数理解析研究所講究録 1337 巻 2003 年 166-175

Adaptive expectation generates the Harrodian Instability

Hiroyuki KATO*
Graduate School of Economics, Keio University

Abstract

This paper studies an instability and a cycle of a capital accumulation path which permits a disequilibrium of labor market. We show that an adaptive expectation plays an essential role for the instability which does not recover. This result also explains why two sided investment properties give rise to a Harrod's knife edge. In addition, we show an existence of business cycles by considering a government expenditure as endogenous variable.

1 Introduction

There are two traditional conflicting view about the working of a market economy. Classical school believes that the economy is by nature well-behaved and stable, unless disturbances from outside–whether originating from the external environment or from policy–are injected into the economy. But Keynesians do not believe in the controllability of the economy because of their serious view of its ill-behavedness. So they believe the persistency of a disequilibrium of a labor market whereas classical school has a belief that markets clear at every time unless exogenous shocks occur. One of the most important difference between the classical school and the Keynesians is the way of an expectation about a future economy. Models of the classical vintage typically assume that every individual's assesment of the future is correct at any moment given his information.
According to the Keynesian view, a change of expectations are likely to occur rather in an unpredictably way and are potentially major sources of endogenous business fluctuations(market phychology, animal spirits). In a classical

*E-mail adress: dk025205@mita.cc.keio.ac.jp

context for example, Friedman and Lucas explain a variation of a national income by a seperation of an expected inflation rate and an actual one and conclude that only a correct expectation supports an equilibrium of a labor market. Grandmont, Azariadis and Guenerie show an existence of cycles even under a perfect foresight or a rational expectation in a monetary economy.
On the other hand, in a Keynesian context, Harrod considered that a cause of a fluctuation of an economy is two sided properties of an inventment. One side of them is an effect of an investment on demand side, the other is on a supply side by a variation of a capital stock. If there is an excess demand, a firm feels capital shortage and increases an investment. Then the demand increases and the excess demand does not recover. If there is a excess supply, the opposite occurs. That is a so called "Harrod's Knife Edge". Nikaido(1975, 1980) and Yosida(1999), for example, describes the above manner and specialize a firm's investment behavior. [1] But these are ambiguous on the reason why the firm behaves such a way.

In this paper, we show an occurrence of the "Harrodian" instability which means that disequilibriums of markets persist essentially depends upon a difficulty of a correct expectation of a future total demand. There are few literatures on the Harrodian instability taking account of an expectation. (except Futagami(1991) and Adachi(2000)) We revice Adachi's model and make clear a relationship of an adaptive expectation and a instability. In addition, we study business cycles by considering a government expenditure as an endogenous variable. For this purpose, we use the Hopf bifurcation theorem which shows sufficient conditions for the existence of nontrivial periodic solutions.

2 The Model

We consider a representative firm which produces a single good, workers(consumers, capital stock-holders) and a government.

[1] In Nikaido(1980),

$$(\frac{\dot{I}}{K}) = \phi(\frac{I}{K} - \frac{sF(K,L)}{K})$$

I, K, L, s, F are intended investment, capital stock, population, rate of saving, production function respectively. ϕ is a signe preserving function and $\dot{}$ is a time derivative

2.1 The Firm's behavior

2.1.1 The determination of capital stock

assumption1.
The production function $F(K,N)$ is in $C^2(R^2_+, R_+)$ and homogenous of degree one. $f'(x)>0,\quad f''(x)<0$ where $f(x)\equiv F(1,\frac{N}{K})(x\equiv\frac{N}{K})$ K and N are a capital stock and a labor.

Let Y^e be a level of demand long term ahead which a representative firm predicts. Let (w^e, r^e) be a factor price vector which the firm forcasts. Let $x^e=\frac{N}{K}(w^e/r^e)$ be an optimal factor employment which satisfies $f'(x^e)=w^e$, $f(x^e)-x^e f'(x^e)=r^e$. Because the production function is homogenity of degree one, x^e is uniquely determined depending on only w^e/r^e. Then a capital stock the firm desired corresponding to Y^e is

$$K^e\equiv\frac{1}{f(x^e)}Y^e\qquad (f(x^e)\equiv F(1,\frac{N}{K}(w^e/r^e))).$$

We consider the firm's investment is determined by the follwing.

$$\dot{K}\equiv I\equiv\epsilon(K^e-K)$$

$\epsilon\in(0,1)$ means the term of expectation. K is a current capital stock. Let $Y=F(K,N)$ be a current productoin level. Therefore

$$I=\epsilon(\frac{1}{f(x^e)}(Y^e-Y)+\frac{1}{f(x^e)}Y-K)$$

$$\frac{I}{K}=\epsilon(\frac{1}{f(x^e)}\frac{Y}{K}\frac{Y^e-Y}{Y}+\frac{1}{f(x^e)}\frac{Y}{K}-1)$$

$$=\epsilon(\frac{f(x)}{f(x^e)}\frac{Y^e-Y}{Y}+\frac{f(x)}{f(x^e)}-1)$$

We consider $\frac{Y^e-Y}{Y}$ as an expected rate of demand and put $\frac{Y^e-Y}{Y}\equiv g_e$. We write

$$\frac{I}{K}=i(x,g_e,x^e)$$

For simplicity we assume that x^e is a fixed number and put $x^e\equiv\bar{x}>0$ because we consider the expected factor prices are not so flexible. Then we write,

$$\frac{I}{K}=i(x,g_e)$$

2.1.2 The determination of labor

For each time t, a representative firm maximize a following problem because a K_t is not a control variable at t (K_t is determined at $t - \Delta$)

$$\max_{N_t}[P_t F(K_t, N_t) - W_t N_t]$$

Then the aggregate supply function at t is

$$AS(K_t, W_t, P_t) = \{F(K_t, N_t) \in R \quad | \quad F_N(K_t, N_t) = \frac{W_t}{P_t}\}$$

2.2 The demand side

2.2.1 The saving function

A total saving consists of a private sector and a government sector. We define as follows.

$$S_p = Y - T - C$$

$$S_g = T - G$$

where Y, T, C, G are a national income, a tax, a consumption, a government expenditure respectively. We specialize a consumption function such as

$$C = \alpha(Y - T) + \bar{C}$$

Let $\tau > 0$ be a rate of tax on Y. We write,

$$T = \tau Y$$

Then the total saving S is

$$\begin{aligned} S &= S_p + S_g \\ &= (Y - T - C) + (T - G) \\ &= (Y - \tau Y - \alpha Y + \alpha\tau Y - \bar{C}) + (\tau Y - G) \\ &= (1 - \alpha + \alpha\tau)Y - (\bar{C} + G) \end{aligned}$$

Put $a \equiv \frac{\bar{C}+G}{K}$, then

$$\begin{aligned} \frac{S}{K} &= (1 - \alpha + \alpha\tau)f(x) - a \\ &\equiv s(x, a) \end{aligned}$$

So the aggregate demand function at t is

$$AD(K_t, K_t^e, G_t) = \{Y_t \in R \quad | \quad Y_t = C(Y_t) + I(K_t, K_t^e) + G_t\}$$

3 The Dynamical system

We assume that W_t/P_t is determined such as $AD(K_t, K_t^e, G_t) = AS(K_t, W_t, P_t)$ at each time because an adjustment of P_t is instantaneous. So at each time,

$$i(x, g_e) = s(x, a)$$

To guarantee the above setting, we have to postulate that the adjustment is stable.

assumption2.
$i_x(x, g_e) \neq s_x(x, a)$ for all (x, g_e, a) with $i(x, g_e) = s(x, a)$

Because the equation $i(x, g_e) = s(x, a)$ is always met, we can apply the implicit function theorem to $i(x, g_e) - s(x, a) = 0$ about x by assumption2. Then there exists locally defined C^2 function $x(\cdot, \cdot)$ which satisfies $i(x(a, g_e), g_e) = s(x(a, g_e), a)$. Now let (K_t, G_t, g_e) be given. If there is an excess demand, that is, $s < i$, P_t increases and a firm increases production by an increase of labor(so x) because W_t/P_t is down. In case of an excess supply, the opposite occurs. Therefore, we need to assume the inequality $i_x < s_x$ for the above adjustment process to be stable. But the function $x(a, g_e)$ is defined only locally because that is led by the implicit function theorem. So, there may be multiple equilibria x for a given (a, g_e) depending on shapes of i and s. But in the following, we only consider equilibria which is stable, that is, x which satisfy $i(x, g_e) = s(x, a)$ and $i_x(x, g_e) < s_x(x, a)$ for a given (a, g_e).
Then, we get

$$x_a(a, g_e) = -\frac{-s_a(x, a)}{i_x(x, g_e) - s_x(x, a)} > 0$$

$$x_{g_e}(a, g_e) = -\frac{i_{g_e}(x, g_e)}{i_x(x, g_e) - s_x(x, a)} > 0$$

We consider a dynamical system of g_e as follows.

$$\dot{g_e} = \mu[i(x, g_e) - i(\bar{x}, g_e)]$$

$\mu > 0$ is an adjustment parameter. The above system means that if a real level of demand is higher(resp. lower) than an expected level of demand, an expected rate of growth g_e is up(resp. down).
We first consider the case without a government expenditure, especially

$a = 0$. A unique solution of $x(0, g_e) = \bar{x}$ is $g_e = \frac{1}{\varepsilon}(1 - \alpha + \alpha\tau)f(\bar{x})$. This is a fixed point of the above dynamical system in case of $a = 0$. We assume the following inequality so that the product market is stable.

assumption3.
$i_x(\bar{x}, \frac{1}{\varepsilon}(1 - \alpha + \alpha\tau)f(\bar{x})) < s_x(\bar{x})$

Theorem1(Harrodian Insability without a government expenditure).
If $a = 0$, the fixed point of the system

$$\dot{g}_e = \mu[i(x(g_e), g_e) - i(\bar{x}, g_e)]$$

is locally unstable.[2]

(proof)
Because this is a one dimensional differntial equation, we only check the derivative of the dynamical system at the fixed point.

$$\frac{d\dot{g}_e}{dg_e}|_{g_e=\frac{1}{\varepsilon}(1-\alpha+\alpha\tau)f(\bar{x})} = \mu i_x x_{g_e} > 0$$

Then the system is unstable around the fixed point. (Q.E.D)

Remark:The conclution of the above theorem essencially depends on two sided inventment properties and a lack of a rationality of a firm. If a real demand is higher than a firm's expectation, the firm revices his expectation upward and increases an investment. But this increases an aggregate demand at the same time, then the seperation of an expected demand and a real become larger. If a real demand is less than a firm's expectation, the opposite will occur. This process leads to an instability of the system.

Next, we consider a case of $a \neq 0$.
We consider about a labor market.[3] A labor demand curve $F_N(K, N) = \frac{W}{P}$ shifts right if K increases because of $F_{NK}(K, N)(= -\frac{N}{K^2}f''(x)) > 0$ by assumption 1. Since $F_N(K, N) = f'(\frac{N}{K})$, if K increases at the rate of $\frac{\dot{K}}{K}$ and

[2]Let x^* be a fixed point of a differential equation $\dot{x} = f(x)$. x^* is locally unstable iff $\exists\delta > 0 \quad \forall\xi(\| \xi - x^* \| < \delta) \quad \exists\bar{t} \geq t_0 \| \phi(t, \xi) - x^* \| \geq \delta \quad \forall t \geq \bar{t}$
where $\phi(t, \xi)$ is a solution of $\dot{x} = f(x)$ with a initial condetion (t_0, ξ)

[3]In this model, we only see an excess supply of labor market, that is, a real wage rate W_t/P_t is sufficiently high because of a shortage of an aggregate demand. Let a capital stock be always fully employed. So $F(K, N)$ is produced with $F_N(K, N) = W/P$.

N also increases at $\frac{\dot{N}}{N} = \frac{\dot{K}}{K}$, $F_N(K,N)$ is invariable. Therefore, the labor demand curve shifts right at the rate of $\frac{\dot{K}}{K} = \frac{\dot{N}}{N}$ If K decreases, the opposite occurs.
We assume a supply of labor is a total population. Let $g_n > 0$ be a rate of population growth.

Since $\frac{\dot{K}}{K} = i(x(g_e,a),g_e)$, we consider the government recoginaizes the following way. If $g_n > i(x(g_e,a),g_e)$, he thinks an involuntary unemployment increases, if $g_n < i(x(g_e,a),g_e)$, the opposite will occur. Now we consider a dynamical system of G.

$$\frac{\dot{G}}{\bar{C}+G} = \beta(g_n - i(x(g_e,a),g_e))$$

So the dynamical system of a is

$$\frac{\dot{a}}{a} = \frac{\dot{G}}{\bar{C}+G} - \frac{\dot{K}}{K}$$

$$= \beta(g_n - i(x(g_e,a),g_e)) - i(x(g_e,a),g_e)$$

Then we get

$$\dot{a} = a(\beta g_n - (1+\beta)i(x(g_e,a),g_e))$$

We assume that $g_n > 0$ is a fixed number. $\beta > 0$ is an adjustment parameter.

4 The existence of Hopf cycles

We consider the following dynamical systems.

$$\begin{cases} \dot{g}_e = \mu[i(x(g_e,a),g_e) - i(\bar{x},g_e)] \\ \dot{a} = a(\beta g_n - (1+\beta)i(x(g_e,a),g_e)) \end{cases}$$

We put the following assumption for the fixed point of the above dynamical system to be in an interior of nonnegative orthant.

assumption4
$(1-\alpha+\alpha\tau)f(\bar{x}) - \frac{\beta}{1+\beta}g_n > 0$

This assumption is met for sufficiently small β.
We write the above dynamical system as

$(\dot{g}_e, \dot{a}) = H(g_e, a, \mu)$
$(a^0 \equiv (1-\alpha+\alpha\tau)f(\bar{x}) - \frac{\beta}{1+\beta}g_n, \frac{\beta}{\varepsilon(1+\beta)}g_n)$ is a fixed point of H beause of the equation $i(\bar{x}, \frac{\beta}{\varepsilon(1+\beta)}g_n) = s(\bar{x}, (1-\alpha+\alpha\tau)f(\bar{x}) - \frac{\beta}{1+\beta}g_n) = \frac{\beta}{1+\beta}g_n$.
We assume the following inequality for the product market to be stable at the fixed point.

assumption5.
$i_x(\bar{x}, \frac{\beta}{\varepsilon(1+\beta)}g_n) < s_x(\bar{x}, (1-\alpha+\alpha\tau)f(\bar{x}) - \frac{\beta}{1+\beta}g_n)$

We use a mathematical theorem of the following form.

Hopf bifurcation theorem(Lorenz[1], Marsden and McCradken[3], Wiggins[6])
Let $\dot{x} = X(x, \mu)$ be in $C^k(R^2 \times R, R^2)(k \geq 2)$. Suppose $X((x_1^*, x_2^*), \mu) = 0$ for all $\mu \in R$ and let $D_x X((x_1^*, x_2^*), \mu)$ have two distinct , complex conjugate eigenvalues $z_{\pm}(\mu) = \alpha(\mu) \pm i\beta(\mu)$ such that $\alpha(\mu^*) = 0$, $\alpha'(\mu^*) \neq 0$ and $\beta(\mu^*) > 0$. Then, there is an $\varepsilon > 0$ and C^{k-1} function $\mu^* : (x_1^* - \varepsilon, x_1^* + \varepsilon) \to R$ such that $\mu^*(x_1^*) = \mu^*$ and such that for each $x_1 \in (x_1^*, x_1^* + \varepsilon)$, the point $(x_1, x_2^*) \in R^2$ lies on a closed orbit of $X(\cdot, \mu^*(x_1))$ with period approximately $2\pi/\beta(\mu^*)$ and which contains the origin in its interior.

Theorem 2(The existence of business cycles).
Put $\mu^* \equiv a^0(1+\beta)x_a/x_{g_e}$
There exists an $\varepsilon > 0$ and a C^1 function μ^*: $(a^0 - \varepsilon, a^0 + \varepsilon) \to R$ such that $\mu^*(a^0) = \mu^*$ and such that for each $a_1 \in (a^0, a^0 + \varepsilon)$, the point $(a_1, \frac{\beta}{\varepsilon(1+\beta)}g_n) \in R^2$ lies on a closed orbit of $H(\cdot, \cdot, \mu^*(a_1))$ which contains the $(a^0, \frac{\beta}{\varepsilon(1+\beta)}g_n)$ in its interior.

(proof)
We only have to check following conditions because of the Hopf bifurcation theorem
1.$H \in C^2$.
Because F is C^2 function, i and s are C^2 functions too. By the implicit function theorem x is C^2 class. Then $H \in C^2$.

2.$H(\frac{\beta}{\varepsilon(1+\beta)}g_n, a^0, \mu) = 0$ for all $\mu \in R$.
By definition of H, it is clear.

3.$D_{(g_e,a)}H(\frac{\beta}{\varepsilon(1+\beta)}g_n, a^0, \mu)$ has two distinct, complex conjugate eigenvalues $z(\mu)$ such that $\mathrm{Re}z(\mu^*) = 0$ and $\frac{d\mathrm{Re}z(\mu)}{d\mu}|_{\mu=\mu^*} \neq 0$.

$$D_{(g_e,a)}H(\frac{\beta}{\varepsilon(1+\beta)}g_n, a^0, \mu)$$
$$= \begin{pmatrix} \mu i_x x_{g_e} & \mu i_x x_a \\ -a^0(1+\beta)(i_x x_{g_e} + i_{g_e}) & (\beta g_n - (1+\beta) i(x(\frac{\beta}{\varepsilon(1+\beta)}g_n, a^0), \frac{\beta}{\varepsilon(1+\beta)}g_n) - a^0(1+\beta)(i_x x_a) \end{pmatrix}$$
$$= \begin{pmatrix} \mu i_x x_{g_e} & \mu i_x x_a \\ -a^0(1+\beta)(i_x x_{g_e} + \varepsilon) & -a^0(1+\beta) i_x x_a \end{pmatrix}$$

The characteristic equation is
$Q(z) = z^2 - [\mu i_x x_{g_e} - a^0(1+\beta) i_x x_a] z + [\{-a^0(1+\beta)\mu(i_x x_{g_e} i_x x_a)\} + \{a^0(1+\beta)\mu(i_x x_{g_e} + \varepsilon) i_x x_a\}]$
So the eigenvalue is
$z(\mu) = \frac{1}{2}\{[\mu(i_x x_{g_e}) - a^0(1+\beta) i_x x_a] \pm \sqrt{\Delta(\mu)}\}$
where $\Delta(\mu) = [\mu(i_x x_{g_e}) - a^0(1+\beta) i_x x_a]^2 - 4[\{-a^0(1+\beta)\mu(i_x x_{g_e}) i_x x_a\} + \{a^0(1+\beta)\mu(i_x x_{g_e} + \varepsilon) i_x x_a\}]$
Because of $\mu^* \equiv a^0(1+\beta) x_a / x_{g_e}$, we get $\mathrm{Re}z(\mu^*) = 0$。
Then $\Delta(\mu^*) = -4[\{-a^0(1+\beta)\mu^*(i_x x_{g_e}) i_x x_a\} + \{a^0(1+\beta)\mu^*(i_x x_{g_e} + \varepsilon) i_x x_a\}]$
$= -4\varepsilon[a^0(1+\beta) x_a]^2 \frac{i_x}{x_{g_e}}$
< 0
So there exists $\delta > 0$ such that for all $\mu > 0$ with $|\mu - \mu^*| < \delta$, $\Delta(\mu) < 0$
Then $z(\mu)$ are two complex conjugate eigenvalues. In addition,
$\frac{dRez(\mu)}{d\mu}|_{\mu=\mu^*} = \frac{1}{2} i_x x_{g_e} \neq 0$
So all conditions of the Hopf bifurcation theorem are satisfied. Therefore the proof is complete. (Q.E.D)

We showed the existence of cycles of (g_e, a) by theorem2, then the following is clear.

Corollary.
A real rate of capital accumuration $\frac{\dot{K}}{K} = i(x(g_e, a), g_e)$ has cycles for each μ which is sufficiently close to $\mu^* \equiv a^0(1+\beta) x_a / x_{g_e}$ and for some initial condition.

References

[1] Hans-Walter Lorenz (1989), *Non linear Dynamical Economics and Chaotic Motion* , Springer Verlag

[2] Harrod(1973) *Economic Dynamics* , London, Macmillan （宮崎義一訳「ハロッド経済動学」丸善，昭和51年）

[3] Marsden and McCradken (1976), *The Hopf Bifurcation and its Applications* , Springer New York

[4] Nikaido (1975) "Factor Substitution and Harrod's Knife-Edge" , *Zeitschrift für Nationalökonomie*, 35, 149-154

[5] Nikaido (1980) "Harrodian Pathology of Neoclassical Growth:The Irrelevance of Smooth Factor Substitution" , *Zeitschrift für Nationalökonomie*, 40, 111-134

[6] Wiggins (1990) *Introduction to Applied Nonlinear Systems and Chaos* , Springer-Verlag

[7] Yosida (1999) "Harrod's "Knife-Edge" Reconsidered: An Application of the Hopf Bifurcation Theorem and Numerical Simulations", *Journal of Macroeconomics*, 21, 537-562

[8] 足立　英之 (2000) 不完全競争とマクロ動学理論， 有斐閣

[9] 二神　孝一　(1991) "準成長循環-Goodwin, Harrod-", *The Economic Studies Quarterly*, 42, 164-173