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Backward Bifurcation in a HIV/AIDS Epidemic Model with Age Structure II:

The Case of General Transmission Rate

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Abstract

In this short note, we discuss the bifurcation problem for endemic steady states in a HIV/AIDS epidemic model with age structure. By using the Lyapunov-Schmidt type technique, we show a condition to determine the type of bifurcation occurring when the basic reproduction number is crossing the unity. For the case of proportionate mixing assumption, a concrete condition for parameters to produce a backward bifurcation is established.

1 Introduction

In many classical epidemic models, the threshold phenomena can be formulated by using the basic reproduction number, denoted by R_0, which is defined as the expected number of secondary cases produced, in a completely susceptible population, by a typical infected individual during its entire period of infectiousness. Then the epidemiological threshold criterion states that the disease can invade if $R_0 > 1$, whereas it cannot if $R_0 < 1$. Moreover, we oftenly state that there exists an endemic steady state with local stability if $R_0 > 1$ and there is only disease-free steady state if $R_0 < 1$. This means that the bifurcation of nontrivial steady state at $R_0 = 1$ is forward one when we take the basic reproduction number as a bifurcation parameter. Nevertheless it has been pointed out by several authors that the backward bifurcation can occur for more complex (realistic) epidemic models.

In a short note (Inaba 2003), we have considered the bifurcation of endemic steady states in an age-structured model for HIV/AIDS epidemic in homosexual

community. Under the assumption of proportionate mixing, we proved that there could exist multiple endemic steady states even if the basic reproduction number is less than one. In this note, we deal with the bifurcation problem without proportionate mixing assumption.

2 The basic model

In the following, we consider an age-structured population of homosexual men with a constant birth rate. For simplicity, we assume that individuals have sexual contacts with each other at random and the duration of a partnership is negligibly short, so we neglect the effect of persistent partnership. We divide the sexually active homosexual population into two groups: S (uninfected but susceptible) and I (HIV infected). We do not introduce a latent class, since the latent period of AIDS is negligibly short in compare with its long incubation period. Thus all of I-individuals are infectious and will develop full-blown AIDS eventually. We assume that infected individuals with fully developed AIDS symptoms are sexually inactive and hence they are removed from the spread process.

Let $S(t,a)$ be the age-density of susceptible population at time t and age a and let B be the birth rate of susceptible population. Let a denote the age at infection for I-individuals and let $I(t,\tau;a)$ be the density of infected population at time t and *disease-age* (duration since infection) τ. Next let a be the age at which infected individuals have developed AIDS. Let $\mu(a)$ be the age-specific natural death rate (or the rate of terminating sexual life), $\gamma(a;\zeta)$ the rate of developing AIDS and let $\lambda(t,a)$ be the infection rate (the *force of infection*). Then the dynamics of the host population is governed by the following system:

$$S_t(t,a) + S_a(t,a) = -(\mu(a) + \lambda(t,a))S(t,a), \tag{2.1}$$

$$I_t(t,\tau;a) + I_\tau(t,\tau;a) = -(\mu(a+\tau) + \gamma(\tau;a))I(t,\tau;a), \tag{2.2}$$

$$S(t,0) = B, \tag{2.3}$$

$$I(t,0;a) = \lambda(t,a)S(t,a), \tag{2.4}$$

where $S_t = \partial S/\partial t$, etc. The force of infection $\lambda(t,a)$ is assumed to have the following expression:

$$\lambda(t,a) = \frac{C(P(t))}{P(t)} \int_0^\omega \int_0^b \beta(a,b,\tau)I(t,\tau;b-\tau)d\tau db, \tag{2.5}$$

where $P(t)$ is the total size of sexually active population $N(t,a) := S(t,a) + \int_0^a I(t,\tau;a-\tau)d\tau$ given by

$$P(t) := \int_0^\omega N(t,a)da = \int_0^\omega \left[S(t,a) + \int_0^a I(t,\tau;a-\tau)d\tau\right] da,$$

and $C(P)$ denotes the mean number of sexual partners an average individual has per unit time when the population size is P. Typical examples for $C(P)$ is given as follows:

$$(i)\ \ C(P) = \alpha_0 P, \quad (ii)\ \ C(P) = \frac{\alpha_0 \alpha_\infty P}{\alpha_0 P + \alpha_\infty}, \quad (iii)\ \ C(P) = \alpha_\infty. \tag{2.6}$$

The saturating contact law (ii) approaches to *mass action type contact law* (i) when $P \to 0$ and become the homogeneous of degree one (scale independent) contact law (iii) if $P \to \infty$.

In order to simplify system (2.1)-(2.5), let us introduce new functions s, i, n by

$$\begin{cases} S(t,a) = s(t,a)B\ell(a), \\ I(t,\tau;a) = i(t,\tau;a)B\ell(a+\tau)\Gamma(\tau;a), \\ N(t,a) = n(t,a)B\ell(a), \end{cases} \tag{2.7}$$

where $\ell(a)$ and $\Gamma(\tau;a)$ are the *survival functions* defined by

$$\ell(a) := \exp(-\int_0^a \mu(\sigma)d\sigma), \quad \Gamma(\tau;a) := \exp(-\int_0^\tau \gamma(\sigma;a)d\sigma).$$

Then $\ell(a)$ is the probability that an individual survives to age a under the natural death rate and $1-\Gamma(\tau;a)$ gives the *incubation distribution* for individuals infected at age a. Now we obtain the new simplified system for (s,i) as follows:

$$s_t(t,a) + s_a(t,a) = -\lambda(t,a)s(t,a), \tag{2.8}$$

$$i_t(t,\tau;a) + i_\tau(t,\tau;a) = 0, \tag{2.9}$$

$$s(t,0) = 1, \tag{2.10}$$

$$i(t,0;a) = \lambda(t,a)s(t,a), \tag{2.11}$$

$$\lambda(t,a) = \frac{C(P(t))}{P(t)} \int_0^\omega db \int_0^b d\tau K(a,b,\tau)i(t,\tau;b-\tau), \tag{2.12}$$

where

$$K(a,b,\tau) := \beta(a,b,\tau)B\ell(b)\Gamma(\tau;b-\tau),$$

$$P(t) = \int_0^\omega B\ell(a)\left[u(t,a) + \int_0^a \Gamma(\tau;a-\tau)i(t,\tau;a-\tau)d\tau\right]da.$$

Existence and uniqueness of solutions for the basic system (2.8)-(2.12) can be proved by semigroup approach or by classical integral equation approach, though we do not discuss its well-posedness here. The reader may refer to Inaba (2002, 2003) for more information about the basic model.

3 Bifurcation of endemic steady states

Let (s^*, i^*) be the steady state for system (2.7)-(2.11) and let $\lambda^*(a)$ be the force of infection in the steady state. Then it follows that

$$s^*(a) = e^{-\int_0^a \lambda^*(\xi)d\xi}, \quad i^*(\tau; a) = \lambda^*(a)s^*(a).$$

It follows from (2.11) that λ^* must satisfy the nonlinear integral equation as follows:

$$\lambda^*(a) = \frac{C(P(\lambda^*))}{P(\lambda^*)} \int_0^\omega db \int_0^b d\tau K(a,b,\tau)\lambda^*(b-\tau)e^{-\int_0^{b-\tau} \lambda^*(\xi)d\xi}, \tag{3.1}$$

where $P(\lambda^*)$ denotes the size of steady state population with force of infection λ^* given by

$$P(\lambda^*) := \int_0^\omega B\ell(a) \left[e^{-\int_0^a \lambda^*(\xi)d\xi} + \int_0^a \Gamma(a-\tau;\tau)\lambda^*(\tau)e^{-\int_0^\tau \lambda^*(\xi)d\xi} d\tau \right] da.$$

It is clear that $\lambda^* = 0$ is a trivial solution corresponding to a disease-free steady state. Let us define a nonlinear positive operator F on $L^1(0,\omega)$ as follows:

$$F(\lambda)(a) := \frac{C(P(\lambda))}{P(\lambda)} \int_0^\omega db \int_0^b d\tau K(a,b,\tau)\lambda(b-\tau)e^{-\int_0^{b-\tau} \lambda(\xi)d\xi}, \quad \lambda \in L^1.$$

where the Frésche derivative of F at $\lambda = 0$, denoted by $F'[0]$, is the *next-generation operator* given by

$$(F'[0]\psi)(a) := \frac{C(P(0))}{P(0)} \int_0^\omega \int_0^b K(a,b,b-\tau)\psi(\tau)d\tau db.$$

The next generation operator transforms a distribution of infected population to the distribution of secondary cases in the initial invasion phase, so the basic reproduction number R_0 is given by the spectral radius of $F'[0]$, denoted by $r(F'[0])$ (Diekmann, et al. 1990, Diekmann and Heesterbeek 2000). Then it is not difficult to show that the disease-free steady state is locally stable if $R_0 < 1$, and it is unstable if $R_0 > 1$.

Our interest here is to see what kind of bifurcation of endemic steady states could occur at $R_0 = 1$. In order to make our mathematical argument possible, we assume that

Assumption 3. 1 *The next generation operator $F'[0]$ is compact and nonsupporting and $R_0 = r(F'[0]) = 1$.*

A positive bounded linear operator T in a Banach space X with positive cone X_+ is called *nonsupporting* if and only if for every pair $\psi \in X_+ \setminus \{0\}$ and $\phi^* \in X_+^* \setminus \{0\}$, there exists a positive integer $p = p(\psi, \phi^*)$ such that $< T^n\psi, \phi^* >> 0$ for all $n \geq p$. If $F'[0]$ is nonsupporting, its spectral radius $r(F'[0])$

is a simple isolated positive eigenvalue whose eigenspace is one-dimensional. The eigenvector (could be called as the Frobenius eigenvector) corresponding to $r(F'[0])$ is a quasi-interior point in X_+ and any other possible eigenvector in X_+ is proportional to the Frobenius eigenvector. The eiegnspace of the adjoint operator $F'[0]^*$ corresponding to $r(F'[0])$ is also one-dimensional subspace of X^* spanned by a strictly positive (eigen)functional. For more detail properties of nonsupporting operator, the reader may refer to Sawashima (1964), Marek (1970) and Inaba (1990, 2002).

Let ϵ be a bifurcation parameter and let us define

$$\Psi(\lambda, \epsilon) := \epsilon F(\lambda) - \lambda, \quad (\lambda, \epsilon) \in L^1(0, \omega) \times \mathbf{R}_+,$$

and we assume that $\Psi(\lambda, \epsilon)$ is analytic with respect to (λ, ϵ).

We are interested in the structure of solution set

$$\Psi^{-1}(0) := \{(\lambda, \epsilon) \in L^1(0, \omega) \times \mathbf{R}_+ : \Psi(\lambda, \epsilon) = 0.\} \tag{3.2}$$

From the Implicit Function Theorem, we can expect a bifurcation from the trivial branch $(0, \epsilon)$ only for those values ϵ such that the linear mapping

$$L(\epsilon) := D_1\Psi(0, \epsilon) = \epsilon F'[0] - I,$$

is not boundedly invertible, where D_1 denotes the Frechet derivative for the first element and I is the identity operator. Since $F'[0]$ has a unique positive eigenvalue one, hence the only possible bifurcation from the trivial branch can occur at $\epsilon = 1$.

Let $\sigma(\epsilon) = \epsilon - 1$ be the simple real strictly dominant eigenvalue of $L(\epsilon)$, $\phi(\epsilon)$ the eigenvector of $L(\epsilon)$ and $\phi^*(\epsilon)$ the eigenvector of $L^*(\epsilon)$ (the adjoint operator of $L(\epsilon)$) associated with $\sigma(\epsilon)$ such that

$$< \phi(\epsilon), \phi^*(\epsilon) >= 1,$$

where $< \phi, \phi^* >$ is the value of ϕ^* at ϕ.

According to Britton (1986), in order to look for the bifurcating steady solution (λ, ϵ) of $\Psi(\lambda, \epsilon) = 0$ around the trivial solution $(0, 1)$, we expand both λ and ϵ in terms of a small parameter α so that

$$\lambda = \sum_{n=1}^{\infty} \alpha^n \lambda_n, \quad \epsilon = 1 + \sum_{n=1}^{\infty} \alpha^n \epsilon_n, \tag{3.3}$$

where we take

$$\alpha =< \lambda, \phi^*(1) >, \quad \lambda_1 = \phi(1).$$

Note that $\phi(1)$ is the Frobenius eigenvector of $F'[0]$ corresponding to the eigenvalue one. This short cut method to construct the bifurcating solution can be justified by the well-known Lyapunov-Schmidt Theory (Temme, 1978).

It follows from the above definition and (3.3) that

$$< \lambda_1, \phi^*(1) >= 1, \quad < \lambda_n, \phi^*(1) >= 0, \ n > 1.$$

Substituting the expansion (3.3) into the equation $\Psi(\lambda, \epsilon) = 0$ and equating power of α, we have

$$D_1\Psi(0,1|\lambda_1) = 0, \tag{3.4}$$

$$D_1\Psi(0,1|\lambda_2) + \epsilon_1 D_1 D_2 \Psi(0,1|\lambda_1) + \frac{1}{2} D_1^2 \Psi(0,1|\lambda_1|\lambda_1) = 0. \tag{3.5}$$

From the Fredholm Alternative, (3.5) has a solution if and only if

$$< \epsilon_1 D_1 D_2 \Psi(0,1|\lambda_1) + \frac{1}{2} D_1^2 \Psi(0,1|\lambda_1|\lambda_1), \phi^*(1) >= 0,$$

where we can observe that

$$< D_1 D_2 \Psi(0,1|\lambda_1), \phi^*(1) >=< F'[0]\phi(1), \phi^*(1) >= 1.$$

Therefore we have

$$\epsilon_1 = -\frac{1}{2} < D_1^2 \Psi(0,1|\lambda_1|\lambda_1), \phi^*(1) > . \tag{3.6}$$

Then we can conclude the following bifurcation result:

Proposition 3. 2 *The bifurcation at* $(0,1)$ *is subcritical if* $\epsilon_1 < 0$, *and it is supercritical if* $\epsilon_1 > 0$.

The partial derivative $D_1^2\Psi(0,1|\lambda_1|\lambda_1)$ can be calculated as follows:

$$D_1^2\Psi(0,1|\lambda_1|\lambda_1) = \frac{\partial^2}{\partial h \partial k} F((h+k)\lambda_1)\Big|_{(h,k)=(0,0)}$$

$$= 2\left[\frac{C'(P(0))}{C(P(0))} - \frac{1}{P(0)}\right] P'(0)\lambda_1 - 2F'[0]\psi,$$

where we have used the fact that $F'[0]\lambda_1 = \lambda_1$ and ψ, $P(0)$ and $P'(0)$ are given by

$$\psi(a) := \lambda_1(a)\exp(-\int_0^a \lambda_1(\sigma)d\sigma),$$

$$P(0) = \int_0^\omega B\ell(a)da,$$

$$P'(0) = -\int_0^\omega B\ell(a)\int_0^a (1 - \Gamma(a-\tau;\tau))\lambda_1(\tau)d\tau da. \tag{3.7}$$

Then the following corollary directly follow from the above proposition:

Corollary 3. 3 *The bifurcation at* $(0,1)$ *is supercritical if*

$$C'(P(0)) \geq \frac{C(P(0))}{P(0)}. \tag{3.8}$$

In particular, if the number of contacts per unit time $C(P)$ *is proportional to the host population size* P *(the mass action law), the bifurcation is supercritical.*

In order to proceed the above calculation, let us assume that $C(P) \equiv C_0$, that is, the average number of contacts is constant C_0. In this case, we obtain

$$D_1^2\Psi(0,1|\lambda_1|\lambda_1) = -2\frac{P'(0)}{P(0)}F'[0]\lambda_1 - 2F'[0]\psi, \tag{3.9}$$

Therefore we have

$$\epsilon_1 = \frac{P'(0)}{P(0)} + < F'[0]\psi, \phi^*(1) > . \tag{3.10}$$

Furthermore, let us assume that the *proportionate mixing assumption* holds, that is, the kernel K is decomposed as $K(a,b,\tau) = k_1(a)k_2(b,\tau)$. For biomathematical roots of this assumption, the reader may refer to Dietz and Schenzle (1985). In this special case, the Frobenius eigenvector corresponding to the eigenvalue one is given by k_1 and the next generation operator is a one-dimensional map given by

$$F'[0]\phi = \left(\frac{C_0}{P(0)}\int_0^\omega \int_0^b k_2(b,b-\tau)\phi(\tau)d\tau db\right)k_1, \tag{3.11}$$

and its spectral radius can be expressed as

$$R_0 = r(F'[0]) = \frac{C_0}{P(0)}\int_0^\omega \int_0^b k_2(b,b-\tau)k_1(\tau)d\tau db. \tag{3.12}$$

Note that by our assumption 3.1, $R_0 = r(F'[0]) = 1$. Then for any $\phi \in L^1$, it follows that

$$< \phi, \phi^*(1) > = < \phi, F'[0]^*\phi^*(1) > = < F'[0]\phi, \phi^*(1) >$$

$$= < k_1, \phi^*(1) > \frac{C_0}{P(0)}\int_0^\omega \int_0^b k_2(b,b-\tau)\phi(\tau)d\tau db.$$

If we denote $\phi^*(1)$ as the adjoint eigenvector of $F'[0]$ corresponding to the eigenvalue one such that $< k_1, \phi^*(1) > = 1$, we have

$$< \phi, \phi^*(1) > = \frac{C_0}{P(0)}\int_0^\omega \int_0^b k_2(b,b-\tau)\phi(\tau)d\tau db. \tag{3.13}$$

That is, we obtain

$$F'[0]\phi =< \phi, \phi^*(1) > k_1. \tag{3.14}$$

By using the above fact, under the assumption of proportionate mixing, we can calculate ϵ_1 as

$$\epsilon_1 = \frac{P'(0)}{P(0)} + < F'[0]\psi, \phi^*(1) >$$

$$= -\frac{1}{P(0)} \int_0^\omega B\ell(a) \int_0^a (1 - \Gamma(a-\tau;\tau))k_1(\tau)d\tau da$$

$$+ \frac{C_0}{P(0)} \int_0^\omega \int_0^b k_2(b, b-\tau)k_1(\tau) \int_0^\tau k_1(\zeta)d\zeta d\tau db.$$

Since $R_0 = 1$ and $P(0) = \int_0^\omega B\ell(a)da$, it follows from (3.12) that

$$B = \frac{C_0}{\int_0^\omega \ell(a)da} \int_0^\omega \int_0^b k_2(b, b-\tau)k_1(\tau)d\tau db.$$

Then using Proposition 3.2, we arrive at the following statement:

Proposition 3. 4 *Suppose that $C(P) \equiv 1$ and the kernel K is decomposed as $K(a, b, \tau) = k_1(a)k_2(b, \tau)$. Then the bifurcation at $(0, 1)$ is subcritical if and only if*

$$\int_0^\omega \frac{\ell(a)}{\int_0^\omega \ell(a)da} \int_0^a (1 - \Gamma(a-\tau;\tau))k_1(\tau)d\tau da$$

$$> \frac{\int_0^\omega \int_0^b k_2(b, b-\tau)k_1(\tau) \int_0^\tau k_1(\zeta)d\zeta d\tau db}{\int_0^\omega \int_0^b k_2(b, b-\tau)k_1(\tau)d\tau db}. \tag{3.15}$$

It is easy to see that the condition (3.15) is the same as the condition given by Proposition 3.1 in Inaba (2003), and this condition is independent from the average number of contacts C_0. By simple calculation, we know that if k_1, k_2 and γ are constant, the condition (3.15) does not hold, so the bifurcation is forward.

4 Discussion

From the above argument, we know that a backward bifurcation of endemic steady states is possible for the HIV/AIDS epidemic model. The presence of a backward bifurcation has practically important consequences for the control of infectious diseases. If the bifurcation of endemic state at $R_0 = 1$ is forward one, the size of infected population will be approximately proportional to the difference $|R_0 - 1|$. On the other hand, in a system with a backward bifurcation,

the endemic steady state that exists for R_0 just above one could have a large infectious population, so the result of R_0 rising above one would be a drastic change in the number of infecteds. Conversely, reducing R_0 back below one would not eradicate the disease, as long as its reduction is not sufficient. That is, if the disease is already endemic, in order to eradicate the disease, we have to reduce the basic reproduction number so far that it enters the region where the disease-free steady state is globally asymptotically stable and there is no endemic steady state. Our results suggest that HIV/AIDS dynamics would be more complex in compare with common airborne diseases.

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