lambda^{-1} : G_0^{\{\rho,\sigma\}}(X, Y) \to G_0^{\{\rho,\sigma\}}(X, Y)$ is bounded. Moreover we have

$$(4.10) \qquad \|(P_1' + P_1'' + P_1''' + P_1'''')\Lambda^{-1}U\|_{X,Y}^{\{\rho,\sigma\}} \leq A_4(X,Y)\|U\|_{X,Y}^{\{\rho,\sigma\}}.$$

Next let us estimate nonlinear terms. Let

$$g(x, y, u) = \sum_{\alpha+\beta\geq 0, r\geq 2} g_{\alpha\beta} x^\alpha y^\beta u^r$$

be the Taylor expansion of $g(x, y, u)$ (recall that $g(x, y, 0) \equiv g_u(x, y, 0) \equiv 0$). Furthermore let us define the formal power series $|g|(x, y, u)$ by

$$|g|(x, y, u) = \sum_{\alpha+\beta\geq 0, r\geq 2} |g_{\alpha\beta}| x^\alpha y^\beta u^r.$$

We may assume that $|g|(x, y, u)$ converges in $\{x \in \mathbf{C}; |x| \leq K\} \times \{y \in \mathbf{C}; |y| \leq L\} \times \{u \in \mathbf{C}; |u| \leq M\}$ for some positive constants K, L and M.

We remark the following. It holds that

$$g_u(x, y, u) = \sum_{\alpha+\beta\geq 0, r\geq 1} (r + 1)g_{\alpha\beta,r+1} x^\alpha y^\beta u^r,$$

and that

$$|g_u|(x,y,u) := \sum_{\alpha+\beta\geq 0, r\geq 1} (r+1)|g_{\alpha\beta,r+1}|x^\alpha y^\beta u^r$$

converges in $\{x \in \mathbf{C}; \ |x| \leq K\} \times \{y \in \mathbf{C}; \ |y| \leq L\} \times \{z \in \mathbf{C}; \ |z| \leq M\}$.

Now it follows from (4.3) and Lemma 3.3, (1) that if $X \leq K$, $Y \leq L$, $U \in G_0^{\{\rho,\sigma\}}(X,Y)$ and $\|U\|_{X,Y}^{\{\rho,\sigma\}} \leq M/MC$, where $\mathsf{M} = \max\{\rho,\sigma\}$, then $g(x,y,\Lambda^{-1}U(x,y))$ belongs to $G_0^{\{\rho,\sigma\}}(X,Y)$. Moreover it holds that

$$(4.11) \qquad \|g(x,y,\Lambda^{-1}U(x,y))\|_{X,Y}^{\{\rho,\sigma\}} \leq \frac{1}{\mathsf{M}}|g|\left(X,Y,\mathsf{MC}\|U\|_{X,Y}^{\{\rho,\sigma\}}\right)$$

$$\leq \frac{1}{\mathsf{M}}|g|\left(K,L,\mathsf{MC}\|U\|_{X,Y}^{\{\rho,\sigma\}}\right) < +\infty.$$

Next by noting

$$g(x,y,u) - g(x,y,v) = (u-v)\int_0^1 g_u(x,y,\theta u + (1-\theta)v)d\theta,$$

we see that if $X \leq K$, $Y \leq L$, $\|U\|_{X,Y}^{\{\rho,\sigma\}}$, $\|V\|_{X,Y}^{\{\rho,\sigma\}} \leq M/2MC$, then we have

$$(4.12) \qquad \|g(x,y,\Lambda^{-1}U(x,y)) - g(x,y,\Lambda^{-1}V(x,y))\|_{X,Y}^{\{\rho,\sigma\}}$$

$$\leq \|U-V\|_{X,Y}^{\{\rho,\sigma\}} \times C|g_u|\left(X,Y,\mathsf{MC}(\|U\|_{X,Y}^{\{\rho,\sigma\}} + \|V\|_{X,Y}^{\{\rho,\sigma\}})\right)$$

$$\leq \|U-V\|_{X,Y}^{\{\rho,\sigma\}} \times C|g_u|\left(K,L,\mathsf{MC}(\|U\|_{X,Y}^{\{\rho,\sigma\}} + \|V\|_{X,Y}^{\{\rho,\sigma\}})\right).$$

Similarly it follows from (4.3) and Lemma 3.3, (2) that if $U \in \widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$ and $\||U\||_{X,Y}^{\{\rho,\sigma\}} \leq M/MC$, where X, Y and M are same as above, then we have $g(x,y,\Lambda^{-1}U(x,y)) \in \widetilde{G}_0^{\{\rho,\sigma\}}(X,Y)$, and that

$$(4.13) \qquad \||g(x,y,\Lambda^{-1}U(x,y))\||_{X,Y}^{\{\rho,\sigma\}} \leq \frac{1}{\mathsf{M}}|g|\left(X,Y,\mathsf{MC}\||U\||_{X,Y}^{\{\rho,\sigma\}}\right)$$

$$\leq \frac{1}{\mathsf{M}}|g|\left(K,L,\mathsf{MC}\||U\||_{X,Y}^{\{\rho,\sigma\}}\right) < +\infty.$$

Moreover if $\||U\||_{X,Y}^{\{\rho,\sigma\}}$, $\||V\||_{X,Y}^{\{\rho,\sigma\}} \leq M/2MC$, we have

$$(4.14) \qquad \||g(x,y,\Lambda^{-1}U(x,y)) - g(x,y,\Lambda^{-1}V(x,y))\||_{X,Y}^{\{\rho,\sigma\}}$$

$$\leq \||U-V\||_{X,Y}^{\{\rho,\sigma\}} \times C|g_u|\left(X,Y,\mathsf{MC}(\||U\||_{X,Y}^{\{\rho,\sigma\}} + \||V\||_{X,Y}^{\{\rho,\sigma\}})\right)$$

$$\leq \||U-V\||_{X,Y}^{\{\rho,\sigma\}} \times C|g_u|\left(K,L,\mathsf{MC}(\||U\||_{X,Y}^{\{\rho,\sigma\}} + \||V\||_{X,Y}^{\{\rho,\sigma\}})\right).$$

Under the above preparations let us take X, Y and $\varepsilon > 0$ as follows: We take $\varepsilon > 0$ such that

$$(4.15) \qquad \frac{1}{\mathsf{M}}|g|(K,L,\mathsf{MC}\varepsilon) < \varepsilon$$

$$(4.16) \qquad |g_u|(K, L, 2MC\varepsilon) < 1.$$

Since $|g|(x, y, u) = O(u^2)$ and $|g_u|(x, y, u) = O(u)$, we can take such $\varepsilon > 0$. Furthermore for this ε let us take X and Y such that the followings hold:

In the case $P_1'''' = 0$:

$$(4.17) \qquad \{A_1(X, Y) + A_2(X, Y)\}\varepsilon + |||g_0|||_{X,Y}^{\{\rho, \sigma\}} + \frac{1}{M}|g|(K, L, MC\varepsilon) \leq \varepsilon$$

and

$$(4.18) \qquad A_1(X, Y) + A_2(X, Y) + C|g_u|(K, L, 2MC\varepsilon) < 1.$$

In the case $P_1'' = 0$:

$$(4.19) \qquad \{A_1(X, Y) + A_3(X, Y)\}\varepsilon + \|g_0\|_{X,Y}^{\{\rho, \sigma\}} + \frac{1}{M}|g|(K, L, MC\varepsilon) \leq \varepsilon$$

and

$$(4.20) \qquad A_1(X, Y) + A_3(X, Y) + C|g_u|(K, L, 2MC\varepsilon) < 1.$$

In the case P_1'', $P_1'''' \neq 0$ and $(\rho, \sigma) \in \widetilde{S} \cap S \cap \widetilde{S}' \cap S''$:

$$(4.21) \qquad \{A_1(X, Y) + A_4(X, Y)\}\varepsilon + |||g_0|||_{X,Y}^{\{\rho, \sigma\}} + \frac{1}{M}|g|(K, L, MC\varepsilon) \leq \varepsilon$$

and

$$(4.22) \qquad A_1(X, Y) + A_4(X, Y) + C|g_u|(K, L, 2MC\varepsilon) < 1.$$

In the case P_1'', $P_1'''' \neq 0$ and $(\rho, \sigma) \in \widetilde{S} \cap S \cap S' \cap \widetilde{S}''$:

$$(4.23) \qquad \{A_1(X, Y) + A_4(X, Y)\}\varepsilon + \|g_0\|_{X,Y}^{\{\rho, \sigma\}} + \frac{1}{M}|g|(K, L, MC\varepsilon) \leq \varepsilon$$

and (4.22).

We can take such X and Y by the fact $g_0(0, 0) = 0$ and the expressions of $A_1(X, Y)$, $A_2(X, Y)$, $A_3(X, Y)$ and $A_4(X, Y)$.

In the case $P_1'''' = 0$ we see that if $U \in \widetilde{G}_0^{\{\rho, \sigma\}}(X, Y)$ and $|||U|||_{X,Y}^{\{\rho, \sigma\}} \leq \varepsilon$, then $TU \in \widetilde{G}_0^{\{\rho, \sigma\}}(X, Y)$ and $|||TU|||_{X,Y}^{\{\rho, \sigma\}} \leq \varepsilon$ by (4.4), (4.7), (4.13) and (4.17). Hence T is well-defined as a mapping from $\widetilde{G}_0^{\{\rho, \sigma\}}(X, Y; \varepsilon)$ to itself. Moreover by (4.4), (4.7), (4.14) and (4.18), we see that $T : \widetilde{G}_0^{\{\rho, \sigma\}}(X, Y; \varepsilon) \to \widetilde{G}_0^{\{\rho, \sigma\}}(X, Y; \varepsilon)$ is a contraction mapping. Similarly in other cases we can prove that $T : G \to G$ is well-defined and that it is a contraction mapping.

Therefore there exists a unique $U(x, y) \in G$ which satisfies $TU(x, y) = U(x, y)$. Lemma 3.1 implies $U(x, y) \in G^{\{\rho, \sigma\}}$. Hence $u(x, y) = \Lambda^{-1}U(x, y)$ also belongs to $G^{\{\rho, \sigma\}}$, and it is easy to see that this $u(x, y)$ is a solution of (2.3). Since we admit the unique existence of the formal solution, the proof is completed. ∎

5 Unique Existence of Formal Solution.

Here we shall prove the unique existence of the formal solution for the equation (2.3).

Let us define the vector space $H(x, y; l)$ which consists of homogeneous polynomials of degree l as follows:

$$H(x, y; l) = \text{(the vector space spanned by } \{x^\alpha y^\beta;\ (\alpha, \beta) \in \mathbf{N}^2,\ \alpha + \beta = l\}).$$

By $\Lambda(x^\alpha y^\beta) = \{\lambda\alpha - f_u(0, 0, 0)\}x^\alpha y^\beta$ and the condition (2.2), the following lemma is obvious.

Lemma 5.1 *For all $l \geq 0$ the linear operator*

$$\Lambda : H(x, y; l) \to H(x, y; l)$$

is bijective. ————

Now in order to solve the equation (2.3) we set

$$u(x, y) = \sum_{l=1}^{\infty} u_l(x, y), \quad g_0(x, y) = \sum_{l=1}^{\infty} g_{0l}(x, y),$$

where $u_l(x, y)$, $g_{0l}(x, y) \in H(x, y; l)$. Then we have the following recursion formula for $\{u_l(x, y)\}_{l=1}^{\infty}$:

$$\Lambda u_1(x, y) = g_{01}(x, y),$$
$$\Lambda u_2(x, y) = g_{02}(x, y)$$
$$+ \text{(homogeneous part of degree 2 of } Q_1 u_1(x, y) + g(x, y, u_1(x, y))),$$
$$\Lambda u_3(x, y) = g_{03}(x, y) + \text{(homogeneous part of degree 3 of}$$
$$Q_1(u_1(x, y) + u_2(x, y)) + g(x, y, u_1(x, y) + u_2(x, y))),$$

$$\cdots$$

$$\Lambda u_l(x, y) = g_{0l}(x, y)$$
$$+ \text{(homogeneous part of degree } l \text{ of}$$
$$Q_1(u_1(x, y) + \cdots + u_{l-1}(x, y)) + g(x, y, u_1(x, y) + \cdots + u_{l-1}(x, y))),$$

$$\cdots \ ,$$

where $Q_1 = \Lambda - P_1$.

Therefore by Lemma 5.1 we can obtain $\{u_l(x, y)\}_{l=1}^{\infty}$ inductively and uniquely. This completes the proof of the unique solvability for the equation (2.3).

References

[1] Hibino, M., Divergence property of formal solutions for singular first order linear partial differential equations. *Publ. Res. Inst. Math. Sci.* **35** (1999), no. 6, 893–919.