数理解析研究所講究録 1336 巻 2003 年 86-98

Structure of formal solutions of nonlinear First Order Singular Partial Differential Equations in Complex Domain

名古屋大学多元数理科学研究科 三宅 正武 (Masatake Miyake)
名古屋大学多元数理科学研究科 白井 朗 (Akira Shirai)

Graduate School of Mathematics, Nagoya University

This note is based on a preprint [MS2]. The proofs of theorem, propositions and lemmas in the below will be omitted or shortened, since we are not permitted enough space to write down the complete proofs. The complete proofs will be found in the web site "*Preprint Series In Mathematical Sciences*" No.2002-4, whose address is

`http://www.math.human.nagoya-u.ac.jp/preprint.html`

1 Introduction

Let $\mathcal{O}_x$ be the ring of germs of holomorphic functions in a neighborhood of origin of $\mathbf{C}_x^n$ and let $\mathcal{M}_x[[t]]$ be a maximal ideal of formal power series with holomorphic coefficients, that is,

$$u(t,x) \in \mathcal{M}_x[[t]] \Longleftrightarrow u(t,x) = \sum\nolimits_{|\alpha|\geq 1} u_\alpha(x) t^\alpha, \quad u_\alpha(x) \in \mathcal{O}_x, \tag{1.1}$$

where $(t,x) = (t_1,\cdots,t_d,x_1,\cdots,x_n) \in \mathbf{C}_t^d \times \mathbf{C}_x^n$ $(d \geq 1, n \geq 0)$, $\alpha \in \mathbf{N}^d$ $(\mathbf{N} = \{0,1,2,\ldots\})$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

We shall study the formal solutions $u(t,x) \in \mathcal{M}_x[[t]]$ of the following nonlinear first order partial differential equation:

$$f(t,x,u,\partial_t u,\partial_x u) = 0 \quad \text{with} \quad u(0,x) \equiv 0, \tag{1.2}$$

where $\partial_t u = (\partial_{t_1} u,\cdots,\partial_{t_d} u)$ and $\partial_x u = (\partial_{x_1} u,\cdots,\partial_{x_n} u)$.

Throughout this paper, we assume the following three assumptions:

[A1] The function $f(t,x,u,\tau,\xi)$ $(\tau = (\tau_j) \in \mathbf{C}^d,\ \xi = (\xi_k) \in \mathbf{C}^n)$ is holomorphic in a neighborhood of the origin. Moreover, $f(t,x,u,\tau,\xi)$ is an entire function in τ variables for any fixed t, x, u and ξ in the definite domain.

[A2] The equation (1.2) is *singular* in t variables in the sense that

$$f(0,x,0,\tau,0) \equiv 0 \ \text{ and } \ \frac{\partial f}{\partial \xi_k}(0,x,0,\tau,0) \equiv 0, \quad (k = 1,2,\ldots,n). \tag{1.3}$$

[A3] The equation (1.2) has a formal solution $u(t,x) \in \mathcal{M}_x[[t]]$.

Our purpose in this paper is to characterize the convergence or the divergence of such a formal solution. In order to state our results we need to prepare some notations.

Let $\varphi_j(x) = \partial_{t_j} u(0,x) \in \mathcal{O}_x$ $(j = 1, \cdots, d)$ and put $\varphi(x) = (\varphi_j(x))$. We differentiate the equation (1.2) by t_i $(i = 1, 2, \cdots, d)$, then we get the following equations for $\{\varphi_i(x)\}$ from the second assumptions in (1.3) of [A2];

$$(1.4)\qquad \begin{aligned} &\frac{\partial}{\partial t_i} f(t, x, u(t,x), \{\partial_{t_j} u(t,x)\}, \{\partial_{x_k} u(t,x)\})\Big|_{t=0} \\ &\equiv \frac{\partial f}{\partial t_i}(0, x, 0, \varphi(x), 0) + \frac{\partial f}{\partial u}(0, x, 0, \varphi(x), 0)\varphi_i(x) = 0 \end{aligned}$$

for $i = 1, 2, \ldots, d$. We take and fix such a solution $\varphi(x)$.

We set $\mathbf{a}(x) = (0, x, 0, \varphi(x), 0)$ for the simplicity of notation. Now we define holomorphic functions $a_{ij}(x)$ $(i, j = 1, 2, \ldots, d)$ by

$$(1.5)\qquad a_{ij}(x) = \frac{\partial^2 f}{\partial t_i \partial \tau_j}(\mathbf{a}(x)) + \frac{\partial^2 f}{\partial u \partial \tau_j}(\mathbf{a}(x))\varphi_i(x),$$

and put $A(x) = (a_{ij}(x))_{i,j=1}^d$. Then our main result is stated as follows:

Theorem 1.1 *Under the assumptions* [A1], [A2] *and* [A3], *we have:*

(i) (Convergent Case) *Let* $\{\lambda_j\}_{j=1}^d$ *be the eigenvalues of the matrix* $A(0)$. *Then if* $\{\lambda_j\}_{j=1}^d$ *satisfies the condition below which we call the Poincaré condition, the formal solution* $u(t,x) \in \mathcal{M}_x[[t]]$ *is convergent in a neighborhood of the origin:*

$$(1.6)\qquad \mathrm{Ch}(\lambda_1, \ldots, \lambda_d) \not\ni 0 \qquad \text{(Poincaré condition)},$$

where $\mathrm{Ch}(\lambda_1, \ldots, \lambda_d)$ *denotes the convex hull of* $\{\lambda_1, \ldots, \lambda_d\}$.

(ii) (Divergent Case) *Suppose that* $A(x)$ *is a nilpotent matrix, and take an integer* N *with* $1 \le N \le d$ *such that* $A^N(x) \equiv \mathbf{O}$, *but* $A^j(x) \not\equiv \mathbf{O}$ *for* $j = 0, \ldots, N-1$, *where* $\mathbf{O}$ *denotes the null matrix. Then if* $f_u(\mathbf{a}(0)) \neq 0$, *the formal solution* $u(t,x) \in \mathcal{M}_x[[t]]$ *diverges in general, and it belongs to the Gevrey class of order at most* $2N$ *in* t *variables, which means that the formal* $2N$*-Borel transform of* $u(t,x)$, $\sum_{|\alpha|\ge 1} u_\alpha(x) t^\alpha / |\alpha|!^{2N-1}$ *is convergent in a neighborhood of the origin.*

The theorem will be proved by reducing the equation (1.2) to an equation which is similar but more general than that studied by Gérard and Tahara [GT] and many others as we shall show below.

We put $v(t,x) = u(t,x) - \sum_{j=1}^d \varphi_j(x) t_j$ $(= O(|t|^2))$. Then by an easy calculation, we can see that $v(t,x)$ satisfies the following nonlinear singular partial differential equation:

$$(1.7)\qquad \left(\sum_{i,j=1}^d a_{ij}(x) t_i \partial_{t_j} + \frac{\partial f}{\partial u}(\mathbf{a}(x))\right) v(t,x) = \sum_{|\alpha|=2} b_\alpha(x) t^\alpha + f_3(t, x, v, \partial_t v, \partial_x v),$$

where $b_\alpha(x) \in \mathcal{O}_x$ and $f_3(t, x, v, \tau, \xi)$ is holomorphic in a neighborhood of the origin with Taylor expansion

$$(1.8) \qquad f_3(t, x, v, \tau, \xi) = \sum_{|\alpha|+2p+|q|+2|r|\geq 3} f_{\alpha pqr}(x) t^\alpha v^p \tau^q \xi^r \in \mathcal{O}_x\{t, v, \tau, \xi\},$$

where $\alpha \in \mathbf{N}^d$, $p \in \mathbf{N}$, $q \in \mathbf{N}^d$, $r \in \mathbf{N}^n$ and $\mathcal{O}_x\{X\}$ denotes the set of convergent series in all variables x and X.

Remark 1.2 (About the assumption [A1]) The assumption that $f(t, x, u, \tau, \xi)$ is an entire function in τ variable is only for the convenience. Once we fix $\varphi(x) = (\varphi_j(x)) \in \mathcal{O}_x{}^d$ which satisfy the equations (1.4), it is sufficient to assume that f is holomorphic in a neighborhood of $(0, 0, 0, \varphi(0), 0)$.

Remark 1.3 (Nonresonance condition) If $f_u(\mathbf{a}(0))$ satisfies the nonresonance condition, that is,

$$(1.9) \qquad \lambda \cdot \alpha + f_u(\mathbf{a}(0)) \neq 0, \quad \text{for all } |\alpha| \geq 2,$$

($\lambda \cdot \alpha = \sum_{j=1}^d \lambda_j \alpha_j$), then the theorem does hold for the formal solution $u(t, x) \in \mathbf{C}[[t, x]]$ if we assume the existence of $\varphi(x) = (\varphi_j(x)) \in \mathcal{O}_x{}^d$.

Remark 1.4 (Singular equation) Our definition [A2] or (1.3) on the singular equation corresponds to the one considered by T. Oshima [O] for linear partial differential equations. Especially, our assumption that $f_{\xi_k}(0, x, 0, \tau, 0) \equiv 0$ $(k = 1, 2, \ldots, n)$ assures that in the reduced equation (1.7) the vector field on the left hand side depends only on ∂_{t_j} $(j = 1, 2 \cdots, d)$. Instead of this assumption, if we assume $f_{\xi_k}(0, 0, 0, \tau, 0) \equiv 0$ $(k = 1, 2 \cdots, n)$, then we get a sigluar equation of another kind that in the reduced equation the terms $b_k(x)\partial_{x_k}$ with $b_k(0) = 0$ $(k = 1, 2, \cdots, n)$ appear in the vector field. For such equations, similar problems have been studied in a series of papers [CT], [CL] and [CLT] by Chen, Luo and Tahara where the reduced type equations were studied under more restricted conditions than ours which they called the singular equations of totally characteristic type. The generalization of their results has been studied by A. Shirai. The convergent result has been obtained in [S2] under the generalized Poincaré condition, and the Maillet type theorem has been studied in a preparing paper [S3].

2 Preparations to Prove Theorem 1.1.

In this section, we shall prepare some notations, definitions and lemmas, which will be used in the proof of Theorem 1.1.

- $D_{z_0}(R) = \{x = (x_1, \ldots, x_n) \in \mathbf{C}^n \ ; \ |x_j - z_0| \leq R, \ j = 1, 2, \ldots, d, \ z_0 \in \mathbf{C}\}$.
- $\mathcal{O}_{z_0}(R)$: the set of holomorphic functions on $x \in D_{z_0}(R)$.
- $\mathbf{C}[t]_L = \{u_L(t) = \sum_{|\alpha|=L} u_\alpha t^\alpha \ ; \ u_\alpha \in \mathbf{C}\}$. (Homogeneous polynomials of order L)

- $\mathcal{O}_{z_0}(R)[t]_L = \{u_L(t,x) = \sum_{|\alpha|=L} u_\alpha(x)t^\alpha \ ; \ u_\alpha(x) \in \mathcal{O}_{z_0}(R)\}$.

Definition 2.1 (s-Borel transform and Gevrey space $\mathcal{G}^s$) Let $\mathbf{R}_{\geq 1} = \{x \in \mathbf{R} \ ; \ x \geq 1\}$. For d dimensional real vector $\mathbf{s} = (s_1, s_2, \ldots, s_d) \in (\mathbf{R}_{\geq 1})^d$ and a formal power series $f(t,x) = \sum_{\alpha \in \mathbf{N}^d} f_\alpha(x)t^\alpha \in \mathcal{O}_x[[t]]$, we define the s-Borel transform $\mathcal{B}^{\mathbf{s}}(f)(t,x)$ of $f(t,x)$ by

$$\mathcal{B}^{\mathbf{s}}(f)(t,x) := \sum_{\alpha \in \mathbf{N}^d} f_\alpha(x) \frac{|\alpha|!}{(\mathbf{s}\cdot\alpha)!} t^\alpha, \tag{2.1}$$

where $\mathbf{s}\cdot\alpha = \sum_{j=1}^d s_j\alpha_j$ and $(\mathbf{s}\cdot\alpha)! = \Gamma(\mathbf{s}\cdot\alpha + 1)$ by the Gamma function.

We say that $f(t,x) \in \mathcal{G}^{\mathbf{s}}$ if $\mathcal{B}^{\mathbf{s}}(f)(t,x) \in \mathbf{C}\{t,x\}$, and s is called the Gevrey order in t variables.

We introduce the s-norm of $u_L(t) = \sum_{|\alpha|=L} u_\alpha t^\alpha \in \mathbf{C}[t]_L$ by

$$\begin{aligned} ||u_L||_{\mathbf{s}} &:= \inf\{C > 0 \ ; \ \mathcal{B}^{\mathbf{s}}(u_L)(t) \ll C(t_1 + \cdots + t_d)^L\} \\ &= \max_{|\alpha|=L} \left\{ |u_\alpha| \frac{\alpha!}{(\mathbf{s}\cdot\alpha)!} \right\}, \qquad (\alpha! = \alpha_1! \cdots \alpha_d!). \end{aligned} \tag{2.2}$$

Lemma 2.2 *Let $f(t,x) = \sum_{\alpha \in \mathbf{N}^d} f_\alpha(x)t^\alpha \in \mathcal{O}_0(R)[[t]]$ and assume $\mathbf{s} = (s, \cdots, s) \in (\mathbf{R}_{\geq 1})^d$. For a regular matrix $Q(x) = (Q_{ij}(x)) \in GL(d, \mathcal{O}_0(R))$, the function $g(\tau, x) := f(\tau Q(x), x)$ belongs to $\mathcal{G}^{\mathbf{s}}$ in τ variables if and only if $f(t,x)$ belongs to $\mathcal{G}^{\mathbf{s}}$ in t variables.*

3 Proof of Theorem 1.1, (i).

We put $v(t,x) = u(t,x) - \sum_{j=1}^d \varphi_j(x)t_j \in \mathcal{M}_x[[t]]$ which satisfies $v(t,x) = O(|t|^2)$. Then, as stated in Introduction, it is easily examined that $v(t,x)$ satisfies the following singular equation:

$$\left(\sum_{i,j=1}^d a_{ij}(x) t_i \partial_{t_j} + c(x) \right) v(t,x) = \sum_{|\alpha|=2} b_\alpha(x)t^\alpha + f_3(t,x,v,\partial_t v, \partial_x v), \tag{3.1}$$

with $a_{ij}(x), c(x), b_\alpha(x) \in \mathcal{O}_x$. Here we remark that $(a_{ij}(0))_{i,j=1}^d$ is a regular matrix with eigenvalues $\{\lambda_j\}_{j=1}^d$ which satisfy the Poincaré condition (1.6), $c(x) = f_u(\mathbf{a}(x))$ and $f_3(t,x,v,\tau,\xi)$ is holomorphic in a neighborhood of the origin with the same Taylor expansion with (1.8)

By the Poincaré condition (1.6), there exists a positive integer $K \geq 2$ such that

$$\left| \sum_{j=1}^d \lambda_j\alpha_j + c(0) \right| \geq C_0 |\alpha|, \quad |\alpha| \geq K \tag{3.2}$$

holds by some positive constant $C_0 > 0$. We take and fix such K.

Once again we set $w(t,x) = v(t,x) - \sum_{|\alpha|=2}^{K-1} u_\alpha(x)t^\alpha$ $(= O(|t|^K))$ as a new unknown function. Then $w(t,x)$ satisfies a singular equation of the following form:

$$\left(\sum_{i,j=1}^{d} a_{ij}(x)t_i\partial_{t_j} + c(x)\right) w = \sum_{|\alpha|=K} d_\alpha(x)t^\alpha + f_{K+1}(t,x,w,\partial_t w,\partial_x w), \tag{3.3}$$

where $d_\alpha(x) \in \mathcal{O}_x$ and $f_{K+1}(t,x,u,\tau,\xi)$ is holomorphic in a neighborhood of the origin with Taylor expansion

$$f_{K+1}(t,x,u,\tau,\xi) = \sum_{|\alpha|+Kp+(K-1)|q|+K|r|\geq K+1} f_{\alpha pqr}(x)t^\alpha u^p \tau^q \xi^r. \tag{3.4}$$

Therefore, the proof of Theorem 1.1, (i) is reduced to prove the following Theorem:

Theorem 3.1 *Under the condition* (3.2), *the equation* (3.3) *with* $w(t,x) = O(|t|^K)$ *has a unique formal solution which converges in a neighborhood of the origin.*

4 Outline of the Proof of Theorem 3.1

By a linear change of t variables which brings $(a_{ij}(0))$ to its Jordan canonical form, the equation (3.3) is reduced to the following one:

$$(\Lambda + \Delta + A)w(t,x) = \sum_{|\alpha|=K} \zeta_\alpha(x)t^\alpha + g_{K+1}(t,x,w,\partial_t w,\partial_x w), \tag{4.1}$$

with $w(t,x) = O(|t|^K)$, where

$$\Lambda = \sum_{j=1}^{d} \lambda_j t_j \partial_{t_j} + c(0), \qquad \Delta = \sum_{j=1}^{d-1} \delta_j t_{j+1}\partial_{t_j}, \tag{4.2}$$

$$A \equiv A(x) = \sum_{i,j=1}^{d} \alpha_{ij}(x)t_i\partial_{t_j} + b(x), \quad (\alpha_{ij}(0) = 0, b(0) = 0),$$

and g_{K+1} is holomorphic in a neighborhood of the origin with the same Taylor expansion with f_{K+1}.

Remark 4.1 In the part Δ, it is normally considered that $\delta_j = 0$ or 1. However, we can take $\{\delta_j\}$ are as small as we want. Indeed, if we take a change of variables by $\widehat{t_j} = \varepsilon^j t_j$, then δ_j is replaced by $\varepsilon\delta_j$.

For the proof our theorem, the following proposition plays an essential role to employ the majorant method.

Proposition 4.2 *Let us consider the linear operator* $P = \Lambda + \Delta + A$.

(i) *For all* $L \geq K$, *the mapping* $P : \mathcal{O}_0(R)[t]_L \longrightarrow \mathcal{O}_0(R)[t]_L$ *is invertible for sufficiently small* $R > 0$.

(ii) *For* $u(t,x) \in \mathcal{O}_0(R)[t]_L$, *if a majorant relation* $u(t,x) \ll W(x)(t_1 + \cdots + t_d)^L$ *does hold by a function* $W(x)$ *with non negative Taylor coefficients, then there exists a positive constant* $F > 0$ *independent of* L *such that*

$$
\begin{aligned}
(4.3) \qquad P^{-1}u(t,x) &\ll \frac{1}{L}\frac{F}{R-X}W(x)(t_1 + \cdots + t_d)^L \\
&= (T\partial_T)^{-1}\frac{F}{R-X}W(x)(t_1 + \cdots + t_d)^L \\
&\ll \frac{F}{R-X}W(x)(t_1 + \cdots + t_d)^L,
\end{aligned}
$$

where $T = t_1 + \ldots + t_d$ *and* $X = x_1 + \cdots + x_n$.

We take a small positive Constant $R > 0$ such that the functions in the equation are holomorphic on $D_0(R)$ and that Proposition 4.2 does hold. By this choice of R we easily see that the formal solution $w(t,x) \in \mathcal{M}_x[[t]]$ with $w(t,x) = O(|t|^K)$ of the equation (4.1) exists uniquely by the invertibility of P on every $\mathcal{O}_0(R)[t]_L$ $(L \geq K)$. Indeed, the formal solution $w(t,x) = \sum_{L \geq K} w_L(t,x)$ $(w_L(t,x) \in \mathcal{O}_0(R)[t]_L)$ are determined inductively on L. Therefore, we have only to prove the convergence of this formal solution $w(t,x)$.

Let $U(t,x) = Pw(t,x)$ be a new unknown function. Then $U(t,x)$ satisfies the following equation by (4.1):

$$
(4.4) \quad U = \sum_{|\alpha|=K} \zeta_\alpha(x)t^\alpha + g_{K+1}(t,x,P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U), \quad U(t,x) = O(|t|^K).
$$

In order to prove the convergence of formal solution $U(t,x)$, we prepare majorant functions (which are convergent) as follows.

$$
\sum_{|\alpha|=K} \zeta_\alpha(x)t^\alpha \ll \frac{A}{(R-X)^K}T^K, \quad (T = t_1 + \cdots + t_d,\ X = x_1 + \cdots + x_n),
$$

$$
\begin{aligned}
g_{K+1}(t,x,u,\tau,\xi) &\ll \sum_{|\alpha+Kp+(K-1)|q|+K|r| \geq K+1} \frac{G_{\alpha pqr}}{(R-X)^{|\alpha|+p+|q|+|r|}} T^{|\alpha|}u^p\tau^q\xi^r \\
&=: G_{K+1}(T,X,u,\tau,\xi).
\end{aligned}
$$

We recall the majorant relations (4.3) in Proposition 4.2, and notice an elementary majorant relation of operators that $\partial_{t_j}(T\partial_T)^{-1} \ll 1/T$. We consider the following equation:

$$
\begin{aligned}
(4.5) \quad W(T,X) = {} & \frac{A}{(R-X)^K}T^K \\
& + G_{K+1}\left(T, X, \frac{F}{R-X}W, \left\{\frac{F}{R-X}\frac{W}{T}\right\}_{j=1}^{d}, \left\{\partial_{x_k}(T\partial_T)^{-1}\frac{F}{R-X}W\right\}_{k=1}^{n}\right)
\end{aligned}
$$

with $W(T,X) = O(T^K)$, where F is the same positive constant in (4.3).

By this construction of the equation, we easily see that the formal solution $W(T,X) \in \mathcal{O}_X[[T]]$ (which is uniquely determined) is a majorant function of $U(t,x)$, that is, $W(t_1 + \cdots + t_d, x_1 + \cdots + x_n) \gg U(t,x)$ holds. Therefore, it is sufficient to prove the convergence of $W(T,X)$. We put $W(T,X) = \sum_{L \geq K} W_L(X) T^L$ and by substituting this into (4.5), we obtain the following recursion formulas:

$$W_K(X) = \frac{A}{(R-X)^K}, \tag{4.6}$$

and for $L \geq K+1$,

$$\begin{aligned} W_L(X) &= \sum_{V(\alpha,p,q,r) \geq K+1} \frac{G_{\alpha pqr}}{(R-X)^{|\alpha|+p+|q|+|r|}} \sum{}' \prod_{l=1}^{p} \frac{F}{R-X} W_{L_l}(X) \\ &\times \prod_{j=1}^{d} \prod_{l=1}^{q_j} \frac{F}{R-X} W_{M_{jl}}(X) \prod_{k=1}^{n} \prod_{l=1}^{r_k} \frac{1}{N_{kl}} \partial_{x_k} \frac{F}{R-X} W_{N_{kl}}(X), \end{aligned} \tag{4.7}$$

where

$$V(\alpha,p,q,r) = |\alpha| + Kp + (K-1)|q| + K|r|, \tag{4.8}$$

and summation $\sum'$ is taken over

$$|\alpha| + \sum_{l=1}^{p} L_l + \sum_{j=1}^{d} \sum_{l=1}^{q_j} (M_{jl} - 1) + \sum_{k=1}^{n} \sum_{l=1}^{r_k} N_{kl} = L. \tag{4.9}$$

By these recursion formulas, we can prove the following lemma:

Lemma 4.3 *The coefficients* $\{W_L(X)\}_{L \geq K}$ *are given by*

$$W_L(X) = \sum_{j=K}^{10L-9K} \frac{W_{Lj}}{(R-X)^j}, \quad \text{by some } W_{Lj} \geq 0. \tag{4.10}$$

By the representation (4.10), we have the following majorant relation:

$$\begin{aligned} \partial_{x_k} (T\partial_T)^{-1} \frac{F}{R-X} W(T,X) &= \sum_{L \geq K} \sum_{j=K}^{10L-9K} \frac{j+1}{L} \frac{F W_{Lj}}{(R-X)^{j+2}} T^L \\ &\ll \frac{10F}{(R-X)^2} W(T,X). \end{aligned} \tag{4.11}$$

As the final step we construct the following functional equation which may be called a majorant (functional) equation to the equation (4.5):

$$\begin{aligned} V(T,X) &= \frac{A}{(R-X)^K} T^K \\ &+ G_{K+1}\left(T, X, \frac{F}{R-X} V, \left\{\frac{F}{R-X} \frac{V}{T}\right\}_{j=1}^{d}, \left\{\frac{10F}{(R-X)^2} V\right\}_{k=1}^{n}\right) \end{aligned} \tag{4.12}$$

with $V(T,X) = O(T^K)$. The existence of unique formal solution $V(T,X)$ which is convergent follows from the classical implicit function theorem, and the above construction of the equation shows that $W(T,X) \ll V(T,X)$ which implies the convergence of $U(t,x)$.

5 Proof of Theorem 1.1, (ii).

We recall the equation we consider is given by

$$(5.1) \qquad \left(\sum_{i,j=1}^{d} a_{ij}(x)t_i\partial_{t_j} + c(x)\right) v(t,x) = \sum_{|\alpha|=2} b_\alpha(x)t^\alpha + f_3(t,x,v,\partial_t v,\partial_x v),$$

where $c(x) = f_u(\mathbf{a}(x))$ with $c(0) \neq 0$ and $A(x) = (a_{ij}(x))_{ij}^d$ is a nilpotent matrix such that $A(x)^N \equiv \mathbf{O}$ but $A(x)^j \not\equiv \mathbf{O}$ for $0 \le j \le N-1$ $(1 \le N \le d)$.

We remark that by the assumption that $c(0) \neq 0$, we may assume $c(x) \equiv 1$ in the above equation by multiplying $c(x)^{-1}$ to the equation which does not change the assumption for $A(x)$.

Let assume the functions in the equation are holomorphic in x on $D_0(R)$ by an $R > 0$. Then we can easily examine the unique existence of the formal solution $v(t,x) = \sum_{|\alpha|\ge 2} v_\alpha(x)t^\alpha$ $(v_\alpha(x) \in \mathcal{O}(R))$. Indeed, under our assumptions the mapping

$$\sum_{i,j=1}^{d} a_{ij}(x)t_i\partial_{t_j} + 1 : \mathcal{O}(R)[t]_L \to \mathcal{O}(R)[t]_L$$

is invertible by the fact that the matrix representation of the part of vector field which we set by $A(x)$ is nilpotent again. Therefore the formal solution is uniquely determined inductively on $L \ge 2$ for $v_L(t,x) = \sum_{|\alpha|=L} v_\alpha(x)t^\alpha \in \mathcal{O}(R)[t]_L$.

Our proof is thus reduced only to estimate the Gevrey order in t variables of the formal solution. Here we recall Lemma 2.2 which guarantees to make a change of variables t by $(\tau_1,\cdots,\tau_d) = (t_1,\cdots,t_d)Q(x)$ by $Q(x) \in GL(d,\mathcal{O}(R))$.

By the assumption of nilpotency for $A(x)$, there exists an invertible matrix $Q(x) = (Q_{ij}(x))$ over the field of meromorphic functions in a neighborhood of the origin such that

$$(5.2) \qquad Q(x)^{-1}(a_{ij}(x))Q(x) = \mathrm{diag}(B_1,\cdots,B_I,O_J) \ : \ \text{Jordan canonical form},$$

where diag$(\cdots)$ denotes the diagonal matrix with the diagonal blocks $(\cdots)$. Here, $B_i^{n_i} = \mathbf{O}$ $(n_i \ge 1)$ and O_J is the zero matrix block of size J with $n_1 + \cdots + n_I + J = d$, and by the assumption we have $\max\{n_1,\ldots,n_I\} = N$.

Now we make a "formal" change of variables by

$$(\tau_1,\ldots,\tau_d) = (t_1,\ldots,t_d)Q(x), \quad y_k = x_k \ (k = 1,\cdots,n).$$

Here the "formal" means that $Q(x)$ may admit meromorphic singular point at the origin, and it is an actual holomorphic change at the points if $Q(x)$ is holomorphically invertible at the origin.

Since $\partial_{t_i} = \sum_{j=1}^{d} Q_{ij}(x)\partial_{\tau_j}$ and $\partial_{x_k} = \sum_{j=1}^{d} t_i\{\partial_{x_k} Q_{ij}(x)\}\partial_{\tau_j} + \partial_{y_k}$, in the reduced equation by this change of variables the vector field is changed by the Jordan canonical form (5.2), and the nonlinear term f_3 is changed to g_3 which satisfies the same condition.

According to the form of (5.2), we make a further change of variables, $y \mapsto x \in \mathbf{C}^n$ (as before), and make a decomposition $\tau = (y, z) \in \mathbf{C}^d$ by

$$(y, z) = (\mathbf{y}^1, \ldots, \mathbf{y}^I, z), \quad \mathbf{y}^i = (y_{i,1}, \ldots, y_{i,n_i}) \in \mathbf{C}^{n_i}, \quad z = (z_1, \ldots, z_J) \in \mathbf{C}^J$$

Now the equation (5.1) is reduced to the following equation:

$$\text{(5.3)} \qquad Pv(y, z, x) = \sum\nolimits_{|\alpha|+|\beta|=2} \zeta_{\alpha\beta}(x) y^\alpha z^\beta + g_3(y, z, x, v, \partial_y v, \partial_z v, \partial_x v),$$

with $v(y, z, x) = O((|y| + |z|)^2)$, where

$$\text{(5.4)} \qquad P = \sum_{i=1}^{I} \sum_{j=1}^{n_i - 1} \delta \; y_{i,j+1} \partial_{y_{i,j}} + 1, \quad \delta \in \mathbf{C},$$

$$\text{(5.5)} \qquad g_3(y, z, x, v, \zeta, \eta, \xi) = \sum_{|\alpha|+|\beta|+2p+|q^1|+|q^2|+2|r| \geq 3} g_{\alpha\beta p q^1 q^2 r}(x) y^\alpha z^\beta v^p \zeta^{q^1} \eta^{q^2} \xi^r,$$

where $q^1 \in \mathbf{N}^{n_1 + \cdots + n_I}$, $q^2 \in \mathbf{N}^J$.

We remark that the constant δ is assumed as small as we want by Remark 4.1.

Here we have to notice that in the reduced equation (5.3) the origin $x = 0$ may be a singular point. Therefore, the proof of the theorem is divided into two steps. In the first step, we prove the theorem under the assumption of holomorphy at $x = 0$. In the second step, we remove such restriction by using the maximum principle for the holomorphic functions from the fact that the equation has a unique formal solution $v(t, x) \in \mathcal{O}(R)[[t]]$ which was mentioned above.

5.1 Holomorphic case.

We assume the equation (5.3) is holomprhic in a neighborhood of the origin and we shall prove that the formal solution $v(y, z, x)$ of (5.3) belongs to $\mathcal{G}^{2N}$ in (y, z) variables with $N = \max\{n_i \; ; \; i = 1, 2, \cdots, I\}$. In order to do that it is sufficient to prove $v(y, z, x)$ belongs to some Gevrey space $\mathcal{G}^{\mathbf{s}}$ in (y, z) variables with $\mathbf{s} = (s_1, s_2, \cdots, s_d)$ such that $\|\mathbf{s}\| = \max\{s_j\} \leq 2N$.

Let us prepare the following lemma:

Proposition 5.1 (i) *For all $L \geq 2$, there exists a radius $R > 0$ independent of L such that the mapping $P : \mathcal{O}_0(R)[y, z]_L \longrightarrow \mathcal{O}_0(R)[y, z]_L$ is invertible.*

(ii) *Let $\widetilde{\mathbf{s}} = (\mathbf{s}_1, \cdots, \mathbf{s}_I, \mathbf{1}_J) \in \mathbf{N}^d$, where*

$$\mathbf{s}_i = (1, 2, \cdots, n_i) \in \mathbf{N}^{n_i}, \quad \mathbf{1}_J = (1, \cdots, 1) \in \mathbf{N}^J,$$

as a manner corresponding to the decomposition $\tau = (y, z)$. *For* $\mathbf{k}_d = (k, \cdots, k) \in \mathbf{N}^d$ *we define* $\widetilde{\mathbf{s}} + \mathbf{k}_d$ (*or* $\widetilde{\mathbf{s}} + k$, *for short*) *by the summation componentwisely.*

For $f(y, z, x) \in \mathcal{O}_0(R)[y, z]_L$, *if* $\mathcal{B}^{\widetilde{\mathbf{s}}+k}(f)(y, z, x) \ll W_L(X)T^L$ $(T = |y| + |z|, X = |x|)$, *then there exists a positive constant* $C > 0$ *independent of* L *such that*

$$\mathcal{B}^{\widetilde{\mathbf{s}}+k}(P^{-1}f)(y, z, x) \ll C\, W_L(X)T^L. \tag{5.6}$$

Remark 5.2 This lemma shows the bijectivity of the mapping $P : \mathcal{G}^{\widetilde{\mathbf{s}}+k} \longrightarrow \mathcal{G}^{\widetilde{\mathbf{s}}+k}$ for all $k \geq 0$. Indeed, let $f(y, z, x) = \sum_{L\geq 1} f_L(y, z, x) \in \mathcal{G}^{\widetilde{\mathbf{s}}+k}$ with $f_L(y, z, x) \in \mathcal{O}_0(R)[y, z]_L$. Since $\mathcal{B}^{\widetilde{\mathbf{s}}+k} f(y, z, x) = \sum_{L\geq 1} \mathcal{B}^{\widetilde{\mathbf{s}}+k} f_L(y, z, x) \in \mathcal{O}_{y,z,x}$, there exist positive constants M and R' such that

$$\mathcal{B}^{\widetilde{\mathbf{s}}+k} f(y, z, x) \ll \frac{M}{(1 - X/R')(1 - T/R')} = \frac{M}{1 - X/R'} \sum_{L\geq 1} \frac{T^L}{R'^L},$$

where T and X are given as above. This means that

$$\mathcal{B}^{\widetilde{\mathbf{s}}+k} f_L(y, z, x) \ll \frac{MT^L}{R'^L(1 - X/R')},$$

and for the formal inverse $P^{-1}f$ we have

$$\mathcal{B}^{\widetilde{\mathbf{s}}+k}(P^{-1}f)(y, z, x) \ll \frac{CM}{(1 - X/R')(1 - T/R')} \in \mathcal{O}_{y,z,x}.$$

We put $U(y, z, x) = Pv(y, z, x)$ as a new unknown function. Then, $U(y, z, x)$ satisfies the following equation:

$$U(y, z, x) = \sum\nolimits_{|\alpha|+|\beta|=2} \zeta_{\alpha\beta}(x) y^\alpha z^\beta + g_3(y, z, x, P^{-1}U, \partial_y P^{-1}U, \partial_z P^{-1}U, \partial_x P^{-1}U) \tag{5.7}$$

with $U(y, z, x) = O((|y| + |z|)^2)$.

Now we apply the $\widetilde{\mathbf{s}}$-Borel transform to the equation (5.7), we obtain

$$\begin{aligned} \mathcal{B}^{\widetilde{\mathbf{s}}}(U)(y, z, x) \;=\; & \sum_{|\alpha|+|\beta|=2} \zeta_{\alpha\beta}(x) \frac{(|\alpha| + |\beta|)!}{(\widetilde{\mathbf{s}} \cdot (\alpha, \beta))!} y^\alpha z^\beta \\ & + \mathcal{B}^{\widetilde{\mathbf{s}}}\{g_3(y, z, x, P^{-1}U, \partial_y P^{-1}U, \partial_z P^{-1}U, \partial_x P^{-1}U)\}. \end{aligned} \tag{5.8}$$

In order to construct a majorant equation for (5.8), we prepare the following lemma:

Lemma 5.3 (i) *The Borel transform of a product* $(uv)(y, z, x)$ *is majorized by*

$$\mathcal{B}^{\widetilde{\mathbf{s}}}(uv)(y, z, x) \ll N\mathcal{B}^{\widetilde{\mathbf{s}}}(|u|)(y, z, x) \times \mathcal{B}^{\widetilde{\mathbf{s}}}(|v|)(y, z, x), \tag{5.9}$$

where $N = \max\{n_1, \ldots, n_I\}$.

(ii) *If $\mathcal{B}^{\tilde{\mathbf{s}}}(u)(y,z,x) \ll W(T,X)$ $(T = |y| + |z|, X = |x|)$, then there exists a positive constant $C_1 > 0$ independent of y, z and x such that the Borel transforms of $\partial_{y_{i,j}} u$, $\partial_{z_k} u$ and $\partial_{x_k} u$ are majorized by*

$$\mathcal{B}^{\tilde{\mathbf{s}}}(\partial_{y_{i,j}} u)(y,z,x) \ll C_1 \partial_T (T\partial_T)^{j-1} W(T,X), \tag{5.10}$$

$$\mathcal{B}^{\tilde{\mathbf{s}}}(\partial_{z_k} u)(y,z,x) \ll C_1 \partial_T W(T,X), \tag{5.11}$$

$$\mathcal{B}^{\tilde{\mathbf{s}}}(\partial_{x_k} u)(y,z,x) \ll C_1 \partial_X W(T,X). \tag{5.12}$$

Now we consider the following equation which is a majorant equation of (5.8):

$$\begin{aligned} W(T,X) = &\left(\sum_{|\alpha|+|\beta|=2} |\zeta_{\alpha\beta}|(\mathbf{X}) \frac{(|\alpha|+|\beta|)!}{(\tilde{\mathbf{s}}\cdot(\alpha,\beta))!} \right) T^2 \\ &+ |g_3| \left(\mathbf{T}, \mathbf{X}, C'W, \left\{ \{ C'\partial_T (T\partial_T)^{j-1} W \}_{j=1}^{n_i} \right\}_{i=1}^{I}, \{C'\partial_T W\}_{k=1}^{J}, \{C'\partial_X W\}_{k=1}^{n} \right), \end{aligned} \tag{5.13}$$

with $W(T,X) = O(T^2)$ where $\mathbf{T} = (T,\dots,T) \in \mathbf{C}^d$, $\mathbf{X} = (X,\dots,X) \in \mathbf{C}^n$ and $C' = C_1 C N$.

Now by the construction of the equation (5.13), we easily see that the formal solution $W(T,X) \in \mathcal{O}_X[[T]]$ is a majorant function of $\mathcal{B}^{\tilde{\mathbf{s}}}(U)(y,z,x)$ of (5.8) by replacing $T = y_{1,1} + \cdots + y_{I,n_I} + z_1 + \cdots + z_J$ and $X = x_1 + \cdots + x_n$.

Here we recall the result in [S1] by Shirai in a special form attached to our case. Let us consider the following equation.

$$V(T,X) = g(X)T^K + h_{K+1}(T,X,V,\{D_T^j V\}_{j=1}^p, D_X V)$$

with $V = O(T^K)$, where $g(X)$ and $h_{K+1}(T,X,V,\tau,\xi)$ $(\tau \in \mathbf{C}^p, \xi \in \mathbf{C})$ are holomorphic in a neighborhood of the origin and

$$h_{K+1}(T,X,V,\tau,\xi) = \sum{}' h_{ab\{c(j)\}d}(X) T^a V^b \prod_{j=1}^{p} \tau_j^{c(j)} \, \xi^d,$$

and the summation $\sum'$ is taken over

$$V(a,b,\{c(j)\},d) := a + Kb + \sum_j (K-j)c(j) + Kd \geq K+1,$$

the left hand side means the order of zeros in T of each monomial by substituting $V(t,x) = O(T^K)$.

Then the formal solution $V(T,X) \in \mathcal{O}_X[[t]]$ which exists uniquely belongs to $\mathcal{G}^{\sigma+1}$ in T variable with

$$\sigma = \max\left\{ \frac{A(a,b,\{c(j)\},d)}{V(a,b,\{c(j)\},d) - K} \; ; \; h_{ab\{c(j)\}d}(x) \not\equiv 0 \right\},$$

by $A(a, b, \{c(j)\}, d)$ ($\in \{0, 1, 2, \cdots, p\}$) which denotes the maximal order of differentiations which appears in the monomial. (This is a special case of Theorem 1 in [S1].)

We return to the equation (5.13). In this case, $K = 2$, $V(a, b, \{c(j)\}, d) - K \geq 1$ and $A(a, b, \{c(j)\}, d) \leq \max\{n_i \,;\, i = 1, 2, \cdots, I\} = N$ which shows that $W(T, X) \in \mathcal{G}^{N+1}$ in T variable. Therefore $\mathcal{B}^{\widetilde{\mathbf{s}}}(U)$ $(U = Pv)$ belongs to the Gevrey space $\mathcal{G}^{N+1}$ in τ variables $\tau(= (y, z))$ variables, which implies $U = Pv \in \mathcal{G}^{\widetilde{\mathbf{s}}+N}$ in τ variables, and hence $v(\tau, x) = P^{-1}U \in \mathcal{G}^{\widetilde{\mathbf{s}}+N}$ in τ variables by Proposition 5.1 and Remark 5.2. Then by Lemma 2.2, we have $v(t, x) \in \mathcal{G}^{2N}$ in t variables, since each component of $\widetilde{\mathbf{s}}$ is estimated by $N = \max\{n_i \,;\, i = 1, 2, \cdots, I\}$. ■

5.2 Meromorphic case.

In this subsection, we shall prove the theorem in the case where $Q(x)$ or $Q(x)^{-1}$ is singular at the origin by the idea used in [M] by Miyake where the inverse theorem of Cauchy-Kowalevski's theorem for general systems was studied. The theorem is an immediate result from the following lemma:

Lemma 5.4 *Assume that $Q(x)$ or $Q(x)^{-1}$ is singular at the origin. We may assume that $Q(x)$ and $Q(x)^{-1}$ are holomorphic on $\prod_{j=1}^{n}\{R_j - \varepsilon \leq |x_j| \leq R_j + \varepsilon\} \subset D_0(R)$ by suitable taking positive constants $R_j > 0$ and $\varepsilon > 0$ $(j = 1, 2, \cdots, n)$ such that $0 < R_j - \varepsilon < R_j + \varepsilon < R$. Then the formal solution $v(\tau, x)$ $(\tau = (y, z))$ of (5.3) belongs to $\mathcal{G}^{2N}$ in τ variables on $\prod_{j=1}^{n}\{|x_j| \leq R_j\}$.*

Proof. We, first, notice that we already know there exists a unique formal solution $v(\tau, x) = \sum_{|\alpha|\geq 2} v_\alpha(x)\tau^\alpha \in \mathcal{O}_x[[\tau]]$, where we may assume that $v_\alpha(x) \in \mathcal{O}_0(R)$ by a small $R > 0$ for all α. We may consider that this R is the one in the statement of the lemma.

Let $\widehat{x} = (\widehat{x}_1, \cdots, \widehat{x}_n) \in \prod_{j=1}^{n}\{|x_j| = R_j\}$ be arbitrary fixed. Then by the assumption, $Q(x)$ is holomorphically invertible on ε neighborhood of $\widehat{x}$. By the result in the previous subsection, we know that the formal solution $v(\tau, x)$ belongs to $\mathcal{G}^{2N}$ in τ variables in a neighborhood of $\widehat{x}$. Therefore there exists a positive constant $r(\widehat{x})$ (which may depend on $\widehat{x}$) such that the following Gevrey estimates hold by positive constants $A_{\widehat{x}}$ and $B_{\widehat{x}}$ which may depend on $\widehat{x}$:

$$(5.14) \qquad \max_{|x_j - \widehat{x}_j| \leq r(\widehat{x})} |v_\alpha(x)| \leq A_{\widehat{x}} B_{\widehat{x}}^{|\alpha|} \{(2N-1)|\alpha|\}!,$$

for all $\alpha \in \mathbf{N}^d$ with $|\alpha| \geq 2$.

Since the polycircle $C(R) = \prod\{|x_j| = R_j\}$ $(R = (R_1, \cdots, R_d))$ is compact, we can take finite number of $\{\widehat{x}^{(k)}\}_k$ on the polycircle so that the union of $r(\widehat{x}^{(k)})$ neighborhood of $\widehat{x}^{(k)}$'s covers the polycircle $C(R)$. Now by taking A the maximum of $A_{\widehat{x}^{(k)}}$'s and B the maximum of $B_{\widehat{x}^{(k)}}$'s, we get the following Gevrey estimates on the polycircle $C(R)$,

$$(5.15) \qquad \max_{x \in C(R)} |v_\alpha(x)| \leq A B^{|\alpha|} \{(2N-1)|\alpha|\}!,$$

for all $\alpha \in \mathbf{N}^d$ with $|\alpha| \geq 2$. Since $v_\alpha(x)$ are all holomorphic on $D_0(R)$, by the maximum principle we get the same Gevrey estimation on the polydisc $\prod_j\{|x_j| \leq R_j\}$, which proves the lemma. ■

References

[CL] Chen H. and Luo Z., On the Holomorphic Solution of Non-linear Totally Characteristic Equations with Several Space Variables, *Preprint* 99/23 *November* 1999, *Institut für Mathematik, Universität Potsdam.*

[CLT] Chen H. and Luo Z. and Tahara H., Formal solutions of nonlinear first order totally characteristic type PDE with irregular singularity, *Ann. Inst. Fourier* (*Grenoble*) **51** (2001), No. 6, 1599—1620.

[CT] Chen H. and Tahara H., On Totally Characteristic Type Non-linear Partial Differential Equations in Complex Domain, *Publ. RIMS, Kyoto Univ.* **35** (1999), 621–636.

[GT] Gérard, R. and Tahara, H., *Singular Nonlinear Partial Differential Equations*, Vieweg Verlag, 1996.

[M] Miyake M., On Cauchy-Kowalevski's theorem for general systems, *Publ. RIMS, Kyoto Univ.*, **15** (1979), 315—337.

[MS] Miyake M. and Shirai A., Convergence of formal solutions of first order singular nonlinear partial differential equations in complex domain, *Annales Polonici Mathmatici*, **74** (2000), 215—228.

[MS2] Miyake M. and Shirai A., Structure of formal solutions of nonlinear First Order Singular Partial Differential Equations in Complex Domain,*Preprint Series In Mathematical Science, Graduate School of Human Informatics, Nagoya Univ.*, No. 2002-4.

[O] Oshima T., On the theorem of Cauchy-Kowalevski for first order linear differential equations with degenerate principal symbols, *Proc. Japan Acad*, **49** (1973), 83—87.

[S1] Shirai A., Maillet type theorem for nonlinear partial differential equations and Newton polygons, *J. Math. Soc. Japan.*, **53** (2001), 565—587.

[S2] Shirai A., Convergence of formal solutions of singular first order nonlinear partial differential equations of totally characteristic type, *Funkcial. Ekvac.*, **45** (2002), 187—208.

[S3] Shirai A., Maillet type theorem for first order singular nonlinear partial differential equations of totally characteristic type, *preprint.*

三宅 正武 (Masatake Miyake)
名古屋大学大学院多元数理科学研究科
Graduate School of Mathematics
Nagoya University
mmiyake@math.nagoya-u.ac.jp

白井 朗 (Akira shirai)
名古屋大学大学院多元数理科学研究科
Graduate School of Mathematics
Nagoya University
m96034q@math.nagoya-u.ac.jp