数理解析研究所講究録 1337 巻 2003 年 1-9

Traveling Waves in Spatially Random Media

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Traveling waves in heterogeneous media are gaining more and more attention in various fields of science such as ecology ([8]), epidemiology, physiology and combustion theory (see the comprehensive survey [9] and references therein). They have also become an important subject of mathematical studies in the past decade. However, most of those theoretical studies have been focused on spatially periodic cases, and little is known about the nature of traveling waves in aperiodically varying media. (This is in marked contrast with the case of temporally varying – but spatially homogeneous – media, for which much is known; see [7].)

In this lecture I will introduce the precise notion of traveling waves in spatially *recurrent* diffusive media – including quasi-periodic and almost periodic ones as special cases – as a natural extension of the conventional notion of traveling waves. I will then discuss the existence, uniqueness and stability of such traveling waves mainly for equations with bistable nonlinearity.

§1. Preliminaries: recurrence and almost periodicity

In order to give a precise definition of traveling waves, let us briefly recall the notion of recurrent functions and almost periodic functions.

A continuous function $b(x):\mathbb{R}\to\mathbb{R}$ is called *almost periodic* (in the sense of Bohr) if for any sequence of real numbers $\ell_1,\ \ell_2,\ \ell_3,\ \cdots$ the sequence of functions $b(x+\ell_1),\ b(x+\ell_2),\ b(x+\ell_3),\ \cdots$ has a uniformly convergent subsequence. In other words, $b(x)$ is almost periodic if its *hull*

$$\mathcal{H}_b := \overline{\{\,\sigma_\ell b \mid \ell\in\mathbb{R}\}}^{L^\infty(\mathbb{R})}$$

is a compact set, where σ_ℓ denotes the shift operator $g(x)\mapsto g(x+\ell)$. For example, if $b(x)$ is periodic (resp. quasi-periodic), then its hull $\mathcal{H}_b$ is homeomorphic to a circle (resp. torus).

The notion of recurrent functions is similar to but wider than that of almost periodic functions. The main difference is that it uses the topology of locally uniform convergence, rather than uniform one. A continuous function $b(x)$: $\mathbb{R}\to\mathbb{R}$ is called *recurrent* if the set

$$\widetilde{\mathcal{H}}_b := \overline{\{\,\sigma_\ell b \mid \ell\in\mathbb{R}\}}^{L^\infty_{loc}(\mathbb{R})}$$

is compact and if

$$\widetilde{\mathcal{H}}_{b^*} = \widetilde{\mathcal{H}}_b \quad \text{for any } b^*\in\widetilde{\mathcal{H}}_b.$$

This second condition means that if $b(x+\ell_1),\ b(x+\ell_2),\ b(x+\ell_3),\ \cdots$ converge locally uniformly to $b^*(x)$, then one can find $\tilde{\ell}_1,\ \tilde{\ell}_2,\ \tilde{\ell}_3,\ \cdots$ such that $b^*(x+$

$\tilde{\ell}_1)$, $b^*(x+\tilde{\ell}_2)$, $b^*(x+\tilde{\ell}_3)$, $\cdots$ converge locally uniformly to $b(x)$. (Note that this second condition is automatically fulfilled if we use the topology of uniform convergence; theorefore almost periodicity defined above implies recurrence.) It is easily seen that any recurrent or almost periodic function is bounded and uniformly continuous on $\mathbb{R}$.

We can also define the notion of almost periodicity or recurrence for multi-variable functions. A function $b(x_1, x_2, \cdots, x_N)$ is called almost periodic in the direction x_N if for any sequence of real numbers ℓ_1, ℓ_2, ℓ_3, $\cdots$ the sequence of functions $b(x+\ell_1 \boldsymbol{e}_N)$, $b(x+\ell_2 \boldsymbol{e}_N)$, $b(x+\ell_3 \boldsymbol{e}_N)$, $\cdots$ has a uniformly convergent subsequence, where $\boldsymbol{e}_N$ denotes the unit vector in the x_N direction. We define the hull $\mathcal{H}_b$ in the same way as above.

§2. Definition of traveling waves

To clarify the underlying idea, we begin with a simple example. Consider a one-dimensional diffusion equation of the form

$$u_t = u_{xx} + b(x)f(u) \qquad (x \in \mathbb{R}, t > 0), \tag{1}$$

where

$$f(0) = f(1) = 0$$

and $b(x)$ is an almost periodic function on $\mathbb{R}$. We will discuss traveling front solutions that connects the state $u = 0$ and $u = 1$. More precisely, we will consider traveling waves $u(x,t)$ satisfying the following conditions at infinity:

$$u(x,t) \to \begin{cases} 1 & \text{as } x \to -\infty, \\ 0 & \text{as } x \to +\infty. \end{cases} \tag{$\spadesuit$}$$

In the homogeneous case, namely the case where $b(x)$ is constant, a traveling wave is defined to be a solution of the form

$$u(x,t) = v(x - ct). \tag{2}$$

Here c is a constant which represents the propagation speed, and the function $v(z)$, $-\infty < z < \infty$, is called the *profile*. Traveling waves in the homogeneous case have two characteristic features:

$\boxed{\text{Property A}}$: the profile remains unchanged;

$\boxed{\text{Property B}}$: the front propagates at a constant speed.

Clearly neither of these properties holds in the inhomogeneous case, as the front encounters varying environments.

In order to cope with varying environments, it is convenient to introduce the notion of "landscape". Let $x = \xi(t)$ denote the position of the front at time t. (The meaning of "front" is vague at this stage, but let's pretend that it is well-defined somehow. Such vagueness will not matter at all in our later argument.)

We define

$$\begin{aligned}\textbf{current landscape} &= b(x+\xi(t)) \quad (=\sigma_{\xi(t)}b),\\ \textbf{current profile} &= u(x+\xi(t),t).\end{aligned}$$

The current landscape tells the shape of the graph of $b(x)$ viewed from the position of the front, and the current profile tells the shape of the solution viewed from that position. We can regard the current landscape $\sigma_{\xi(t)}b$ as a point on the hull $\mathcal{H}_b$. In other words, $\mathcal{H}_b$ is the set of landscapes which the front encounters as it proceeds. $\mathcal{H}_b$ also contains all the "virtual" landscapes obtained as the limit of sequences of real landscapes $\sigma_{\xi(t)}b$. Hereafter we call $\mathcal{H}_b$ the *configuration space* for equation (1).

Note that there is no distinction between the "real" and "virtual" landscapes if $b(x)$ is either constant or periodic. However, they can be different in non-periodic cases, and it is important to consider virtual landscapes as well as real ones when we discuss the nature of traveling waves.

If $b(x)$ is periodic, traveling waves can be characterized as follows:

$\boxed{\text{Property A'}}$: the current profile restores its original shape each time the front encounters the same landscape.

In the non-periodic case, however, the above characterization does not make sense, as the same landscape is never repeated. Nonetheless, similar landscapes appear again and again, since $\sigma_{\xi(t)}b$ forms a dense orbit in the configuration space $\mathcal{H}_b$. Thus we are led to the following characterization of traveling waves, which is a natural extension of Properties A and A':

$\boxed{\text{Property A"}}$: **the current profile depends continuously on the current landscape.**

This property means that the profile restores a similar shape each time the front encouters a similar landscape. Mathematically the above property can be stated as follows:

Denifition 2.1 A solution $u(x,t)$ of (1) satisying (♠) is called a *traveling wave* if there exists a continuous map

$$w(z,s) : \mathcal{H}_b \times \mathbb{R} \to \mathbb{R}$$

and a function $\xi(t) : \mathbb{R} \to \mathbb{R}$ such that

$$u(x+\xi(t),t) = w(\sigma_{\xi(t)}b, x) \quad (x \in \mathbb{R},\ t \in \mathbb{R}),$$

$$w(z,s) \to \begin{cases} 1 & \text{as } s \to -\infty \\ 0 & \text{as } s \to +\infty \end{cases} \quad \text{uniformly in } z \in \mathcal{H}_b\,.$$

The above expression can be rewritten as follows:

$$\boxed{u(x,t) = w(\sigma_{\xi(t)}b, x-\xi(t)).} \tag{3}$$

It is easily seen that this expression agrees with the existing definition of traveling waves for the homogeneous and the periodic cases. We also remark that an analogue of Property B also follows from Property A" if $b(x)$ is almost periodic. In fact, we have the following:

Proposition 2.2 Let $b(x)$ be almost periodic, and let $u(x,t) = w(\sigma_{\xi(t)}b, x-\xi(t))$ be a traveling wave. Then $\xi(t)$ has an average speed in the sense that

$$\frac{\xi(t+T)-\xi(t)}{T} \to c \quad \text{as } T \to \infty \quad \text{uniformly in } t \in \mathbb{R}.$$

(Outline of Proof.) This proposition follows immediately by observing that $\xi(t)$ satisfies a differential equation of the form

$$\dot{\xi}(t) = g(\xi(t)),$$

where $g(y)$ is an almost periodic function. □

The above definition of traveling waves can be adopted to equations of a more general form with little modification. For example, consider the equation

$$u_t = \{d(x,u)u_x\}_x + a(x,u)u_x + f(x,u) \qquad (x \in \mathbb{R},\ t > 0),$$

where $d(x,u), a(x,u), f(x,u)$ are recurrent in $x \in \mathbb{R}$ and have bounded derivatives. In this case, we consider the triplet $(\,d(x,u), a(x,u), f(x,u)\,)$ and define the configuration space as the closure of the entire set of translations $(\,d(x+\ell,u), a(x+\ell,u), f(x+\ell,u)\,)$, $\ell \in \mathbb{R}$.

The same argument applies to higher dimensional problems such as

$$u_t = \nabla\cdot(d(x,u)\nabla u) + a(x,u)\cdot\nabla u + f(x,u) \quad (x \in \Omega,\ t > 0),$$

where the domain Ω, as well as the coefficients a, b, f, is recurrent or almost periodic in some direction, say e_N. A typical example of such a domain Ω is a cylinder with undulating boundary. We can also impose inhomogeneous boundary conditions. In those cases, we pick up all the inhomogeneous quantities and define the configuration space $\mathcal{H}$ as the closure of the entire set of translations of those quantities as we have done before.

§3. Uniqueness and stability

Here and in what follows we only deal with bistable nonlinearity. The equation we consider is of the form

$$u_t = \nabla\cdot(d(x)\nabla u) + b(x)f(u) \quad (x \in \Omega,\ t > 0), \tag{4}$$

where Ω is a domain in $\mathbb{R}^N$ and $d(x), b(x)$ are smooth functions on Ω such that

(Rc) [recurrence] $d(x), b(x), \Omega$ are recurrent in the direction e_N and there exists a constant $\eta > 0$ such that $d(x),\ b(x) \geq \eta \ \ (x \in \Omega)$.

(Bs) [bistability] $f(0) = f(\alpha) = f(1) = 0$ for some $\alpha \in (0,1)$ and

$$f(u) < 0 \ (0 < u < \alpha), \quad f(u) > 0 \ (\alpha < u < 1),$$

$$f'(0), f'(1) < 0, \qquad \int_0^1 f(u)\,du > 0.$$

The last condition in (Bs) means that the state $u = 1$ has a lower potential energy than the state $u = 0$; therefore any traveling wave connecting $u = 0$ and $u = 1$ should move toward the direction that expands the region $u \approx 1$.

Next we introduce a condition on the front size. Define

$$I_\delta(t) := \{\, \boldsymbol{e}_N \cdot x \mid x \in \Omega,\ \delta \le u(x,t) \le 1-\delta \} \ \subset \mathbb{R}$$

(FS) [front-size condition] For any $\delta > 0$

$$\sup_{t\in\mathbb{R}} (\max I_\delta(t) - \min I_\delta(t)\,) < \infty.$$

Theorem 1 Let $\tilde{u}(x,t)$ be an entire solution of (4) (that is, a solution defined for all $t \in \mathbb{R}$) satisfying (♠) along with the front-size condition (FS) and that $\tilde{u}_t(x,t) > 0$. Then

(i) $\tilde{u}$ is a traveling wave;

(ii) any traveling wave satisfying (♠) coincides with $\tilde{u}$ up to time shift;

(iii) for any solution $u(x,t)$ of (4) whose initial data satisfies

$$\liminf_{x\to-\infty} u_0(x) > \alpha, \quad \limsup_{x\to+\infty} u_0(x) < \alpha, \tag{♣}$$

there exists $\tau \in \mathbb{R}$ such that

$$\|u(\cdot,t) - \tilde{u}(\cdot, t+\tau)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty\,.$$

(Outline of proof.) The above theorem can be proved by the following steps:

Step 1 (Liapunov stability) By constructing a suitable family of super- and subsolutions one can show local stability of $\tilde{u}(x,t)$ in the topology of $L^\infty(\Omega)$.

Step 2 (asymptotic stability) By using an argument similar to that of Proposition B2 in [6], one can show that stability of $\tilde{u}(x,t)$ implies its stability with asymptotic phase (which proves (iii)).

Step 3 (uniqueness) Asymptotic stability implies uniqueness of such entire solution up to time shift (which proves (ii)).

Step 4 (continuity) To prove that $\tilde{u}(x,t)$ is a traveling wave, it remains to show that the current profile depends continuously on the current landscape. But this is an easy consequence of the uniqueness result in Step 3 and a compactness argument. □

Remark 3.1 Obviously, condition (FS) is saisfied by any traveling wave. Theorem 1 (i) shows that the converse is also true. Note that the converse does not necessarily hold for the Fisher-KPP type equations (like $f(u) = u(1-u)$) even in the homogeneous case. In fact, those equations have highly peculiar entire solutions that satisfy both (♠), (FS) and behave like a traveling wave with speed

c_1 near $t = -\infty$ while behaving like a traveling wave with speed c_2 near $t = +\infty$ with $c_1 < c_2$ (see [2]). Clearly such a solution is not a traveling wave as it does not have Property A".

The following theorem, which follows easily from Theorem 1, is useful in real applications. To state this theorem, we need a weeker version of condition (FS):

(FS)$_+$ For any $\delta > 0$

$$\limsup_{t\to\infty} (\max I_\delta(t) - \min I_\delta(t)\,) < \infty.$$

Theorem 2 Let $u(x,t)$ be a solution of (4) satisfying (♣), (FS)$_+$ and

$$\lim_{t\to\infty} u(x+ct,t) = 1 \qquad \text{locally uniformly in } \Omega$$

for some $c > 0$. Then a traveling wave exists for (4) and $u(x,t)$ converges to this traveling wave (or its time-shift) as $t \to \infty$ uniformly in Ω.

§4. Exinstence in one space dimension

In this section we discuss the existence of traveling waves in one space dimension. Consider the equation

$$u_t = \{d(x)u_x\}_x + b(x)f(u) \qquad (x \in \mathbb{R},\ t > 0), \tag{5}$$

where we assume (Rc), (Bs) with $\Omega = \mathbb{R}$.

By using comaprison arguments, it is not difficult to show that the front-size condition (FS)$_+$ holds for any solution of (5) whose initial data satisfies (♣). Combining this and Theorem 2, we obtain the following results:

Theorem 3 (Existence criterion) Equation (5) has a traveling wave with condition (♠) if and only if there is a subsolution of positive speed satisfying (♠).

Corollary A traveling wave with condition (♠) exists for (5) if

$$\left(\sqrt{b(x)d(x)}\right)' \le c_0\, b(x) - \delta \quad (x \in \mathbb{R}) \tag{6}$$

for some $\delta > 0$, where c_0 is the traveling wave speed for the homogeneous equation $u_t = u_{xx} + f(u)$.

(Proof) Let $\phi(z)$ be the profile of the traveling wave for the above homogeneous equation. That is, $\phi(z)$ satisfies

$$\phi" + c_0\phi' + f(\phi) = 0 \quad (z \in \mathbb{R})$$

along with the condition (♠). Then condition (6) implies that

$$w(x,t) = \phi\Big(\int_{ct}^{x} \sqrt{b(y)/d(y)}\Big)$$

is a subsolution of speed c provided that we choose $c > 0$ sufficiently small. □

Remark 4.1 The above result is new even for the periodic case. Previously the existence of traveling waves for a bistable equation was known only in two extreme cases: **(Rc)** the nearly constant case (see Xin [9] and the references therein) and **(Bs)** near the homogenization limit (see Heinze [3]). Their results are both based on the implicit function theorem. On the other hand, our condition (6) allows $b(x)d(x)$ to have rather a large oscillation without coming close to the homogenization limit.

Theorem 4 (Classification of long-time behavior) Let $u(x,t)$ be a solution of (5) whose initial data satisfies (♣). Then one of the following three situations occurs as $t \to \infty$:

(Rc) $u(x,t)$ approaches a traveling wave satisfying (♠);
(Bs) $u(x,t)$ converges to a stationary solution satisfying (♠) [**blocking**];
(c) $u(x,t)$ converges locally uniformly to 1, but its front travels at the average speed 0 [**virtual blocking**].

It is not difficult to show that the situation (Bs) or (c) occurs if and only if there exists some $(d^*, b^*) \in \mathcal{H}$ such that the following equation has a stationary solution satisfying (♠):

$$u_t = \{d^*(x)u_x\}_x + b^*(x)f(u). \tag{5'}$$

In other words, propagation is blocked (really or virtually) by the presence of stationary fronts.

Remark 4.2 In the homogeneous or the periodic case, there is no distinction between the real blocking and the virtual one. In other words, "zero average speed" implies blocking. This can be shown by using the fact that any point $(d^*, b^*) \in \mathcal{H}$ is a shift of (d, b) in the homogeneous or the periodic case.

The location of the stationary fronts that cause real or virtual blocking is called *blocking site.* A blocking site had better be understood as a position in the configuration space $\mathcal{H}$ rather than one on the real line. That is to say, a blocking site refers to the landscape viewed from the front of a stationary solution of (5'). This interpretation allows us to locate blocking sites even for a virtual blocking. In the case of virtual blocking, the speed of the traveling front is significantly slowed down each time the current landscape (that is, the position of the traveling front in the configuration space) passes near a blocking site.

§5. Existence in higher space dimensions

In this section we consider equation (4) under the Neumann boundary conditions $\partial u/\partial n = 0$ $(x \in \partial\Omega,\, t > 0)$ and assume (Rc) and (Bs).

In higher dimensional problems, the front size condition $(\mathrm{FS})_+$ does not always hold and this makes the existence question more complicated and intriguing. There are two types of situations that hinder full propagation of fronts:

- Blocking: real or virtual, as in the one-dimensional case; the blocked solution satisfies $(\mathrm{FS})_+$;
- Front breakup: this means that the solution does not satisfy $(\mathrm{FS})_+$; such a situation happens when there is large regional imbalance within the cross-section of Ω.

To explain what the latter means, let us consider the case where $d(x) = 1$ and Ω is a cylindrical domain whose cross-section is an $(N-1)$-dimensional dumbell-shaped domain D :

$$\Omega = D \times \mathbb{R}, \quad D = D_1 \cup R_\epsilon \cup D_2.$$

Here D_1, D_2 are disjoint regions and R_ϵ is a narrow channel connecting D_1 and D_2. Now let $b(x)$ take constant values b_1, b_2 on D_1, D_2, respectively, where $b_1 > b_2$. If the channel R_ϵ is not present, then Ω consists of two disjoint cylindrical domains

$$\Omega_1 = D_1 \times \mathbb{R}, \quad \Omega_2 = D_2 \times \mathbb{R},$$

and any solution with initial data satisfying ($\clubsuit$) develops into planar traveling waves with speed $C\sqrt{b_1}$ and $C\sqrt{b_2}$, respectively in Ω_1 and Ω_2, where C is some constant. Consequently no traveling wave with ($\spadesuit$) exists in the combined region $\Omega_1 \cup \Omega_2$, as the front-size condition $(\mathbf{FS})_+$ is violated. Qualitatively the same story holds if the connecting channel R_ϵ is present but very narrow. (One can prove it rigorously by using comparison arguments similar to those in [5].) It follows that no traveling wave satisfying ($\spadesuit$) exists in Ω. In the mean while, one can also show that neither blocking nor virtual blocking happens, since the region $\{u \approx 1\}$ expands at speeds not less than some positive constant.

Theorem 5 Suppose that equation (4) possesses a solution satisfying ($\clubsuit$) such that $(\mathrm{FS})_+$ does not hold. Then there exists a stationary solution $v(x)$ of (4) having the following properties:

(i) $0 < v(x) < 1$ for $x \in \Omega$;
(ii) $v(x)$ is recurrent in the direction e_N;
(iii) there exists an entire solution $\tilde{u}(x,t)$ of (4) such that $\tilde{u}_t > 0$ and that

$$\tilde{u}(x,t) \to \begin{cases} 0 & \text{as } t \to -\infty \\ v(x) & \text{as } t \to +\infty. \end{cases}$$

The situation described in Theorem 5 may be called *partial penetration*. Partial penetration can occur, for example, in a cylindrical domain whose cross-section is dumbell-shaped, as we have discussed earlier in this section.

Thus, in order to prove the existence of a traveling wave, it suffices to show that neither blocking (real or virtual) nor partial penetration occurs. This is a question concerning the non-existence of stationary solutions having certain properties. In this lecture, if I have time, I will discuss the existence and non-existence of such stationary solutions.

References

[1] X. Chen, *Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations*, Adv. Differential Equations, **2** (1997), 125–160.

[2] F. Hamel and N. Nadirashvili, *Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$*, to appear in Arch. Rat. Mech. Anal.

[3] S. Heinze, *Wave solutions to reaction-diffusion systems in perforated domains*, preprint.

[4] H. Matano, *Traveling waves in spatially inhomogeneous diffusive media with bistable nonlinearity I*, in preparation.

[5] H. Matano and M. Mimura, *Pattern formation in competiton-diffusion systems in nonconvex domains*, Publ. Res. Inst. Math. Sci., **19** (1983), 1049–1079.

[6] T. Ogiwara and H. Matano, *Monotonicity and convergence results in order-preserving systems in the presence of symmetry*, Discrete and Continuous Dynamical systems, **5** (1999), 1-34.

[7] W. Shen, Traveling waves in time almost periodic structures governed by bistable nonlinearities I,II, J. Differential Equations, **159** (1999), 1–54 (I), 55-101 (II).

[8] N. Shigesada, K. Kawasaki and E. Teramoto, *Traveling periodic waves in heterogeneous environments*, Theor. Polulation Biol., **30** (1986), 143–146.

[9] J. Xin, *Front propagation in heterogeneous media*, SIAM Review, **42** (2000), 161–230.